Anomalies, Gauss laws, and Page charges in M-theory

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Strings 2004, Paris

Related works:

Witten 9609122,9610234,9812012,9912086

Diaconescu, Moore, and Witten 00

Diaconescu, Freed, and Moore hep-th/0312069

Freed & Moore, to appear

Moore, to appear

Introduction

In 1978 Cremmer, Julia, and Scherk found the action for 11-dimensional supergravity. The action is simple, but contains a very subtle Chern-Simons term.

This talk is about that Chern-Simons term.

Y: 11-dimensional, oriented, spin.

In topologically trivial situations

$$C \in \Omega^3(Y) \qquad\qquad G := dC \in \Omega^4(Y).$$

The action for the theory is (schematically):

$$\exp\left[-\int_Y \mathrm{vol}(g)\mathcal{R}(g) + G \wedge *G + \bar{\psi}\not{D}\psi\right]\Phi(C)$$

$$\Phi(C) = \exp\left(2\pi i \int_Y \frac{1}{6}CG^2 - CI_8(g)\right)$$

Defining The Chern-Simons term

But we also want $[G] \neq 0$.

Existence of M2 branes $\Rightarrow$ (Witten 96)

$$[G] = \bar{a} - \frac{1}{2}\lambda, \qquad \bar{a} \in \bar{H}^4(Y;\mathbb{Z})$$

If $\partial Y = \emptyset$ the usual definition of a Chern-Simons term involves an extension to a bounding 12-manifold Z:

$$\Phi(C) \sim \exp\left(2\pi i \int_Z \frac{1}{6}G^3 - GI_8(g)\right)$$

$$[I_8(g)] = \frac{p_2 - \lambda^2}{48}$$

a priori only defined up to a 96^{th} root of unity.

Witten 96: $a \in H^4(Y,\mathbb{Z}) \leftrightarrow$ Principal E_8 bundle $P(a)$.

Identify

$$[G] = [\mathrm{tr}F^2 - \frac{1}{2}\mathrm{tr}R^2]$$

$$\frac{1}{6}G^3 - GI_8 = \frac{1}{2}i(\not{D}_A) + \frac{1}{4}i(\not{D}_{RS}) + d(*)$$

Index theory $\Rightarrow$ Φ is well-defined up to a ± 1.

The sign cancels against

$$\mathrm{Pf}(\not{D}_{RS}) = \sqrt{\det \not{D}_{RS}}$$

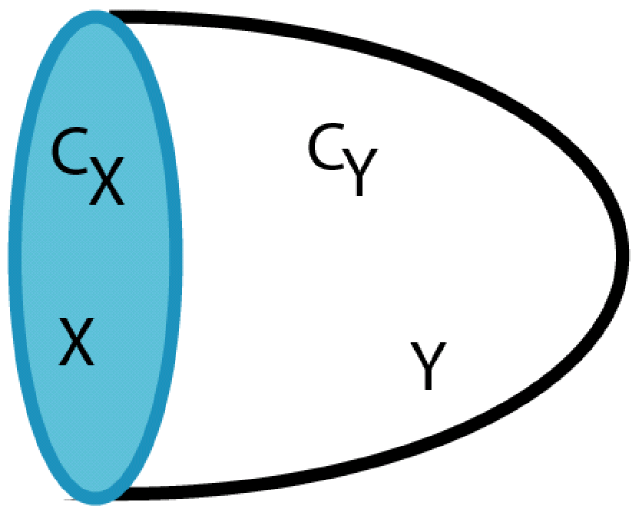
Boundaries

The extension to the case with boundary is nontrivial.

$\partial Y = X$: We must distinguish

- Temporal boundaries

- Spatial boundaries



In either case we need a "model" for the C-field.

Models for the C-field

For the C-field what we know for sure is the isomorphism class of C. It is fixed by the "Wilson loops"

$$\Sigma \to e^{2\pi i \int_\Sigma C}$$

$\Rightarrow$ isomorphism classes labeled by Cheeger-Simons characters, $\check{H}^4(Y, U(1))$.

But the proper way to express this in terms of redundant variables is not really known.

Abstractly,...

