Three Projects Using Lattices, Modular Forms, String Theory & K3 Surfaces

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Part I

Desperately Seeking Moonshine

a project with Jeff Harvey

still in progress…

Part II Consistency Conditions For Asymmetric Orbifolds

with Jeff Harvey and Nati Seiberg

still in progress…

Part III

Holography & Zamolochikov Volumes Of Moduli Spaces of Calabi-Yau Manifolds

N. Benjamin, M. Cheng, S. Kachru, G. Moore, and N. Paquette,
``Elliptic Genera and 3d Gravity,'' arXiv:1503.04800

G. Moore,
``Computation Of Some Zamolodchikov Volumes, With An Application,''
arXiv:1508.05612

Part I Outline

1. Brief Background On Moonshine

2. Heterotic Strings On Tori

3. Crystal Symmetries Of Toroidal Compactification

4. Desperately Seeking Mooneshine

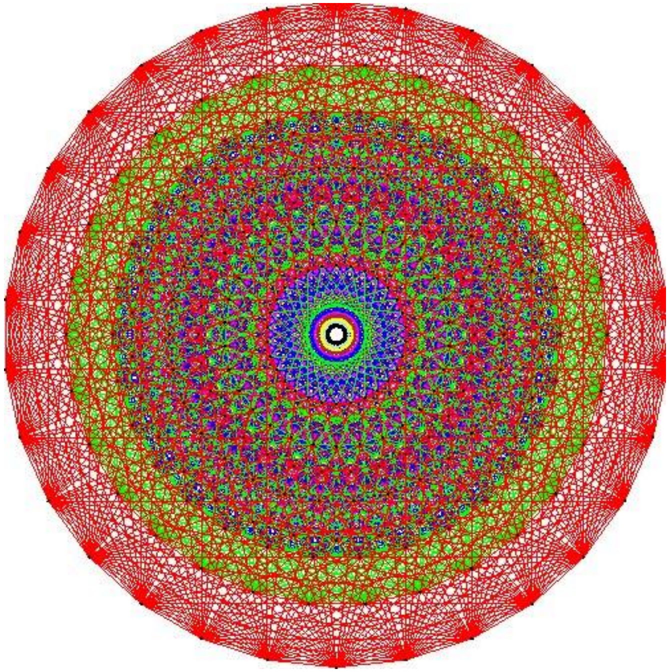
Philosophy

We can divide physicists into two classes:

Our world is a random choice drawn from a huge ensemble:

Philosophy – 1/2

The fundamental laws of nature are based on some beautiful exceptional mathematical structure:

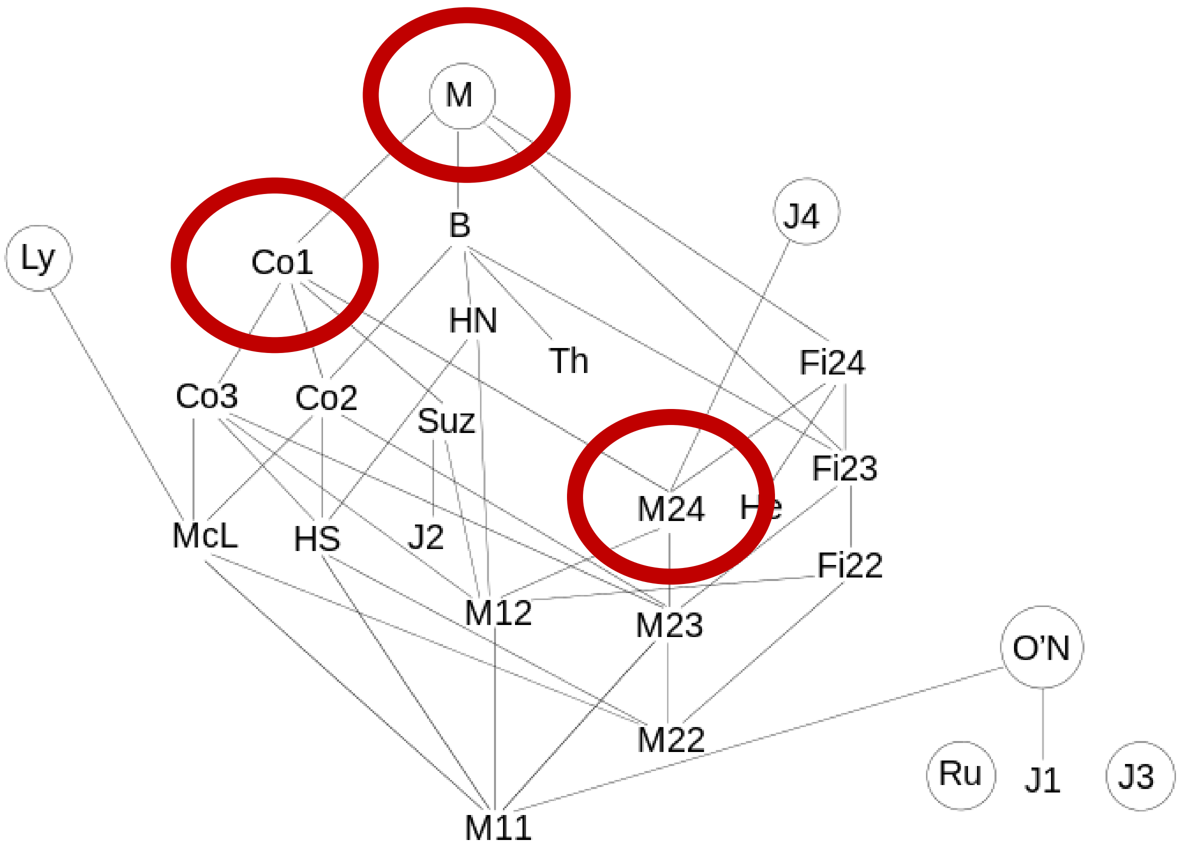


Part 1: String compactifications with unusual and exceptional symmetries.

