Global Anomalies In Six-Dimensional Supergravity

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1 Introduction & Summary Of Results

2 Six-dimensional Sugra & Green-Schwarz Mech.

3 Quantization Of Anomaly Coefficients.

4 Geometrical Anomaly Cancellation, $\eta$-Invariants & Wu-Chern-Simons

5 Technical Tools & Future Directions

6 F-Theory Check

7 Concluding Remarks
Motivation

Relation of apparently consistent theories of quantum gravity to string theory.

From W. Taylor’s TASI lectures:

State of art summarized in Brennan, Carta, and Vafa 1711.00864
Brief Summary Of Results

Focus on 6d sugra

(More) systematic study of global anomalies

Result 1: NECESSARY CONDITION:
unifies & extends all previous conditions

Result 2: NECESSARY & SUFFICIENT:
A certain 7D TQFT $Z_{TOP}$ must be trivial.

But effective computation of $Z_{TOP}$ in
the general case remains open.

Result 3: Check in F-theory:
(Requires knowing the global form of the
identity component of the gauge group.)
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7. Concluding Remarks
(Pre-) Data For 6d Supergravity

(1,0) sugra multiplet + vector multiplets + hypermultiplets + tensor multiplets

VM: Choose a (possibly disconnected) compact Lie group $G$.

HM: Choose a quaternionic representation $\mathcal{R}$ of $G$

TM: Choose an integral lattice $\Lambda$ of signature $(1,T)$

Pre-data: $\quad (G, \mathcal{R}, \Lambda)$
6d Sugra - 2

Can write multiplets, Lagrangian, equations of motion. [Riccioni, 2001]

Fermions are chiral (symplectic Majorana-Weyl)

2-form field strengths are (anti-)self dual

<table>
<thead>
<tr>
<th>Multiplet</th>
<th>Field Content</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gravity</td>
<td>( (g_{\mu\nu}, \psi_\mu^+, B_{\mu\nu}^+) )</td>
</tr>
<tr>
<td>Tensor</td>
<td>( (B_{\mu\nu}^-, \chi^-, \phi) )</td>
</tr>
<tr>
<td>Vector</td>
<td>( (A_\mu, \lambda^+) )</td>
</tr>
<tr>
<td>Hyper</td>
<td>( (\psi^-, 4\phi) )</td>
</tr>
<tr>
<td>Half-hyper</td>
<td>( (\psi^-_R, 2\phi) )</td>
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</tbody>
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The Anomaly Polynomial

Chiral fermions & (anti-)self-dual tensor fields $\Rightarrow$ gauge & gravitational anomalies.

From $(G, R, \Lambda)$ we compute, following textbook procedures,

$$I_8 \sim (\dim_{\mathbb{H}}(R) - \dim(G) + 29T - 273)Tr(R^4) + \cdots$$

$$+ (9 - T)(Tr R^2)^2 + (F^4-\text{type}) + \cdots$$

6d Green-Schwarz mechanism requires

$$I_8 = \frac{1}{2} \mathcal{Y}^2 \quad \mathcal{Y} \in \Omega^4(\mathcal{W}; \Lambda \otimes \mathbb{R})$$
Standard Anomaly Cancellation

Interpret $Y$ as background magnetic current for the tensor-multiplets $\Rightarrow$

$$dH = Y$$

$\Rightarrow B$ transforms under diff & VM gauge transformations...

Add counterterm to sugra action

$$e^{iS} \rightarrow e^{iS} e^{-2\pi i \frac{1}{2} \int BY}$$
So, What’s The Big Deal?
Definition Of Anomaly Coefficients

Let's try to factorize:

\[ I_8 = \frac{1}{2} Y^2 \quad Y \in \Omega^4(\mathcal{W}; \Lambda \otimes \mathbb{R}) \]

\[ g = g_{SS} \oplus g_{Abel} \cong \bigoplus_i g_i \oplus I \ u(1) \]

General form of \( Y \):

\[ Y = \frac{a}{4} p_1 - \sum_i b_i c_i^2 + \frac{1}{2} \sum_{ij} b_{ij} c_1^i c_1^j \]

\[ p_1 := \frac{1}{8\pi^2} \text{Tr}_{vec} R^2 \]

\[ c_2^i := \frac{1}{16\pi^2 h_i^\nu} \text{Tr}_{adj} F_i^2 \]