Gauge potentials = objects in a category

gauge transformations = morphisms

global gauge transformations = automorphisms

Different models for the C-field correspond to *equivalent categories.*

E_8 Model for the C-field

Definition: A *"C-field" on Y with characteristic class a* is a pair (A, c) in

$$\mathcal{C}(Y) := \mathrm{Conn}(P(a)) \times \Omega^3(Y)$$

Gauge invariant fieldstrength:

$$G = \mathrm{tr}F^2 - \frac{1}{2}\mathrm{tr}R^2 + dc$$

- Morally speaking

$$C = CS(A) - \frac{1}{2}CS(g) + c$$

- Note that G depends on metric.

Chern-Simons term

$$\Phi(C) = \exp\left[2\pi i\left\{\frac{1}{4}\eta(\not{D}_A) + \frac{1}{8}\eta(\not{D}_{RS})\right\} + 2\pi i I_{\rm local}\right]$$

$$I_{\rm local} = \int_Y \left(c(\frac{1}{2}G^2 - I_8) - \frac{1}{2}cdcG + \frac{1}{6}c(dc)^2\right)$$

The same formula applies with or without boundary

Φ is a section of a *line bundle*:

$$\mathcal{L} \to \mathcal{C}(Y) \times Met(Y)$$

When $\partial Y = X$, $\mathcal{L}$ has a connection:

$$\mathcal{A} = 2\pi \int (\frac{1}{2}G^2 - I_8)\delta C$$

with nonzero curvature:

$$\mathcal{F} = \pi \int_X G\delta C\delta C$$

Anomalies

The theory of the C-field by itself is quantum inconsistent. The theory of the gravitino by itself is quantum inconsistent. But the product is consistent.

No boundary

$\mathrm{Pf}(\not{D}_{RS})$, a section of

$$L := \mathrm{PF}(\not{D}_{RS}) \to Met(Y)$$

L is a *complex line bundle* with real structure $\Rightarrow$ holonomies $= \pm 1$: The gravitino has a global anomaly.

A natural isomorphism $L \cong \mathcal{L} \Rightarrow$ global anomaly cancelation:

$$\mathrm{Pf}(\not{D}_{RS}) \cdot \Phi$$

is a well-defined function on $\mathcal{C}(Y) \times Met(Y)/\mathcal{G}$

Temporal boundary :

Φ is a section of a line bundle over $Met(Y)$, *NOT* over $Met(X)$.

$\Rightarrow$ Obstruction to defining a well-defined Hilbert space of states.

Nevertheless, with APS bc's on fermions

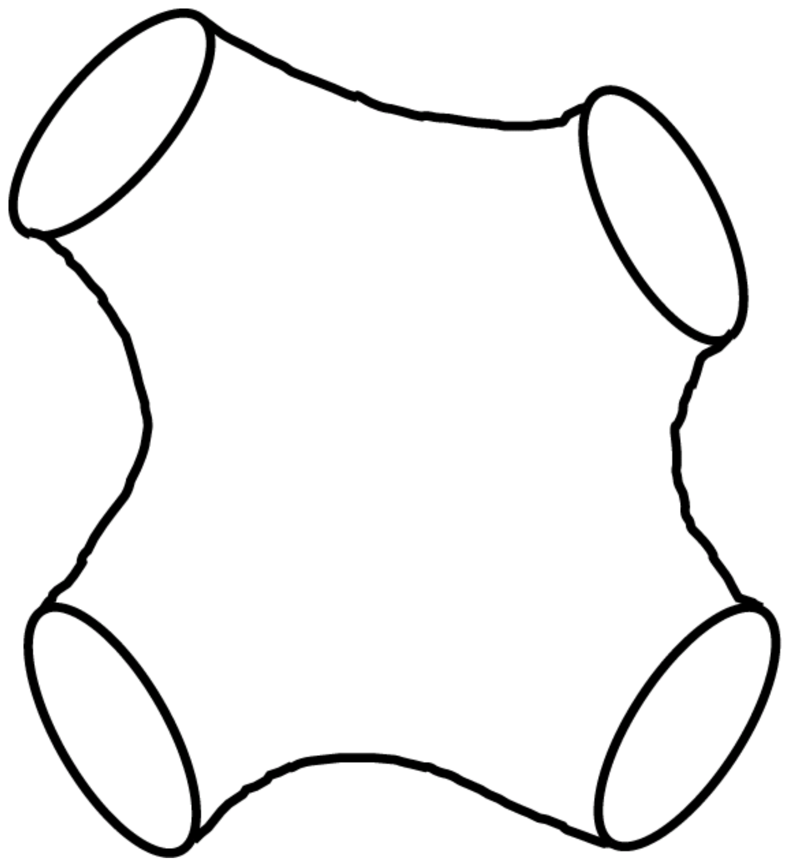
$$\mathrm{Pfaff}(\not{D}_{RS}) \cdot \Phi \in \Gamma\Big(\mathrm{PF}(\not{D}_A) \to \mathcal{C}(X) \times Met(X)\Big)$$

