Homework 1, Quantum Mechanics 501, Rutgers

September 18, 2016

1) Prove Schwartz inequality:

$$|\langle v|w\rangle| \leq |v||w| \tag{1}$$

and triangle inequality

$$|v+w| \leq |v| + |w| \tag{2}$$

2) a) Do functions defined on the interval $[0...L]$ and that vanish at the end points $x = 0$ and $x = L$ form a vector space?
   b) How about periodic functions obeying $f(L) = f(0)$?
   c) How about all functions with $f(0) = 4$?

3) Consider the vector space $\mathbf{V}$ spanned by real 2×2 matrices.

   a) What is its dimension?
   b) What would be a suitable basis?
   c) Consider three example vectors from this space:

$$|1\rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; |2\rangle = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; |3\rangle = \begin{pmatrix} -2 & -1 \\ 0 & -2 \end{pmatrix} \tag{3}$$

   Are they linearly independent? Support your answer with details.

4) Consider the two vectors $\vec{A} = 3\hat{i} + 4\hat{j}$ and $\vec{B} = 2\hat{i} - 6\hat{j}$ in the 2-dimensional space of the x-y plane. Do they form a suitable set of basis vectors? (Explain.) Do they form an orthonormal basis set? If not, use Gram-Schmidt algorithm to turn them into an othomormal set.

5) Assume the two operators Ω and Λ are Hermitian. What can you say about

   a) $\Omega\Lambda$
   b) $\Omega\Lambda + \Lambda\Omega$

c) $[\Omega, \Lambda]$

6) Consider the matrix

$$\Omega = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{4}$$

a) Is it Hermitian?

b) Find its eigenvalues and eigenvectors.

c) Verify that $U^{\dagger}\Omega U$ is diagonal, U being the matrix formed by using each normalized eigenvector as one of its columns. (Show that U is unitary!)

d) Calculate $\exp(i\Omega)$ and show it is unitary.

7) Consider the "Theta-funcion"

$$\theta(x - x') = \begin{cases} 1 & x \geq x' \\ 0 & otherwise \end{cases}$$

Show that $\delta(x - x') = \frac{d\theta(x-x')}{dx}$ by multiplaying on the r.h.s with an arbitrary square-integrable function $f(x)$ and integrating over all x.

8) Consider a ket space spanned by the eigenkets $\{|a_i\rangle\}$ and eigenvalues $\{a_i\}$ of a Hermitian operator $\mathbf{A}$ of dimension n. There is no degeneracy.

a) Prove that operator

$$\prod_{i=1}^{n}(A - a_i)$$

is a null operator $|0\rangle$ in this space.

b) What type of projector is this operator

$$\prod_{j=1, j\neq i}^{n} \frac{1}{a_i - a_j}(A - a_j) \quad ?$$