Anomaly coefficients:

\[ a, b_i, b_{ij} \in \Lambda \otimes \mathbb{R} \]
The Data Of 6d Sugra

The very existence of a factorization $I_8 = \frac{1}{2} Y^2$ puts constraints on $(G, \mathcal{R}, \Lambda)$. These have been well-explored. For example:...

$$\dim_{\mathbb{H}} \mathcal{R} - \dim G + 29 T - 273 = 0$$

$$a^2 = 9 - T, \ldots$$

Also: There are multiple choices of anomaly coefficients $(a, b_i, b_{IJ})$ factoring the same $I_8$

Full data for 6d sugra:

$$(G, \mathcal{R}, \Lambda) \ \ \ \text{AND} \ \ \ a, b_i, b_{IJ} \in \Lambda \otimes \mathbb{R}$$
For any \((G, R, \Lambda, a, b)\) adding the GS term cancels all perturbative anomalies.

All is sweetness and light...
There are solutions of the factorizations conditions that cannot be realized in F-theory!

Global anomalies?

Does the GS counterterm even make mathematical sense?
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Global anomalies have been considered before. We have just been a little more systematic.

To state the best result we note that $b = (b_i, b_{IJ})$ determines a $\Lambda \otimes \mathbb{R}$-valued quadratic form on $g$:

Vector space $Q$ of such quadratic forms arises in topology: $Q \cong H^4(BG_1; \Lambda \otimes \mathbb{R})$

$H^4(BG_1; \Lambda) \subset Q$

$\frac{1}{2} b \in H^4(BG_1; \Lambda)$
A Derivation

A consistent sugra can be put on an arbitrary spin 6-fold with arbitrary gauge bundle.

Cancellation of background string charge in compact Euclidean spacetime \( \Rightarrow \forall \Sigma \in H_4(\mathcal{M}_6; \mathbb{Z}) \)

\[
\int_{\Sigma} Y \in \Lambda
\]

Because the background string charge must be cancelled by strings.

This is a **NECESSARY** but not (in general) **SUFFICIENT** condition for cancellation of all global anomalies…
6d Green-Schwarz Mechanism Revisited

Goal: Understand Green-Schwarz anomaly cancellation in precise mathematical terms.

Benefit: We recover the constraints:

\[ \frac{1}{2} b \in H^4(BG_1; \Lambda) \quad a \in \Lambda \quad \Lambda^v \cong \Lambda \]

and derive a new constraint:

\[ a \text{ is a characteristic vector:} \]

\[ \forall v \in \Lambda \quad v \cdot v = v \cdot a \mod 2 \]
What’s Wrong With Textbook Green-Schwarz Anomaly Cancellation?

What does $B$ even mean when $\mathcal{M}_6$ has nontrivial topology? ($H$ is not closed!)

How are the periods of $dB$ quantized?

Does the GS term even make sense?

$$\frac{1}{2} \int_{\mathcal{M}_6} B \ Y = \frac{1}{2} \int_{\mathcal{U}_7} dB \ Y$$

must be independent of extension to $\mathcal{U}_7$!
But it isn’t ….

Even for the difference of two B-fields,

$$d(H_1 - H_2) = 0$$

we can quantize $[H_1 - H_2] \in H^3(\mathcal{U}_7; \Lambda)$

$$\exp\left( 2\pi i \frac{1}{2} \int_{\mathcal{U}_7} (H_1 - H_2)Y \right)$$

is not well-defined because of the factor of $\frac{1}{2}$. 
Quantization Of Anomaly Coefficients.

Six-dimensional Sugra & Green-Schwarz Mech.