Background: Finite-Simple Groups

Jordan-Holder Theorem: Finite simple groups are the atoms of finite group theory.

$\mathbb{Z}_p$ p = prime $\quad A_n \; n \geq 5 \quad SL_n(\mathbb{F}_p)$ etc.



Background: McKay & Conway-Norton 1978-1979

$$J = \sum_n J_n q^n = q^{-1} + 196884\, q + 21493760\, q^2 + 864299970\, q^3 + \cdots$$

Now list the dimensions of irreps of $\mathbb{M}$

R_n = 1, 196883, 21296876, 842609326, 18538750076, 19360062527, 293553734298,…. , $3^{12} \cdot 5^7 \cdot 13^3 \cdot 17 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \sim 2.6 \times 10^{26}$

$$J_{-1} = R_1 \qquad J_1 = R_1 + R_2$$

$$J_2 = R_1 + R_2 + R_3 \qquad J_3 = 2R_1 + 2R_2 + R_3 + R_4$$

A way of writing J_n as a positive linear combination of the R_j for all n is a ``***solution of the Sum-Dimension Game***.''

There are infinitely many such solutions!!

Background: Monstrous Moonshine

Which, if any, of these solutions is interesting?

Every solution defines an infinite-dimensional $\mathbb{Z}$-graded representation of $\mathbb{M}$

$$V = q^{-1} R_1 \oplus q(R_1 \oplus R_2) \oplus q^2(R_1 \oplus R_2 \oplus R_3) \oplus \cdots$$

Now for every $g \in \mathbb{M}$ we can compute the character:

$$\chi(q; g) \coloneqq Tr_V\, g\, q^N$$

A solution of the Sum-Dimension game is *modular* if the $\chi(q; g)$ is a modular function in $\Gamma_0(m)$ where $g^m = 1$.

There is a unique modular solution of the Sum-Dimension game! Moreover the $\chi(q; g)$ have very special properties.

FLM Construction -1/3

String theory compactification of chiral bosonic string on a torus:

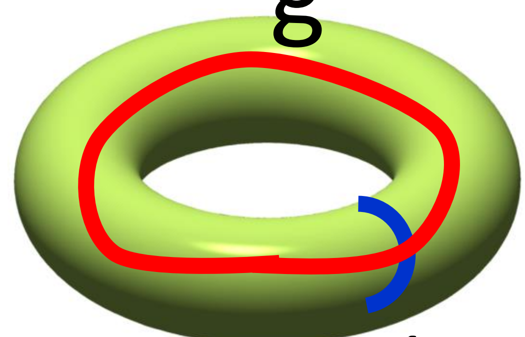
$$\partial_z x^\mu = -i \sum_n \alpha_n^\mu e^{inz} \qquad \mu = 0, \ldots, 25$$
$$z = \sigma + \tau$$

$$[\alpha_n^\mu, \alpha_m^\nu] = n\eta^{\mu\nu}\delta_{n+m,0} \qquad \alpha_0^\mu = p^\mu \in \Lambda \oplus U$$

OPE of conformal fields form a VOA: $\wp(\partial^* x) e^{i\, p\cdot x}$

g 
1

$$:= Tr_{\mathcal{H}}\, g q^{L_0 - \frac{c}{24}} =$$

g

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Modularity

FLM Construction – 2/3

Symmetries include $Aut(\Lambda) = Co_0$

Now ``orbifold'' by $\vec{x} \rightarrow -\vec{x}$ for $\vec{x} \in \mathbb{R}^{24}/\Lambda$

"Orbifold by a symmetry G of a CFT":

Gauge the symmetry

Different G-bundles over the circle $\Rightarrow$

$\mathcal{H}_{untwisted}$ $\quad\quad \Phi(\sigma + 2\pi) = \Phi(\sigma)$

$\mathcal{H}_{[g]-twisted}$ $\quad\quad \Phi(\sigma + 2\pi) = \mathrm{g} \cdot \Phi(\sigma)$

FLM Construction – 3/3

Automorphisms of the OPE algebra of the quotient theory = $\mathbb{M}$

Heisenberg group of translations on 24 fixed points + Co_1 + a ``quantum symmetry'' generate the Monster.

This is the gold standard for the conceptual explanation of Moonshine-modularity

(But a truly satisfying conceptual explanation of genus zero properties remains elusive.)

New Moonshine: Mathieu Moonshine

Eguchi, Ooguri, Tachikawa 2010

Start with a CFT, rather than a group: K3 susy σ − model.

$$\mathcal{E}_{\mathcal{C}}(z,\tau) = Tr_{\mathcal{H}_{RR}}(-1)^F q^{L_0 - c/24} e^{2\pi i z J_0} \bar{q}^{\tilde{L}_0 - c/24}$$

$$\mathcal{E}_{K3}(z,\tau) =$$

$$8\left[\left(\frac{\vartheta_2(z)}{\vartheta_2(0)}\right)^2 + \left(\frac{\vartheta_3(z)}{\vartheta_3(0)}\right)^2 + \left(\frac{\vartheta_4(z)}{\vartheta_4(0)}\right)^2\right]$$

$$= \sum_{h,\ell} d_{h,\ell}\,\mathrm{ch}_{h,\ell}(z,\tau)$$

$$\mathcal{E}_{K3}(z,\tau) = 24\mathrm{ch}_{\frac{1}{4},0} + 2\sum_{n=0}^{\infty} d_n \mathrm{ch}_{n+\frac{1}{4},\frac{1}{2}}$$

$$\{d_n\} = \{-1, 45, 231, 770, 2277, \dots\}$$

Dimensions of irreps of the sporadic finite simple group M24 !

∃ unique modular solution of the sum-dimension game!

[Gaberdiel, Hohenegger, Volpato 2011; Gannon 2012]

But there is no known analog of the FLM construction.

Proposal: It is related to the ``algebra of BPS states.''

Something like: M24 is a distinguished group of automorphisms of the algebra of spacetime BPS states in some string compactification using K3.

