Final Exam, Quantum Mechanics 501, Rutgers

December 15, 2015

1. (a) Construct the spin singlet ($S = 0$) state and the spin triplet ($S = 1$) states of a two electron system.

   (b) In the experiment we have two electrons, which are in the spin-singlet state. They move in the opposite direction along the y-axis, and two observers A and B measure the spin state of each electron. A measures the spin component along the z axis, and B measures the spin component along an axis making an angle θ with the z axis in the xz-plane. Suppose that A's measurement yields a spin down state and subsequently B makes a measurement. What is the probability that B's measurement yields an up spin (measured along an axis making an angle θ with the z-axis)?

   The explicit formula for the representation of the rotation operator $\exp(-i\mathbf{S}\cdot\hat{\mathbf{n}}\theta/\hbar)$ in the spin space is given by the spin 1/2 Wigner matrix

$$D^{(1/2)}(\hat{\mathbf{n}},\theta) = \begin{pmatrix} \cos(\theta/2) - in_z\sin(\theta/2) & (-in_x - n_y)\sin(\theta/2) \\ (-in_x + n_y)\sin(\theta/2) & \cos(\theta/2) + in_z\sin(\theta/2) \end{pmatrix} \tag{1}$$

   and $\hat{\mathbf{n}} = n_x\vec{e}_x + n_y\vec{e}_y + n_z\vec{e}_z$ ($|\hat{\mathbf{n}}| = 1$) is the axis of rotation.

2. The Wigner-Eckart theorem s given by

$$\langle n'j'm'|T_q^{(l)}|njm\rangle = \langle j'm'|lq, jm\rangle\,\frac{\langle\langle n'j'|T^{(l)}|nj\rangle\rangle}{\sqrt{2j+1}} \tag{2}$$

   (a) Explain the meaning of the two terms on the right hand side.

   (b) The interaction of the electromagnetic field with a charged particle is given by

$$\Delta H = \frac{e}{2m}\mathbf{A}\cdot\mathbf{p}$$

   If the electromagnetic fields are in the form of a plane wave, then $\mathbf{A} = A_0\ \hat{\varepsilon}e^{i\mathbf{kr}}$, where $\hat{\varepsilon}$ is the polarization of the plane wave. Assuming that the wavelength $\lambda = 2\pi/k$ is much larger than the atomic size, we may write

$$\mathbf{A} = A_0\hat{\varepsilon}(1 + i\mathbf{k}\cdot\mathbf{r} + \cdots)$$

   such that

$$\Delta H \approx \frac{e}{2m}A_0\ \hat{\varepsilon}\cdot\mathbf{p}(1 + i\mathbf{k}\cdot\mathbf{r})$$

Her we kept both the dipole (the first term), and the quadrupole terms (the second term).

If the field is polarized along the x-axis ($\hat{\varepsilon} = \vec{e}_x$), and the wave propagation is along the z-axis ($\mathbf{k} = k\vec{e}_z$) express the Hamiltonian in terms of spherical harmonics. Note that $\mathbf{p}$ is a vector operator, and transforms under rotation as $\mathbf{r}$. For symmetry consideration you may therefore replace $\mathbf{p}$ by $C\mathbf{r}$

(c) For the above configuration, derive the selection rules for the dipole and the quadrupole transitions, by considering the transition probability matrix elements $|\langle\psi_f|\Delta H|\psi_i\rangle|^2 = |\langle l_f m_f|\Delta H|l_i m_i\rangle|^2$. Note: selection rules state under which conditions is a transition possible.

The explicit expressions for the spherical harmonics for $l = 1, 2$ are given by

$$Y_{1,1} = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\frac{x+iy}{r} \qquad Y_{1,0} = \frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{z}{r} \tag{3}$$

$$Y_{2,2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\frac{(x+iy)^2}{r^2} \qquad Y_{2,1} = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\frac{(x+iy)z}{r^2} \qquad Y_{2,0} = \frac{1}{4}\sqrt{\frac{5}{\pi}}\frac{2z^2-x^2-y^2}{r^2} \tag{4}$$

and $Y_{l,-m} = (-1)^m Y^*_{l,m}$.

3. A particle of reduced mass $\mu = 200\, MeV/c^2$ is moving in a spherical potential well of range a and depth $V_0 = -150\, MeV$. [$V(\mathbf{r}) = V_0$ for $|\mathbf{r}| < a$ and $V(\mathbf{r}) = 0$ for $|\mathbf{r}| > a$].

   The particle is bound in the $1s$ ground state with binding energy $E = -5\, MeV$. (This is supposed to be a very simple model of the deuteron). Note: $\hbar c = 197.327\, MeV\, fm$.

   (a) Solve the Schroedinger equation for both $r < a$ and for $r > a$.

   (b) Using the boundary conditions at $r = a$, extract the size of the "potential range" a.

   (c) Calculate the probability that a measurement of r will find $r > a$, i.e. the particle is outside the range of the potential (which is of course forbidden classically).