Geometrical Anomaly Cancellation, \( \eta \)-Invariants & Wu-Chern-Simons

Technical Tools & Future Directions

F-Theory Check

Concluding Remarks
Geometrical Formulation Of Anomalies

Space of all fields in 6d sugra is fibered over nonanomalous fields:

$$\mathcal{B} = \text{Met}(\mathcal{M}_6) \times \text{Conn}(\mathcal{P}) \times \{\text{Scalar fields}\}$$

Partition function:

$$\Psi_{\text{Anomaly}}(A, g_{\mu\nu}, \phi) := \int_{\mathcal{B}/\mathcal{G}} e^{S_0 + S_{\text{Fermi}} + B}$$

You cannot integrate a section of a line bundle over $\mathcal{B}/\mathcal{G}$ unless it is trivialized.
Approach Via Invertible Field Theory

Definition [Freed & Moore]:
An invertible field theory $Z$ has

Partition function $\in \mathbb{C}^*$

One-dimensional Hilbert spaces of states ...

satisfying natural gluing rules.

Freed: Geometrical interpretation of anomalies in d-dimensions = Invertible field theory in (d+1) dimensions
Invertible Anomaly Field Theory

Interpret anomaly as a 7D invertible field theory $Z_{Anomaly}$ constructed from $G, \mathcal{R}, \Lambda, \mathcal{B}$

Data for the field theory: $G$-bundles $\mathcal{P}$ with gauge connection, Riemannian metric, spin structure $\varsigma$. (it is NOT a TQFT!)

Varying metric and gauge connection $\Rightarrow$

$Z_{Anomaly}(\mathcal{M}_6)$ is a LINE BUNDLE

$\Psi_{Anomaly}$ is a SECTION of $Z_{Anomaly}(\mathcal{M}_6)$
Anomaly Cancellation In Terms of Invertible Field Theory

1. Construct a "counterterm"
   7D invertible field theory $Z_{CT}$

   \[ Z_{CT}(\mathcal{M}_6) \cong Z_{Anomaly}(\mathcal{M}_6)^* \]

2. Using just the data of the local fields in six dimensions, we construct a section:

   \[ \Psi_{CT} \in Z_{CT}(\mathcal{M}_6) \]

   Then:

   \[ \int_{Fermi+B} e^{S_{Fermi+B}} \Psi_{CT} \]

   is canonically a function on $\mathcal{B}/\mathcal{G}$
Dai-Freed Field Theory

$D$: Dirac operator in ODD dimensions.

$\xi(D) := \frac{\eta(D) + \dim \ker(D)}{2}$

$e^{2\pi i \xi(D)}$ defines an invertible field theory

[Dai & Freed, 1994]

If $\partial \mathcal{U} = \emptyset$ then $Z_{DaiFreed}(\mathcal{U}) = e^{2\pi i \xi(D)}$

If $\partial \mathcal{U} = \mathcal{M} \neq \emptyset$ then $e^{2\pi i \xi(D)}$ is a section of a line bundle over the space of boundary data.

Suitable gluing properties hold.
Anomaly Field Theory For 6d Sugra

On 7-manifolds $\mathcal{U}_7$ with $\partial \mathcal{U}_7 = \emptyset$

$$Z_{\text{Anomaly}}(\mathcal{U}_7) = \exp[2\pi i \left( \xi(D_{\text{Fermi}}) + \xi(D_{B\text{-field}}) \right)]$$

On 7-manifolds with $\partial \mathcal{U}_7 = \mathcal{M}_6$:

The sum of $\xi$–invariants defines a unit vector $\Psi_{\text{Anomaly}}$ in a line $Z_{\text{Anomaly}}(\mathcal{M}_6)$
Simpler Expression When \((U_7, P)\) Extends To Eight Dimensions

In general it is impossible to compute \(\eta\)-invariants in simpler terms.

But if the matter content is such that \(I_8 = \frac{1}{2} Y^2\)

\textbf{AND if} \((U_7, P)\) is bordant to zero:

\[ Z_{Anomaly}(U_7) = \exp\left(2\pi i \left(\frac{1}{2} \int_{W_8} Y^2 - \frac{\text{sign}(\Lambda)\sigma(W_8)}{8}\right)\right) \]