Newer Moonshine? : String-Math, 2014

Today's story begins in Edmonton, June 11, 2014. Sheldon Katz was giving a talk on his work with Albrecht Klemm and Rahul Pandharipande [“KKP”]

He was describing how to count BPS states for type II strings on a K3 surface taking into account the so(4) = su(2) + su(2) quantum numbers of a particle in six dimensions.

(O(4) is the automorphism group of the orthogonal to $M^{1,1}$ in six-dimensional Minkowski space.)

Slide # 86 said ….

Refined K3 BPS Invariants

- Define the *refined K3 BPS invariants* $R^{h\geq 0}_{j_L,j_R}$ by

$$\sum_{h=0}^{\infty}\sum_{j_L}\sum_{j_R} R^{h}_{j_L,j_R}\,[j_L]_u[j_R]_y q^h =$$

$$\prod_{n=1}^{\infty}\frac{1}{(1-u^{-1}y^{-1}q^n)(1-u^{-1}yq^n)(1-q^n)^{20}(1-uy^{-1}q^n)(1-uy}$$

where $[j]_x = x^{-j}+\ldots+x^{j}$

- For $h\leq 2$ the nonvanishing invariants are

$$R^{0}_{0,0}=1\,,$$

$$R^{1}_{0,0}=20\,,\quad R^{1}_{\frac{1}{2},\frac{1}{2}}=1\,,$$

$$R^{2}_{0,0}=231\,,\quad R^{2}_{\frac{1}{2},\frac{1}{2}}=21\,,\quad R^{2}_{1,1}=1$$

Heterotic/Type II Duality

KKP compute (roughly) the cohomology groups of the moduli spaces of objects in $D^b(K3)$ with fixed K-theory invariant and stable wrt a stability condition determined by a complexified Kahler class.

IIA/K3 = Het/T4

Physically: D4-D2-D0 boundstates

Dabholkar-Harvey states: Perturbative heterotic BPS states

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Heterotic String Crash Course

$$\mathbb{M}^{1,1+d} \times T^{8-d}$$

$$\partial_z x^\mu = -i \sum_n \alpha_n^\mu e^{inz} \qquad \mu = 0, \dots, 25 \qquad z = \sigma + \tau$$

$$\partial_{\tilde{z}} \tilde{x}^a = -i \sum_n \tilde{\alpha}_n^a e^{in\tilde{z}} \qquad a = 0, \dots, 9 \qquad \tilde{z} = \sigma - \tau$$

$$\tilde{\psi}^a(\tilde{z}) = \sum_n \tilde{\psi}_n^a \, e^{in\tilde{z}} \qquad \{\tilde{\psi}_n^a, \tilde{\psi}_m^b\} = \eta^{ab}\delta_{n+m,0}$$

$$\left(\alpha_0^i; \tilde{\alpha}_0^a\right) \coloneqq (P_L; P_R) \qquad H = P_L^2 + P_R^2 + H_{osc}$$

Heterotic Toroidal Compactifications

Quantization of momentum/winding implies

$$P = (P_L; P_R) \in \Gamma^{n_L;n_R}$$

$$n_L = 24 - d \quad +$$
$$n_R = 8 - d \quad -$$

Even, unimodular lattice.

Abstractly, there is a unique even unimodular lattice of this signature, but physics depends on the embedding:

$$II^{n_L;n_R} \hookrightarrow \Gamma^{n_L;n_R} \subset \mathbb{R}^{n_L;n_R}$$

Narain moduli space of CFT's:

$$O_{\mathbb{Z}}(II^{n_L;n_R}) \backslash O_{\mathbb{R}}(n_L;n_R)/(O_{\mathbb{R}}(n_L) \times O_{\mathbb{R}}(n_R))$$

We pause for a small advertisement

Algebra Of BPS States -1/2

DH/Perturbative BPS States $V_L(z) \otimes V_R(\tilde{z})$

$$V_R(\bar{z}) = \varepsilon \cdot \tilde{\psi} e^{\mathrm{i}|P_R|\tilde{x}^0 + \mathrm{i}P_R\cdot\tilde{x}}$$ SYM ground state at rest

$$V_L(z) = \wp(\partial^* x) e^{\mathrm{i}|P_R|x^0 + \mathrm{i}P_L\cdot x}$$

Vector space of DH states is graded by $\Gamma^{n_L;n_R}$

$V_L(z)$ is a dimension one Virasoro primary

Mutually local holomorphic dimension one Virasoro primaries form a Lie algebra:

$$V_1(z_1)V_2(z_2) = \cdots + \frac{1}{z_{12}}(V_1 * V_2)(z_2) + \cdots$$

Algebra Of BPS States -2/2

Problem: $V_L(P_1)$ and $V_L(P$ … P_1 and P_2 are in general not mutually local.

Solution: Choose a sp … ere are unique boosts along this direction s …

$$\Lambda_1(|P_{R,1}|,\vec{0}) + \Lambda \quad \ldots \quad (|P_{R,1} + P_{R,2}|,\vec{0})$$

Observe: For the booste … ere is a term in the OPE:

$$V_1(z_1)V_2(z_2) = \quad \ldots \quad (V_1 * V_2)(z_2) + \cdots$$

$z_{12} \; z_{12}$

V_1*V_2 is the product in the algebra of BPS states.

