Homework 6, Quantum Mechanics 501, Rutgers

December 6, 2016

1) Using the matrix elements of the operator L_x in the subspace for $l = 1$ derived in the previous homework, show that the matrix for arbitrary rotations around the x-axis is given by

$$D_{mm'}(\theta) = \exp(-i\theta L_x/\hbar) = \begin{pmatrix} \frac{1}{2}\cos\theta + \frac{1}{2} & -\frac{i}{\sqrt{2}}\sin\theta & \frac{1}{2}\cos\theta - \frac{1}{2} \\ -\frac{i}{\sqrt{2}}\sin\theta & \cos\theta & -\frac{i}{\sqrt{2}}\sin\theta \\ \frac{1}{2}\cos\theta - \frac{1}{2} & -\frac{i}{\sqrt{2}}\sin\theta & \frac{1}{2}\cos\theta + \frac{1}{2} \end{pmatrix} \tag{1}$$

Show that applying this matrix for the case of $\theta = \pi$ on the eigenfunction $|l = 1, m = 1\rangle$ gives the same result as rotating explicitly the function $Y_{1,1}(\theta, \phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}$ by 180-degrees around the x-axis.

2) A hydrogen-like atom with atomic number Z is in its ground state when, due to nuclear processes (operating at a time scale much shorter than the characteristic time scale of the H atom), its nucleus is modified to have the atomic number increased by one unit, i.e. to $Z + 1$. The electronic state of the atom does not change during this process. What is the probability of finding the atom in the new ground state at a later time? Answer the same question for the new first excited state.

3) Consider the delta-shell potential model, which is a very simple model of the force experienced by a neutron interacting with a nucleus. In this model, the force experienced by *neutron* has the form

$$V(r) = -\frac{\hbar^2 g^2}{2\mu}\delta(r - a) \tag{2}$$

Here r is written in spherical coordinates.

Investigate the existence of bound states in the case of negative energy.

a) Write down the Schroedinger equation for $u_l(r)$ in spherical coordinates using potential $V(r)$.

b) What are solutions for free particles ($V = 0$)? Which solution can be used for interior part ($r < a$) and which for exterior part ($r > a$)?

c) Integrating around the point $r = a$, determine the discontinuity condition, and hence equation for the eigenstates.

d) Assuming that $g^2 a = 2$, solve (possibly numerically) for bound state energy at $l = 0$.

4) A beam of composite particles is subject to a simultaneous measurement of the spin operators S^2 and S_z. The measurement gives pairs of values $s = m_s = 0$ and $s = 1$, $m_s = 1$ with probabilities 3/4 and 1/4 respectively.

(a) Reconstruct the state of the beam immediately before the measurement.

(b) The particles in the beam with $s = 1, m_s = 1$ are separated out and subjected to a measurement of S_x. What are the possible outcomes and their probabilities?

(c) For the purpose of understanding the symmetry of the wave function, it is convenient to replace spin operators with corresponding orbital angular momentum operators, i.e., $S_x \to L_x$ and $S^2 \to L^2$. Write down the spatial wave functions of the states that arise from the second measurement if the operator was orbital angular momentum operatore L_x. Give the x, y, z dependence of such wave functions.
Hint: First figure out the decomposition of the measured states in terms of $|l, m_l\rangle$ states. Using spherical harmonics, express the resulting wave function in real space.