Final Exam, Quantum Mechanics 501, Rutgers

December 17, 2014

1) This problem concerns Clebsch-Gordan coefficients $\langle jm|j_1m_1, j_2m_2\rangle$.

   a) What are allowed values of total j for the addition of angular momenta $j_1 = 3$ and $j_2 = 1$?
   b) Explain why $\langle 44|33, 11\rangle = 1$.[Notation $\langle jm|j_1m_1, j_2m_2\rangle$]
   c) Find $\langle 43|32, 11\rangle$ (Hint: Use the spin lowering operator).

2) A system is in a state described by the wavefunction $\psi(\mathbf{r}) = f(r)(x + iy + z)$, where $f(r)$ is a radial wave function. If L_z is measured, what are the possible values of the measurement, and their probabilities?

   Note that $Y_{00} = \sqrt{\frac{1}{4\pi}}$, $Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$ and $Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$.

3) Consider a system of two non-identical fermions, each with spin 1/2. One is in a state with $S_{1y} = \frac{\hbar}{2}$, while the other is in a state with $S_{2x} = -\frac{\hbar}{2}$. What is the probability of finding the system in a state with total spin quantum numbers $s = 0$?

4) A particle of reduced mass $\mu = 470\, MeV/c^2$ is moving in a spherical potential well of range a and depth $V_0 = -76.73\, MeV$. $[V(\mathbf{r}) = V_0$ for $|\mathbf{r}| < a$ and $V(\mathbf{r}) = 0$ for $|\mathbf{r}| > a]$.

   The particle is bound in the $1s$ ground state with binding energy $E = -2.225\, MeV$. (This is supposed to be a very simple model of the deuteron). Note: $\hbar c = 197.327\, MeV fm$.

   a) Solve the Schroedinger equation for both $r < a$ and for $r > a$.
   b) Using the boundary conditions at $r = a$, extract the size of the "potential range" a.
   c) Calculate the probability that a measurement of r will find $r > a$, i.e. the particle is outside the range of the potential (which is of course forbidden classically).

5) Two elementary particles of spin s_1 and s_2 are bound by an attractive spin-dependent potential, as specified by the Hamiltonian

$$H = \frac{p^2}{2\mu} + U(r) + V(r)\mathbf{S}_1 \cdot \mathbf{S}_2 \tag{1}$$

   where $\mathbf{r}$ and $\mathbf{p}$ are relative coordinate and momentum; μ is the reduced mass; $U(r)$ and $V(r)$ are two different spherically symmetric potentials; and $\mathbf{S}_1$ and $\mathbf{S}_2$ are the spin operators for two particles. (We ignore the center-of-mass motion).

a) The Hamiltonian can also be written as

$$H = [\frac{p^2}{2\mu} + U]^{(c)} \otimes I^{(s)} + V^{(c)} \otimes [\frac{1}{2}(S^2 - S_1^2 - S_2^2)]^{(s)} \tag{2}$$

Briefly explain the notation used above, explain why certain terms appear before or after the '$\otimes$' and show how the last terms involving spins was obtained, in which $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$

b) Show that a vector of the form $|\psi_{nsm}\rangle = |\chi_{ns}\rangle \otimes |sms_1s_2\rangle$ is an eigenvector of H if $|\chi_{ns}\rangle$ obeys the effective Schroedinger equation

$$[-\frac{\hbar^2\nabla^2}{2\mu} + U + VC_s]\,|\chi_{ns}\rangle = E\,|\chi_{ns}\rangle \tag{3}$$

with $C_s = (\hbar^2/2)[s(s+1) - s_1(s_1+1) - s_2(s_2+1)]$. Here the state $|sms_1s_2\rangle$ is bulit from states $|s_1m_1, s_2m_2\rangle$ according to the usual rules for addition of angular momenta.