End of advertisement

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Crystal Symmetries Of Toroidal Compactifications

Construct some heterotic string compactifications on tori with large interesting symmetries.

$$G \subset \mathrm{Aut}(\Gamma^{24-d;8-d})$$

$$G = G_L \times G_R$$

$$G_L \subset O_{\mathbb{R}}(24-d) \qquad G_R \subset O_{\mathbb{R}}(8-d)$$

Then G is a *crystal symmetry* of the CFT

Example: Enhanced Gauge Symmetry

$$d = 7 : \quad \mathbb{M}^{1,9} \times T^1$$

Gaussian model:

$$S = \frac{R^2}{4\pi} \int \partial x \tilde{\partial} x \qquad x \sim x + 2\pi$$

$$e^{\frac{i}{\sqrt{2}}\left(\frac{n}{R}+w\,R\right)x}(z) \otimes e^{\frac{i}{\sqrt{2}}\left(\frac{n}{R}-w\,R\right)\tilde{x}}(\tilde{z})$$

Narain lattice: $(P_L\,; P_R\,) = n\,e + w\,f$

$$e = \frac{1}{\sqrt{2}}\left(\frac{1}{R};\frac{1}{R}\right)$$

$$f = \frac{1}{\sqrt{2}}(R;-R)$$

$R \to 1/R$

$$n \leftrightarrow w$$

&

$$(x, \tilde{x}) \to (x, -\tilde{x})$$

Narain moduli space:

Example: Enhanced Gauge Symmetry

At R=1 we have a theory equivalent to the $SU(2)_1$ WZW model

(”Nonabelian bosonization” or “Frenkel-Kac-Segal cnstn”)

$$J^3(z) = \frac{1}{\sqrt{2}} \partial x(z), J^{\pm}(z) = e^{\pm i\sqrt{2}x}(z)$$

$$\tilde{J}^3(\tilde{z}) = \frac{1}{\sqrt{2}} \partial \tilde{x}(\tilde{z}), \tilde{J}^{\pm}(\tilde{z}) = e^{\pm i\sqrt{2}\tilde{x}}(\tilde{z})$$

Gives an $su(2)_L \oplus su(2)_R$ current algebra.

Crystal symmetry is $O_{\mathbb{Z}}(1,1) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

This example generalizes to Weyl group symmetries of enhanced YM gauge theories for d > 1.

Conway Subgroup Symmetries

But there are other interesting crystal symmetries:

Start with a distinguished d=0 compactification:

$$\mathbb{M}^{1,1} \times T^8$$

$$\Gamma^{24,8} = (\Lambda; 0) \oplus (0; \Gamma_8)$$

(Can replace Λ by any Niemeier lattice.)

Crystal symmetry:

$$\mathrm{Co}_0 \times W(E_8)$$

Note that Co_0 is not a Weyl group symmetry of any enhanced Yang-Mills gauge symmetry.

There are related examples for $\mathbb{M}^{1,d+1} \times T^{8-d}$

A Lattice Lemmino

$\mathfrak{F}_L \subset \Lambda$ & $\mathfrak{F}_R \subset \Gamma_8$ Primitively embedded. Isometric of rank d.

Then ***<u>there exists embedded even unimodular lattice</u>***

$$\Gamma^{24-d;8-d} \hookrightarrow \mathbb{R}^{24-d;8-d}$$

such that, if

$$G_L := \mathrm{Fix}(\mathfrak{F}_L) \subset \mathrm{Aut}(\Lambda) = \mathrm{Co}_0$$

$$G_R := \mathrm{Fix}(\mathfrak{F}_R) \subset \mathrm{Aut}(\Gamma_8) = W(E_8)$$

$\Gamma^{24-d;8-d}$ has crystal symmetry

$$G_L \times G_R \subset O(24-d) \times O(8-d)$$

Easy Proof

Uses standard ideas of lattice theory.

$$\mathcal{D}_+(\mathfrak{F}_L^\perp) \cong \mathcal{D}_-(\mathfrak{F}_L) \;\cong\; \mathcal{D}_-(\mathfrak{F}_R) \cong \mathcal{D}_+(\mathfrak{F}_R^\perp)$$

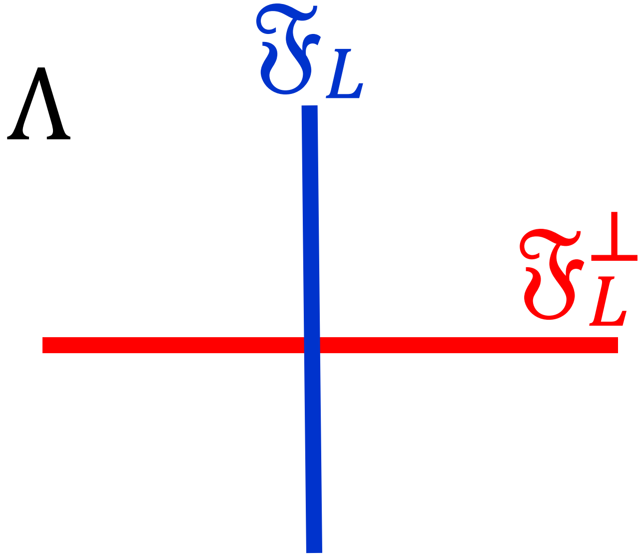
$$\Gamma \subset (\mathfrak{F}_L^\perp)^\vee \oplus (\mathfrak{F}_R^\perp)^\vee \subset \mathbb{R}^{24-d;8-d}$$

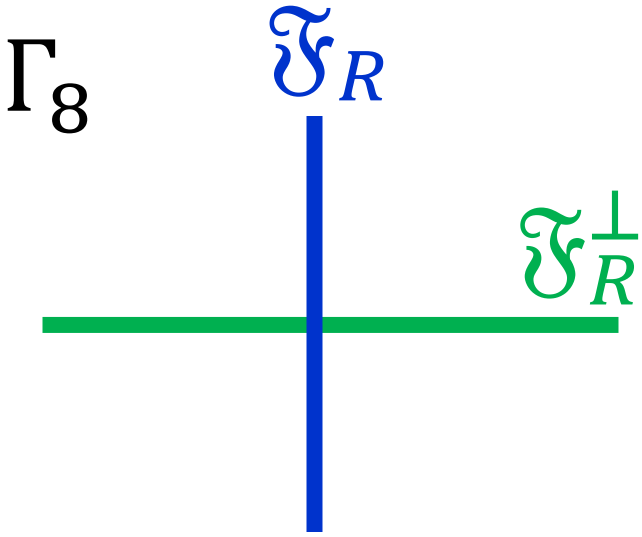
$$\Gamma = \{(x,y) | \bar{x} \cong \bar{y}\}$$

$$g : (x;y) \mapsto (g_L x; g_R y) \qquad \overline{g_L x} = \overline{x} \qquad \overline{g_R y} = \overline{y}$$