When can you extend \(U_7\) and its gauge bundle \(P\) to a spin 8-fold \(W_8\)??
Spin Bordism Theory

$$\Omega^\text{spin}_7 = 0 :$$

Can always extend spin $U_7$ to spin $W_8$

$$\Omega^\text{spin}_7 (BG) : \text{Can be nonzero: There can be obstructions to extending a } G\text{-bundle } P \to U_7 \text{ to a } G\text{-bundle } \tilde{P} \to W_8$$

$$\Omega^\text{spin}_7 (BG) = 0 \text{ for many groups, e.g. products of } U(n), SU(n), Sp(n). \text{ Also } E_8$$

But for some $G$ it is nonzero!
When 7D data extends to $\mathcal{W}_8$ the formula

$$Z_{\text{Anomaly}}(\mathcal{U}_7) = \exp\left( 2\pi i \left( \frac{1}{2} \int_{\mathcal{W}_8} Y^2 - \frac{\text{sign}(\Lambda)\sigma(\mathcal{W}_8)}{8} \right) \right)$$

$\Rightarrow$ clue to constructing $Z_{CT}$:

$$Z_{\text{Anomaly}}(\mathcal{U}_7) = \exp\left( 2\pi i \int_{\mathcal{W}_8} \frac{1}{2} X(X + \lambda') \right)$$

$$X = Y - \frac{1}{2} \lambda' \quad \lambda' = \alpha \otimes \lambda \quad \lambda \equiv \frac{1}{2} \rho_1$$

Thanks to our quantization condition on $b$, $X \in \Omega^4(\mathcal{W}_8; \Lambda)$ has coho class in $H^4(\mathcal{W}_8; \Lambda)$
This is the partition function of a 7D topological field theory known as “Wu-Chern-Simons theory.”

\[ \exp(2\pi i \int_{\mathcal{W}_8} \frac{1}{2} X(X + \lambda') ) \]

is independent of extension ONLY if

\[ a \in \Lambda \text{ is a characteristic vector:} \]
\[ \forall \nu \in \Lambda \quad \nu^2 = \nu \cdot a \mod 2 \]

This is the partition function of a 7D topological field theory known as “Wu-Chern-Simons theory.”
Wu-Chern-Simons Theory

Generalizes spin-Chern-Simons to \( p \)-form gauge fields.

Developed in detail in great generality by Samuel Monnier arXiv:1607.0139

Our case: 7D TFT \( Z_{WCS} \) of a (locally defined) 3-form gauge potential \( C \) with fieldstrength \( X = dC \)

\[
[X] \in H^4(\cdots; \Lambda)
\]

Instead of spin structure we need a ``Wu-structure'': A trivialization \( \omega \) of:

\[
\nu_4 = w_4 + w_3 w_1 + w_2^2 + w_1^4
\]
Wu-Chern-Simons

In our case $n_4 = w_4$ will have a trivialization in 6 and 7 dimensions, but we need to choose one to make sense of $Z_{WCS}(U_7)$ and $Z_{WCS}(M_6)$

$$Z_{WCS}(U_7) = \exp\left(-2\pi i \int_{W_8} \frac{1}{2} X(X + \lambda') \right)$$

$$\lambda' = a \otimes \lambda$$

$a$ must be a characteristic vector of $\Lambda$

$$\Lambda^v \cong \Lambda$$
Defining $Z_{CT}$ From $Z_{WCS}$

To define the counterterm line bundle $Z_{CT}$ we want to evaluate $Z_{WCS}$ on $(\mathcal{M}_6, Y)$.

Problem 1: $Y$ is shifted: $[Y] = \frac{1}{2} a \otimes \lambda + [X]$

$[X] = \sum b_i c^i_2 \quad + \quad \frac{1}{2} \sum b_{IJ} c^I_1 c^J_1 \in H^4(\cdots; \Lambda)$

Problem 2: $Z^\omega_{WCS}$ needs a choice of Wu-structure $\omega$.

!! We do not want to add a choice of Wu structure to the defining set of sugra data $(G, R, \Lambda, a, b)$
Defining $Z_{CT}$ From $Z_{WCS}$

Solution: Given a Wu-structure $\omega$ we can shift $Y$ to $X = Y - \frac{1}{2} \nu(\omega)$, an unshifted field, such that $Z_{WCS}^{\omega}(\ldots; Y - \frac{1}{2} \nu(\omega))$ is independent of $\omega$

$$Z_{CT}(\ldots; Y) := Z_{WCS}^{\omega}(\ldots; Y - \frac{1}{2} \nu(\omega))$$

Thus, $Z_{CT}$ is independent of Wu structure $\omega$: So no need to add this extra data to the definition of 6d sugra.

