Homework 7, Quantum Mechanics 501, Rutgers

December 11, 2016

1) Consider a system of two non-identical fermions, each with spin 1/2. One is in a state with $S_{1x} = \frac{\hbar}{2}$, while the other is in a state with $S_{2y} = -\frac{\hbar}{2}$. What is the probability of finding the system in a state with total spin quantum numbers $s = 1$, $m_s = 0$, where m_s refers to the z-component of the total spin?

   a) First, find the eigenstate of the operator S_{1x} with the eigenvalue $\frac{\hbar}{2}$. Also find the eigenstate of S_{2y} with the eigenvalue $-\frac{\hbar}{2}$.

   b) Using the rules for sumation of angular momenta, find the expression for the state $|s = 1, m_s = 0\rangle$.

   c) Calculate the probability.

2) Consider two spin-1 particles that occupy the state

$$|s_1 = 1, m_1 = 1; s_2 = 1, m_2 = 0\rangle \,.$$

What is the probability of finding the system in an eigenstate of the total spin S^2 with quantum number $s = 1$? What is the probability for $s = 2$?

3) a) Construct the spin singlet ($S = 0$) state and the spin triplet ($S = 1$) states of a two electron system.

   b) In the experiment we have two electrons, which are in the spin-singlet state. They move in the opposite direction along the y-axis, and two observers A and B measure the spin state of each electron. A measures the spin component along the z axis, and B measures the spin component along an axis making an angle θ with the z axis in the xz-plane. Suppose that A's measurement yields a spin down state and subsequently B makes a measurement. What is the probability that B's measurement yields an up spin (measured along an axis making an angle θ with the z-axis)?

   The explicit formula for the representation of the rotation operator $\exp(-i\mathbf{S}\cdot\hat{\mathbf{n}}\theta/\hbar)$ in the spin space is given by the spin 1/2 Wigner matrix

$$D^{(1/2)}(\hat{\mathbf{n}}, \theta) = \begin{pmatrix} \cos(\theta/2) - in_z\sin(\theta/2) & (-in_x - n_y)\sin(\theta/2) \\ (-in_x + n_y)\sin(\theta/2) & \cos(\theta/2) + in_z\sin(\theta/2) \end{pmatrix} \tag{1}$$

   and $\hat{\mathbf{n}} = n_x\vec{e}_x + n_y\vec{e}_y + n_z\vec{e}_z$ ($|\hat{\mathbf{n}}| = 1$) is the axis of rotation.

4) The Wigner-Eckart theorem s given by

$$\langle n'j'm'|T_q^{(l)}|njm\rangle = \langle j'm'|lq,jm\rangle \frac{\langle\langle n'j'|T^{(l)}|nj\rangle\rangle}{\sqrt{2j+1}} \tag{2}$$

a) Explain the meaning of the two terms on the right hand side.

b) The interaction of the electromagnetic field with a charged particle is given by

$$\Delta H = \frac{e}{2m}\mathbf{A}\cdot\mathbf{p}$$

If the electromagnetic fields are in the form of a plane wave, then $\mathbf{A} = A_0\,\hat{\varepsilon}e^{i\mathbf{kr}}$, where $\hat{\varepsilon}$ is the polarization of the plane wave. Assuming that the wavelength $\lambda = 2\pi/k$ is much larger than the atomic size, we may write

$$\mathbf{A} = A_0\hat{\varepsilon}(1 + i\mathbf{k}\cdot\mathbf{r} + \cdots)$$

such that

$$\Delta H \approx \frac{e}{2m}A_0\,\hat{\varepsilon}\cdot\mathbf{p}(1 + i\mathbf{k}\cdot\mathbf{r})$$

Here we kept both the dipole (the first term), and the quadrupole terms (the second term).

If the field is polarized along the x-axis ($\hat{\varepsilon} = \vec{e}_x$), and the wave propagation is along the z-axis ($\mathbf{k} = k\vec{e}_z$) express the Hamiltonian in terms of spherical harmonics. Note that $\mathbf{p}$ is a vector operator, and transforms under rotation as $\mathbf{r}$. For symmetry consideration you may therefore replace $\mathbf{p}$ by $C\mathbf{r}$

c) For the above configuration, derive the selection rules for the dipole and the quadrupole transitions, by considering the transition probability matrix elements $|\langle\psi_f|\Delta H|\psi_i\rangle|^2 = |\langle l_f m_f|\Delta H|l_i m_i\rangle|^2$. Note: selection rules state under which conditions is a transition possible.

The explicit expressions for the spherical harmonics for $l = 1, 2$ are given by

$$Y_{1,1} = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\frac{x+iy}{r} \qquad Y_{1,0} = \frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{z}{r} \tag{3}$$

$$Y_{2,2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\frac{(x+iy)^2}{r^2} \qquad Y_{2,1} = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\frac{(x+iy)z}{r^2} \qquad Y_{2,0} = \frac{1}{4}\sqrt{\frac{5}{\pi}}\frac{2z^2-x^2-y^2}{r^2} \tag{4}$$

and $Y_{l,-m} = (-1)^m Y_{l,m}^*$.