CSS Compactifications

This construction defines points of moduli space with
Conway **S**ubgroup **S**ymmetry:
call these CSS compactifications.





$$\Gamma^{n_L;n_R} = (\mathfrak{F}_L^\perp;0) \oplus (0;\mathfrak{F}_R^\perp) + gluevectors$$

CSS Crystal Symmetries

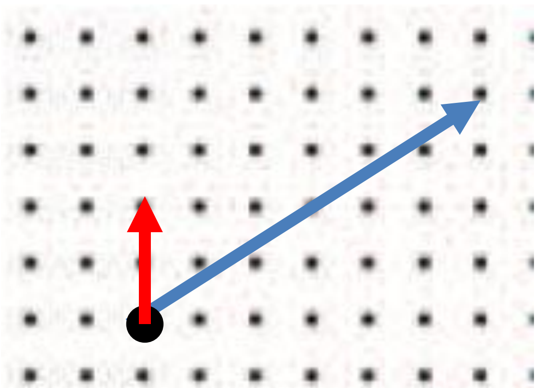
The crystal symmetry of the CSS compactification is

$$G_L \times G_R = Fix(\mathfrak{T}_L) \times Fix(\mathfrak{T}_R) \subset Co_0 \times W(E_8)$$

What crystal symmetries can you get?

In general, a sublattice preserves none of the crystal symmetries of the ambient lattice.

Consider, e.g., the lattice generated by (p,q) in the square lattice in the plane.



Fixed Sublattices Of The Leech Lattice

The culmination of a long line of work is the classification by Hohn and Mason of the 290 isomorphism classes of fixed-point sublattices of the Leech lattice:

221	3	24	$[2^3 3]$ (#3)	8_3^{+3}	0	4	16	1	1	3	Mon_b*
222	2	9196830720	$U_6(2)$	$2_{II}^{-2}3^{+1}$	0	1	1	1	1	-	S*
223	2	898128000	McL	$3^{-1}5^{-1}$	1	1	1	1	1	-	S*
224	2	454164480	$2^{10}.M_{22}$	4_2^{+2}	0	1	1	1	1	-	Mon_a*
225	2	44352000	HS	$2_2^{-2}5^{+1}$	0	1	1	1	1	-	S*
226	2	20643840	$2^9.L_3(4).2$	$4_1^{+1}8_1^{+1}$	0	1	2	1	1	-	Mon_a
227	2	10200960	M_{23}	23^{+1}	1	1	1	1	2	1	M_{23}*

Symmetries For Het/T4

99	4	245760	$2^8{:}M_{20}$	$2_{\mathrm{II}}^{-2}4_{\mathrm{II}}^{-2}$	0	1	1	1	1	-	Mon_a*
100	4	30720	$[2^9].A_5$	$2_{\mathrm{II}}^{-4}5^{-1}$	0	1	1	1	1	-	Mon_a*
101	4	29160	$3^4{:}A_6$	$3^{+2}9^{+1}$	1	1	1	1	1	-	S*
102	4	20160	$L_3(4)$	$2_{\mathrm{II}}^{-2}3^{-1}7^{-1}$	2	1	1	1	2	1	M_{23}*
103	4	12288	$[2^{12}3]$	$2_{\mathrm{II}}^{+2}4_3^{+1}8_1^{+1}$	0	1	2	1	1	-	Mon_a

These discrete groups will be automorphisms of the algebra of BPS states at the CSS points.

Het/II duality implies the space of D4D2D0 BPS states on K3 will naturally be in representations of these subgroups of Co_0 .

Symmetries Of IIA/K3

Theorem [Gaberdiel-Hohenegger-Volpato]: The symmetry groups of K3 sigma models are precisely the subgroups of Co_0 fixing a sublattice with rank ≥ 4.
(``Quantum Mukai Theorem'')

Interpreted by Huybrechts in terms of the bounded derived category of K3 surfaces

$$G \cong \mathrm{Aut}_{H^{2,0} \bigoplus H^{0,2}}(D^b(K3)) \cap \mathrm{Aut}_{B+\mathrm{i}J}(D^b(K3))$$

We are confirming that using Het[T4]=IIA[K3] duality, and the relation of D-branes to the derived category.

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But Is There Moonshine In KKP Invariants?

99	4	245760	$2^8{:}M_{20}$	$2_{\mathrm{II}}^{-2}4_{\mathrm{II}}^{-2}$	0	1	1	1	1	-	Mon_a*
100	4	30720	$[2^9]{:}A_5$	$2_{\mathrm{II}}^{-4}5^{-1}$	0	1	1	1	1	-	Mon_a*
101	4	29160	$3^4{:}A_6$	$3^{+2}9^{+1}$	1	1	1	1	1	-	S*
102	4	20160	$L_3(4)$	$2_{\mathrm{II}}^{-2}3^{-1}7^{-1}$	2	1	1	1	2	1	M_{23}*
103	4	12288	$[2^{12}3]$	$2_{\mathrm{II}}^{+2}4_3^{+1}8_1^{+1}$	0	1	2	1	1	-	Mon_a

So the invariants of KKP will show ``Moonshine'' with respect to this symmetry……

But we want M24, not M20 !!

IS THERE MORE GOING ON ??

So we played the sum dimension game in all possible ways for lowest levels of the generating function of DH states: –

$$\prod_n \frac{1}{(1-q^n)^{20}(1-yzq^n)(1-y^{-1}zq^n)(1-yz^{-1}q^n)(1-y^{-1}z^{-1}q^n)}$$

$$= \sum_{j_L, j_R} q^n \, D^n_{j_L, j_R} \, \chi_{j_L}(y) \chi_{j_R}(z)$$

M24 irreps!

$$D^n_{j_L, j_R} = \sum_a n_a \, R_a$$

The number of possible decompositions grows rapidly with level q^n

For each such decomposition we calculated the graded character of involutions 2A and 2B in M24

The resulting polynomial in q is supposed to be the leading terms of SOME modular form of SOME weight with SOME multiplier system....