Anomaly Cancellation for spatial boundaries

With local (chiral) b.c.'s on fermions one can still define elliptic operators and study geometric invariants (D. Freed's student M. Scholl is studying the general case, filling a gap in the math literature.).

$\Rightarrow$ Rigorous proof of anomaly cancellation in the Horava-Witten model.

The cancellation is completely *local*, e.g. multiple boundaries with $\epsilon_i = \pm$ on each boundary is allowed topologically:



$$G|_{X_i} = \epsilon_i(\mathrm{tr}F^2(A_i) - \tfrac{1}{2}\mathrm{tr}R^2(g_i))$$

$$\Phi: \qquad \mathcal{F} = \sum_i \epsilon_i \int_{X_i} \left(\frac{1}{2} i(\not{D}_A) + \frac{1}{4} i(\not{D}_{RS})\right)$$

$$\mathrm{Pfaff}(\not{D}_{A_i}): \qquad \mathcal{F} = -\epsilon_i \int_{X_i} \frac{1}{2} i(\not{D}_{A_i})$$

$$\mathrm{Pfaff}(\not{D}_{RS}): \qquad \mathcal{F} = -\sum_i \epsilon_i \int_{X_i} \frac{1}{4} i(\not{D}_{RS})$$

The Gauge group $\mathcal{G}$

Our next goal is to write the Gauss law for C-field gauge invariance.

In the E_8 model $C = (A, c)$.

- Small gauge transformations: $c \to c + d\Lambda, \quad \Lambda \in \Omega^2(Y)$
- "All" gauge transformations: $c \to c + \omega, \quad \omega \in \Omega^3_{\mathbb{Z}}(Y)$

Does not properly account for global gauge transformations "$\Lambda \sim$ constant."

Actually, $\mathcal{G}$ is an extension:

$$0 \to H^2(Y, U(1)) \to \mathcal{G} \to \Omega^3_{\mathbb{Z}}(Y) \to 0$$

$H^2(Y, U(1))$ = global gauge transformations,

acts as a group of automorphisms: If $\alpha \in H^2(Y, U(1))$ then

$$g_\alpha : (A, c) \to (A, c)$$

But it still has nontrivial physical effects.

Gauss law

Physical wavefunctions should be gauge invariant:

$$g \cdot \Psi(C) = \Psi(g \cdot C) \qquad \forall g \in \mathcal{G}$$

To formulate the Gauss law we define a lift:

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\mathcal{G}} & \mathcal{L} \\ \downarrow & & \downarrow \\ \mathcal{C}(X) & \xrightarrow{\mathcal{G}} & \mathcal{C}(X) \end{array}$$

$$g \cdot \Psi(C_X) = \varphi(C_X, g) \cdot \exp(\int_{C_X}^{g \cdot C_X} \mathcal{A}) \cdot \Psi$$

Definition of the "lifting phase" $\varphi(C_X, g)$:

Given C_X & $g \Rightarrow$ twisted C-field $C_{X,g}$ on $Y = X \times S^1$:

$$C_{X,g}(x, 1) = g \cdot C_{X,g}(x, 0)$$

$$\varphi(C_X, g) := \Phi(C_{X,g})$$

$$\varphi(C_X, g_1)\varphi(g_1 \cdot C_X, g_2) = e^{-i\pi \int_X G\omega_1\omega_2}\varphi(C_X, g_1 g_2)$$

Writing out the Gauss law

Recall $\mathcal{G}$ is an extension:

$$0 \to H^2(Y, U(1)) \to \mathcal{G} \to \Omega^3_{\mathbb{Z}}(Y) \to 0$$

So the Gauss law consists of two statements:

1. Law for $g = g_\alpha$, $\alpha \in H^2(X, U(1))$

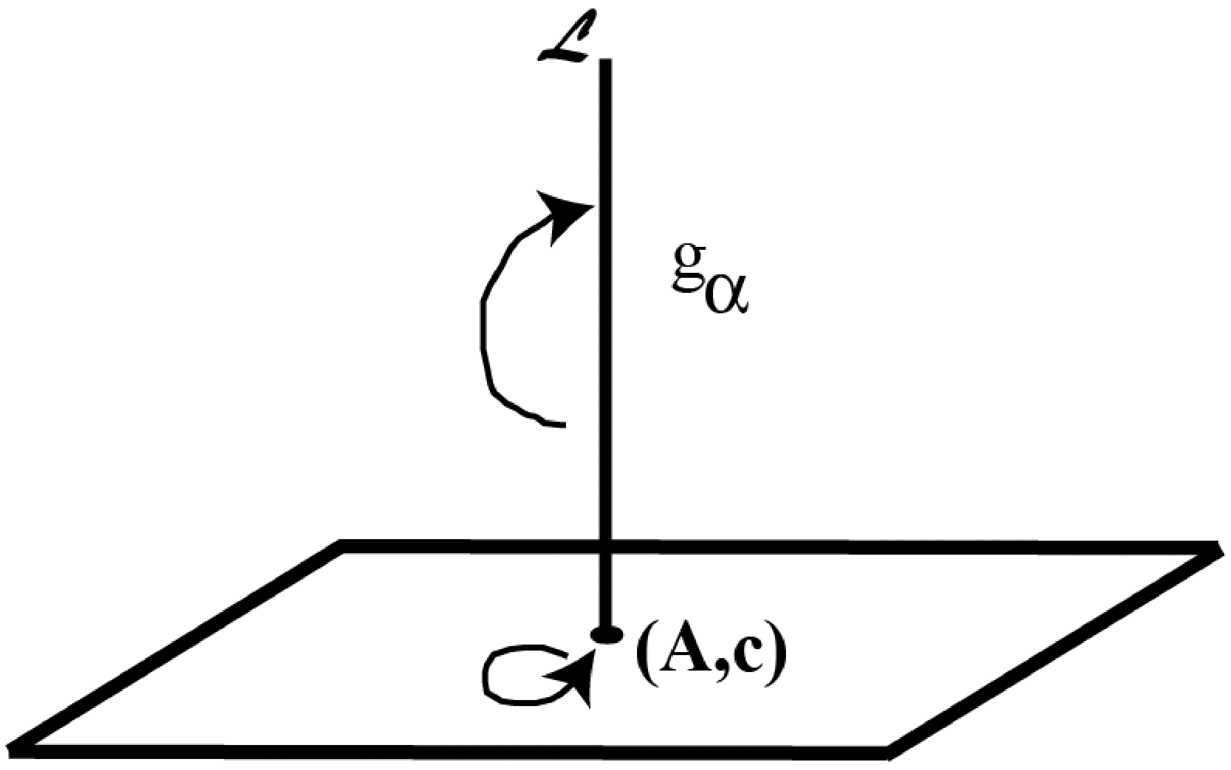
$\Rightarrow$ C-field electric charge tadpole condition.

2. Law for $g \in \Omega^3_{\mathbb{Z}}(X)$

$\Rightarrow$ quantization of Page charge.

The Tadpole Condition

A global gauge transformation g_α, $\alpha \in H^2(X, U(1))$ acts nontrivially on quantum wavefunctions:



If $\Psi \in \mathcal{L}_{A,c}$ then

$$g_\alpha \cdot \Psi = \exp\big[2\pi i \langle Q, \alpha\rangle\big]\ \Psi$$

$Q \in H^8(X, \mathbb{Z})$ is the C-field electric charge.

Gauss Law: $Q \neq 0 \Rightarrow \Psi = 0$.

From the definition of the group lift we get a formula for Q.

It is easy to show that

$$\bar{Q} = [\frac{1}{2}G^2 - I_8]_{DR}$$

but there is indeed a torsion component to the usual tadpole condition.