$Z_{CT}$ transforms properly under B-field, diff, and VM gauge transformations: $Z_{CT}(\mathcal{M}_6) \cong Z_{Anomaly}(\mathcal{M}_6)^*$
Anomaly Cancellation

\[ Z_{TOP} := Z_{Anomaly} \times Z_{CT} \] is a 7D topological field theory that is defined bordism classes of \( G \)-bundles.

7D partition function is a homomorphism:

\[ Z_{Anomaly} \times Z_{CT} : \Omega_{7}^{spin}(BG) \to U(1) \]

If this homomorphism is trivial then

\[ Z_{Anomaly} \times Z_{CT}(\mathcal{M}_6) \cong 1 \]

is canonically trivial.
Anomaly Cancellation

Suppose the 7D TFT is indeed trivializable

Now need a section, $\Psi_{CT}(\mathcal{M}_6)$ which is local in the six-dimensional fields. This will be our Green-Schwarz counterterm:

$$\int e^{S_{Fermi+B}} \Psi_{CT}(\mathcal{M}_6; A, g_{\mu\nu}, B)$$

The integral will be a function on $\mathcal{B}/\mathcal{G}$
Checks & Hats: 
Differential Cohomology

\[ H^k(X) \rightarrow \tilde{H}^k(X) \]
Checks & Hats: Differential Cohomology

Precise formalism for working with $p$-form fields in general spacetimes (and $p$-form global symmetries)

Three independent pieces of gauge invariant information:

- Wilson lines
- Fieldstrength
- Topological class

Differential cohomology is an infinite-dimensional Abelian group that precisely accounts for these data and nicely summarizes how they fit together.

Exposition for physicists: Freed, Moore & Segal, 2006
Construction Of The Green-Schwarz Counterterm:

\[ \Psi_{CT} = \exp 2\pi i \int_{\mathcal{M}_6}^{E,\omega} gst \]

\[ gst = \left( \frac{1}{2} \left[ \left( \tilde{H} - \frac{1}{2} \tilde{\eta} \right) \cup \left( \tilde{Y} + \frac{1}{2} \tilde{\nu} \right) \right] \right)_{hol}, h_2 - \frac{1}{2} \eta \]

Section of the right line bundle & independent of Wu structure \( \omega \).

Locally constructed in six dimensions, but makes sense in topologically nontrivial cases.

Locally reduces to the expected answer
Conclusion: All Anomalies Cancel:

for \((G, R, \Lambda, a, b)\) such that:

\[ I_8 = \frac{1}{2} Y^2 \]

\[ a \in \Lambda \cong \Lambda^Y \text{ is characteristic & } a^2 = 9 - T \]

\[ \frac{1}{2} b \in H^4(BG_1; \Lambda) \]

\[ \Omega_{7}^{spin}(BG) = 0 \]
Except,...
What If The Bordism Group Is Nonzero?

We would like to relax the last condition, but it could happen that

\[ Z_{Anomaly} \times Z_{CT} : \Omega^\text{spin}_7(BG) \to U(1) \]

defines a nontrivial bordism invariant.

For example, if \( G = O(N) \), for suitable representations, the 7D TFT might have partition function

\[ \exp 2\pi i \int_{U_7} w_1^7 \]

Then the theory would be anomalous.
Future Directions

Understand how to compute

$$Z_{TOP} := Z_{Anomaly} \times Z_{CT}$$

When $\Omega_7^{spin}(BG)$ is nonvanishing there will be new conditions.
(Examples exist!!)

Finding these new conditions in complete generality looks like a very challenging problem...
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And What About F-Theory?
$g$ is determined from the discriminant locus  [Morrison & Vafa 96]

In order to check $\frac{1}{2} b \in H^4(BG_1; \Lambda)$
we clearly need to know $G_1$.

We found a way  F-theory passes
to determine $G_1$.  this test.

We believe a very similar argument also gives
the (identity component of) 4D F-theory.
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