Characters Of An Involution

$$Z_{2A} = 8 + 1/q + 36q + 144q^2 + 282q^3$$ + ...

$$= 8 + 1/q + 36q + 144q^2 + 426q^3$$

$$= 8 + 1/q + 36q + 144q^2 + 218q^3$$

$$= 8 + 1/q + 36q + 144q^2 + 362q^3$$

$$= 8 + 1/q + 36q + 144q^2 + 266q^3$$

$$= 8 + 1/q + 36q + 144q^2 + 410q^3$$

$$= 8 + 1/q + 36q + 144q^2 + 202q^3$$

$$= 8 + 1/q + 36q + 144q^2 + 346q^3$$

$$= 8 + 1/q + 36q + 144q^2 + 378q^3$$

$$= 8 + 1/q + 36q + 144q^2 + 522q^3$$

$$= 8 + 1/q + 36q + 144q^2 + 314q^3$$

$$= 8 + 1/q + 36q + 144q^2 + 458q^3$$

Should be modular form for $\Gamma_0(2)$.

Weight?

(assumed half-integral)

Multiplier system?

A Trick

$\Gamma_0(2)$ is generated by T and ST^2S

ST^2S has an ``effective fixed point''

$$ST^2S \cdot \tau = \frac{\tau}{1-2\tau}$$

$$\tau_0 = \frac{1}{2}(1+\mathrm{i}) \qquad (ST^2S)\tau_0 = \tau_0 - 1$$

One can deduce the multiplier system from the weight.

What Is Your Weight?

$$\tau = \tau_0 + \delta\tau$$

$$ST^2S \cdot \tau = \tau_0 - 1 - \delta\tau$$

$$w = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \log \left| \frac{Z(\tau_0 - \mathrm{i}\epsilon)}{Z(\tau_0 + \mathrm{i}\epsilon)} \right|$$

$$q_0 = e^{2\pi \mathrm{i} \tau_0} = -e^{-\pi} = -0.04...$$

Convergence is good so can compute the weight numerically.
For Z_{2A} it converges to -8.4...... Not pretty. Not half-integral !!

No positive combination of reps is modular. No M24 Moonshine.

Part I Conclusions

So, what can we say about Mathieu Moonshine?

GHV: Quantum Mukai theorem:
Symmetries of K3 sigma models are not subgroups of M24

This talk:
M24 does not govern symmetries of nonperturbative spacetime BPS states of type IIA K3 compactifications.

Still leaves the possibility: Algebra of BPS states of the PERTURBATIVE BPS states of IIA on, say, K3 x S1.

Part II
Consistency Conditions For Asymmetric Orbifolds

with Jeff Harvey and Nati Seiberg

still in progress…

Part II Outline

1. Motivation: T2 x K3 Orbifolds Of CSS Points
2. A Mysterious Mod-Two Condition

Application To Heterotic-Type II Duality

Existence of CSS points have some interesting math predictions using string-string duality:

$$\mathrm{Het}/T^2 \times K3' \Longleftrightarrow \mathrm{IIA}/\mathfrak{X}$$

[Ferrara, Harvey, Strominger, Vafa; Kachru, Vafa (1995)]

$\mathcal{X}$ = K3-fibered CY3

Perturbative heterotic BPS states $\Longleftrightarrow$ Vertical D4-D2-D0 boundstates

Generalized Huybrechts Theorem

So, if we can make a suitable orbifold of CSS compactifications of the heterotic string on T6

$$T^6/G \cong T^2 \times K3$$

And if there is a type II dual then we can conclude:

$$\mathrm{Aut}_\sigma(D^b(\mathfrak{X})_{\mathrm{vertical}})$$

is the subgroup of Co_0 fixing the rank two sublattice $\mathfrak{T}_L$ and centralizing the orbifold action.

An Explicit Example -1/2

For simplicity: $\mathbb{Z}_2$ orbifold

$$X \to RX + \delta$$

$$R = (g_L; g_R) \in G_L \times G_R$$

6=8-d for d=2 so we are looking for d=2 isometric sublattices $\mathfrak{F}_L \subset \Lambda$ and $\mathfrak{F}_R \subset \Gamma_8$ fixed by large subgroups of $Co_0 \times W(E_8)$

Gram matrix of $\mathfrak{F}_L \cong \mathfrak{F}_R \ = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

There exists g_R in W(E8) fixing $\mathfrak{F}_R$ with ev's $+1^4$, -1^4. Mod out by this on the right.

An Explicit Example – 2/2

Need to choose involution g_L.

There are four conjugacy classes of involutions in Co_0.

We can choose:

A $$g_L \sim \mathrm{Diag}\{-1^8, +1^{16}\}$$

B $$g_L \sim \mathrm{Diag}\{-1^{16}, +1^8\}$$

C $$g_L \sim \mathrm{Diag}\{-1^{12}, +1^{12}\}$$

The subgroup that centralizes g_L will be a symmetry.

Part II Outline

1. Motivation: T2 x K3 Orbifolds Of CSS Points

2. A Mysterious Mod-Two Condition

Consistency Conditions For Asymmetric Orbifolds

These are examples of asymmetric orbifolds.
Such orbifolds can have anomalies.

So, are they quantum consistent?

Narain, Sarmadi, Vafa (1987) described some general consistency conditions for asymmetric orbifolds. But one condition was a little mysterious:

$$p \cdot gp = 0 \mod 2$$ for all Narain vectors p.

OK for C; Not for A & B

Do We Need The Mod Two Condition?

(Discussion on this includes Nati Seiberg.)

Simplest example of a model that is one-loop-modular but violates the NSV mod two condition:

Consider the product of N circles at self-dual radius.

$\mathbb{Z}_2$ Orbifold group: $R_i \to 1/R_i$

These violate the NSV mod two condition for all N.