Part 2 of the Gauss law

When $Q = 0$ we can have nonzero gauge invariant wavefunctions

$$\Psi(C) \in \Gamma(\mathcal{L})$$

There is still further information in the statement of gauge invariance.

Trivialize $\mathcal{L}$.

$\Rightarrow$ Choose a basepoint $C = C_\bullet + c$,

$\Psi(C) \to \psi(c)$

Remainder of the Gauss law says:

$$\psi(c + \omega) = e_\omega(c)\psi(c) \qquad \forall \omega \in \Omega^3_{\mathbb{Z}}(X)$$

$$e_\omega(c) := \varphi(C, \omega)^* e^{-2\pi i \int_X \left(\frac{1}{2}G - \frac{1}{6}dc\right)c\omega}$$

Quantization of Page Charge

The equation

$$\psi(c+\omega) = e_\omega(c)\psi(c)$$

is equivalent to:

$$\exp\left(2\pi i \int \omega P\right)\psi = \psi \qquad \forall \omega \in \Omega^3_{\mathbb{Z}}(X)$$

$$P = \frac{1}{2\pi}\Pi + \left(\frac{1}{2}G_\bullet c + \frac{1}{6}cdc\right) + T_\bullet$$

where $dT_\bullet = \frac{1}{2}{G_\bullet}^2 - I_8$ & $\Pi =$ momentum,

P is the "Page charge," or electric flux, (expressed in a Hamiltonian formalism).

Roughly speaking, $P = dC_6$.

If $[G] = 0$, the quantum Gauss law for large C-field gauge transformations $\Rightarrow$

$$[P] \in \bar{H}^7(X;\mathbb{Z})$$

This is the expected electro-magnetic dual to the quantization of magnetic flux:

$$[G] \in \bar{H}^4(X;\mathbb{Z})$$

Page charges don't commute

However, when $[G] \neq 0$, things are quite different.

For $\phi \in H^3_{DR}(X)$ define

$$P(\phi) = \int_X \phi \wedge P$$

Easy computation:

$$[P(\phi_1), P(\phi_2)] = \frac{i}{2\pi} \int \phi_1 \wedge \phi_2 \wedge G$$

$[P]$ is not gauge invariant:

$$U(\omega) P(\phi) U(\omega)^{-1} = P(\phi) - \int \omega \phi G$$

$$U(\omega) := \exp[2\pi i \int \omega P]$$

The Page Charge Group

When $[G] \neq 0$ part of the lattice $\bar{H}^7(X, \mathbb{Z})$ collapses to a *finite group*.

The gauge invariant objects are

$$W(\phi) := e^{2\pi i P(\phi)}$$

for ϕ such that:

$$\int \phi \omega G \in \mathbb{Z} \qquad \forall \omega \in H^3(X, \mathbb{Z})$$

These $W(\phi)$ form the "Page charge group."

N.B. This group is nonabelian!

$$\begin{aligned} W(\phi_1)W(\phi_2) &= e^{-i\pi \int \phi_1\phi_2 G} W(\phi_1 + \phi_2) \\ &= e^{-2\pi i \int \phi_1\phi_2 G} W(\phi_2)W(\phi_1) \end{aligned}$$

Analogy to 3D Chern-Simons

A closely related theory is 3d massive abelian CS:

$$S = \int_{\Sigma\times\mathbb{R}} -\frac{1}{2e^2}F*F + 2\pi\int_{\Sigma\times\mathbb{R}} kAdA$$

On $\Sigma\times\mathbb{R}$ the dynamics of the topological (flat) modes of A is that of an electron on a torus $H^1(\Sigma;U(1))$ in a constant magnetic field.

Similarly - for harmonic modes of C: $c = \sum_a c_a\omega^a$,

$$H_{\text{eff}} = h_{ab}\big(-i\frac{\partial}{\partial c_a} - \pi\mathcal{B}^{aa'}c_{a'}\big)\big(-i\frac{\partial}{\partial c_b} - \pi\mathcal{B}^{bb'}c_{b'}\big)$$

$$h^{ab} = \int \omega^a * \omega^b \qquad \mathcal{B}^{ab} = \int_X G\omega^a\omega^b$$

$$k \leftrightarrow \tfrac{1}{2}[G]$$

Landau levels $\leftrightarrow$ Wavefunction $\psi(c)$

Magnetic translation operator $\leftrightarrow$ Page charge

Verlinde,'t Hooft, Wilson operator $\leftrightarrow$ $W(\phi)$

A derivation of the 5-brane partition function

As an application, we derive Witten's prescription for the 5-brane partition function.