Claim 1: The model is anomalous for N = 1,2,3 mod 4

Claim 2: The model is sensible for N = 0 mod 4

Both claims contradict several papers in the literature.

Can You Orbifold By T-Duality?

Would be a nice way to construct ``monodrofolds’’ and ``T-folds’’

[Hellerman & Walcher 2005]

Example: Consider a periodic scalar at the self-dual radius:

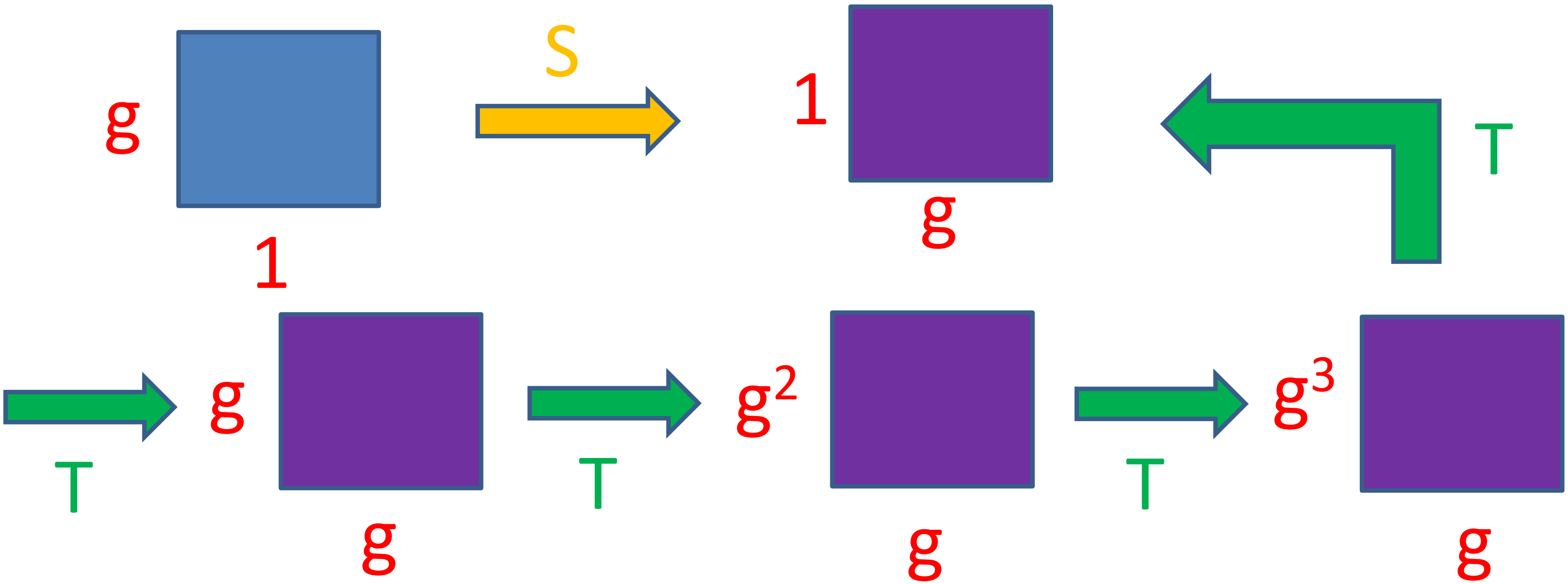
$$g : X_L \to X_L \qquad g : X_R \to -X_R$$

One loop modular anomaly in g-twisted sector:

$$E_L = \frac{1}{2}p_L^2 + \mathbb{Z} \in \frac{1}{4}\mathbb{Z} \qquad E_R = \frac{1}{16} + \frac{1}{2}\mathbb{Z}$$

But *four* copies of the Gaussian model has no one-loop anomaly.

Still, even with level matching satisfied, something peculiar is going on:



So, in the twisted sector, g is actually order four!

Gaussian model at self-dual point has $(SU(2)_L \times SU(2)_R)/\mathbb{Z}_2$ symmetry.

T-duality is a 180 degree rotation in $SO(3)_R$:

$$J^3 \to -J^3 \qquad J^{\pm} \to J^{\mp}$$

Therefore lifts to an order ***four*** element in $SU(2)_R$
-- even in the untwisted sector!

Now the method of orbits gives a sensible Z

Re-Interpreting The NSV Condition

The NSV condition is not a true consistency condition: It means the group acting on the CFT is really $\mathbb{Z}_4$

Conclusion: You can orbifold by simultaneous T-duality of N copies of the Gaussian model for any $N = 0 \; mod \; 4$

You can also show this by constructing proper cocycles for the vertex operators needed for a consistent OPE: The lifted action of the $\mathbb{Z}_2$ automorphism of the Narain lattice lifts to an *order four* automorphism of the CFT.

We are exploring generalizations of this to other orbifolds of toroidal compactifications, together with an interpretation in terms of Chern-Simons theory.

Part III

Holography & Zamolochikov Volumes Of Moduli Spaces of Calabi-Yau Manifolds

N. Benjamin, M. Cheng, S. Kachru, G. Moore, and N. Paquette,
``Elliptic Genera and 3d Gravity,'' arXiv:1503.04800

G. Moore,
``Computation Of Some Zamolodchikov Volumes, With An Application,''
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AdS3/CFT2

$$\mathcal{C}_M = Sym^M(K3)$$

The large M limit of these CFT's exist:

Holographically dual to IIB strings on

$$AdS_3 \times S^3 \times K3$$

(Gives one route to application of Rademacher series and mock modular forms to string theory.)

[Dijkgraaf, Moore, Maldacena, Verlinde 2000; Manschot & Moore 2007]

Some recent activity has centered on question:

``Do more general sequences $\{\mathcal{C}_M\}$ have holographic duals with weakly coupled gravity? "

Put necessary conditions (e.g. existence of a Hawking-Page phase transition) on partition functions $Z(\mathcal{C}_M)$ for a holographic dual of an appropriate type to exist.