$X = D_6 \times S^4$ at conformal infinity for an asymptotically AdS space

$$ds^2 \to (k^{2/3}\ell^2)\left[dr^2 + e^{2r}ds^2_{D_6} + \frac{1}{4}ds^2_{S^4}\right]$$

$$G \to G_\infty = k\omega_{S^4} + \bar{G}$$

AdS/CFT $\Rightarrow$ $(k \gg 1)$

$$Z\big[\mathrm{M-theory}/Y\big] = Z\big[U(k) \qquad (2,0)-\mathrm{theory}/D_6\big]$$

We can say something about the "singleton sector"

$$U(k) = \frac{SU(k) \times U(1)}{\mathbb{Z}_k}$$

$U(1) \sim$ C.O.M. for k 5-branes

- couples to the harmonic modes $c_{\infty,h}$.

Page charge is dual to 't Hooft sector

To write the dependence on $c_{\infty,h}$ we use the symplectic splitting

$$H^3(D_6, \mathbb{Z}) = \Lambda_1 \oplus \Lambda_2$$

$$\langle \omega_1, \omega_2 \rangle = \int_{D_6} \omega_1 \wedge \omega_2$$

Then

$$Z[U(k) \qquad (2,0) - \text{theory}] = \sum_{\beta \in \Lambda_1 / k\Lambda_1} \zeta^\beta \Psi_\beta(c_{\infty,h})$$

- ζ^β is the contribution of the $SU(k)/\mathbb{Z}_k$ $(0,2)$ theory.
- β labels "'t Hooft sectors" of the $SU(k)/\mathbb{Z}_k$ theory (Witten).
- A formula for Ψ_β in terms of theta functions shows that:

$$W(\phi_1)\Psi_\beta = \Psi_{\beta+\phi_1} \qquad \phi_1 \in \Lambda_1/k\Lambda_1$$
$$W(\phi_2)\Psi_\beta = e^{2\pi i k \langle \phi_2, \beta \rangle} \Psi_\beta \qquad \phi_2 \in \Lambda_2/k\Lambda_2$$

- Page charge group = a discrete Heisenberg group on $H^3(D_6, \mathbb{Z}/k\mathbb{Z})$.

'''t Hooft sector = "Page charge."'''

Formula for Ψ_β

Ψ_β are derived from chiral splitting of

$$\sum_{\omega\in\mathcal{H}^3_{\mathbb{Z}}(D_6)} \varphi(C_\bullet,\omega)\exp\left\{-\frac{\pi k}{2}\int_{D_6}(c+\omega)*(c+\omega)-i\pi k\int_{D_6}c\wedge\omega\right\}$$

$$= \sum_{\beta\in\Lambda_1/k\Lambda_1}\zeta^\beta\Psi_\beta(c)$$

Explicitly:

$$\Psi_\beta(c) = e^{-\frac{\pi k}{2}\int_{D_6} c*c+c^{(1,0)}\mathrm{Im}\tau c^{(1,0)}}\Theta_{\beta,k/2}(c^{(1,0)}+\theta+\tau\phi,\tau)$$

The characteristics follow from $\varphi(C_\bullet,\omega)$ *and depend on the the metric.*

$$\varphi(C_\bullet,\omega)\sim\exp[2\pi i\int_{D_6}\omega\wedge CS(g)]$$

Conclusions

• The E_8 formalism for the C-field can be useful, but it has shortcommings.

Is there a better model for the C-field?

How do we formulate M-theory on unoriented manifolds?

• The naive classification of electric+ magnetic fluxes:

$$H^4(X,\mathbb{Z}) \oplus H^7(X,\mathbb{Z})$$

is replaced by a *nonabelian* group

• This has implications for the K-theoretic classification of type II string fluxes.

• Five-brane partition functions have metric-dependent characteristics.

• This can lead to suppression of instanton effects.