Keller; Hartman, Keller, Stoica; Haehl, Rangamani; Belin, Keller, Maloney; ….

Shamit's Question

``How *likely* is it for a sequence of CFT's $\{ \mathcal{C}_M \}$ to have this HP phase transition? "

A complicated question
and hard to make precise.

There is a natural probability distribution on some ensembles of CFT's.

Zamolodchikov Metric

Space of CFT's is thought to have a topology. So we can speak of continuous families and connected components.

At smooth points the space is thought to be a manifold and there is a canonical isomorphism:

$$\Psi : \mathrm{ExactMarg}(\mathcal{C}) \to T_{\mathcal{C}}\mathcal{M}$$

$$\frac{\partial}{\partial t}|_0 S[t] = \int \mathcal{O} \qquad \Psi(\mathcal{O}) = \frac{\partial}{\partial t}|_0 = v \in T_{\mathcal{C}}\mathcal{M}$$

$$\langle \mathcal{O}(z_1)\mathcal{O}(z_2)\rangle := g_{\mathrm{Z}}(v,v)\frac{d^2z_1 d^2z_2}{|z_1-z_2|^4}$$

Remarks

Canonical normalization

Moduli spaces arising in string compactifications and AdS3/CFT2 holography have *finite* Zamolodchikov volumes! [Horne & Moore, 1994; Douglas 2004; Douglas & Lu, 2005]

Moduli space of K3 σ-models:

$$O_{\mathbb{Z}}(II^{20;4})\backslash O_{\mathbb{R}}(20;4)/\left(O_{\mathbb{R}}(20)\times O_{\mathbb{R}}(4)\right)$$

Also for $Sym^m(K3)$ – a similar double-coset

Volumes

Using results from number theory, especially the ``mass formulae'' of Carl Ludwig Siegel, one can -- with some nontrivial work -- compute the Z-volumes of these spaces. For example:

$$\mathrm{vol}_Z(K3) =$$

$$\pi^{-40}\frac{(131)(283)(593)(617)(691)^2(3617)(43867)}{2^{40}\cdot 3^{34}\cdot 5^{15}\cdot 7^{9}\cdot 11^{5}\cdot 13^{4}\cdot 17^{3}\cdot 19^{3}\cdot 23}$$

$$\cong 1.66\times 10^{-61}$$

$$\mathrm{vol}_Z(S^m K3) = ????$$

Much harder, but from C.L. Siegel we get:

$$\frac{\mathrm{vol}(O_{\mathbb{Z}}(Q_{r,s},m)\backslash O_{\mathbb{R}}(Q_{r,s},m))}{\mathrm{vol}(O_{\mathbb{Z}}(Q_{r,s})\backslash O_{\mathbb{R}}(Q_{r,s}))} = \prod_{p<\infty} \alpha_p(m)$$

$$A(d,m,p^t) := \#\{v \mod p^t | Q_{r,s}(v) = 2m \mod p^t\}$$

$$\alpha_p(m) := \lim_{t\to\infty} \frac{A(d,m,p^t)}{p^{t(2d-1)}}$$

$$d = r + 4s$$

$$A(d,m,p^t) := \#\{v \mod p^t | Q_{r,s}(v) = 2m \mod p^t\}$$

$$A(d,p^e,p^t) := p^{t(2d-1)}(1-p^{-d})\frac{1-p^{-(e+1)(d-1)}}{1-p^{-(d-1)}}$$

For p an odd prime & t > e

$$A(d,2^e,2^t) := 2\cdot 2^{t(2d-1)}(1-2^{-d})\frac{1-2^{-(e+1)(d-1)}}{1-2^{-(d-1)}}$$

$$\mathrm{vol}_Z(S^m K3) = \rho m^{42} f_{13}(m)$$

$$f_{13}(m) = \prod_{p|m} \frac{1 - p^{-12-12e_p(m)}}{1 - p^{-12}}$$

$$\rho = \pi^{-42} \frac{(103)(131)(283)(593)(617)(691)(3617)(43867)(2294797)}{2^{51} \cdot 3^{35} \cdot 5^{15} \cdot 7^{10} \cdot 11^{5} \cdot 13^{4} \cdot 17^{3} \cdot 19^{3} \cdot 23^{2}}$$

$$\cong 5.815 \times 10^{-63}$$

An Answer To Shamit's Question

Consider the ensemble of CFT's of the form

$$\mathcal{E} := \{ S^{m_1}K3 \times S^{m_2}K3 \times \cdots \times S^{m_k}K3 \}$$

Have central charge $c = 6(m_1 + \cdots + m_k)$

In a paper with N. Benjamin, M. Cheng, N. Paquette, and S. Kachru we found *necessary* conditions on the elliptic genera of a sequence of CFTs $\mathcal{C}_n$ drawn from this ensemble to have a weakly-coupled gravity dual in the limit of $n \to \infty$

I will call one of these conditions property $\wp$

Rather than put a probability distribution on sequences $\{ \mathcal{C}_n \}$ of CFT's drawn from $\mathcal{E}$ we consider the relative volumes:

$$\frac{vol(\wp; \mathcal{E}_M)}{vol(\mathcal{E}_M)}$$

where $\mathcal{E}_M \subset \mathcal{E}$ is the sub-ensemble of central charge $c = 6M$

Using the above results one can show:

$$\lim_{M \to \infty} \frac{vol(\wp; \mathcal{E}_M)}{vol(\mathcal{E}_M)} = 0$$

Conclusion: Almost every sequence $\{\mathscr{C}_n\}$ does not have a holographic dual

Summary

Part 1: There are heterotic compactifications with large crystal symmetries related to the Conway group, but there is no M24 Moonshine in KKP's refined K3 BPS invariants.

Part 2: Consistency conditions for asymmetric orbifolds should be re-examined. Sometimes one *can* orbifold by T-duality, contrary to standard wisdom.

Part 3: Zamolodchikov volumes of certain moduli spaces of CFT's are interesting. The set of sequences of CFT's with weakly coupled gravity duals is measure zero (suitably defined).