On the role of six-dimensional (2,0) theories in recent developments in Physical Mathematics

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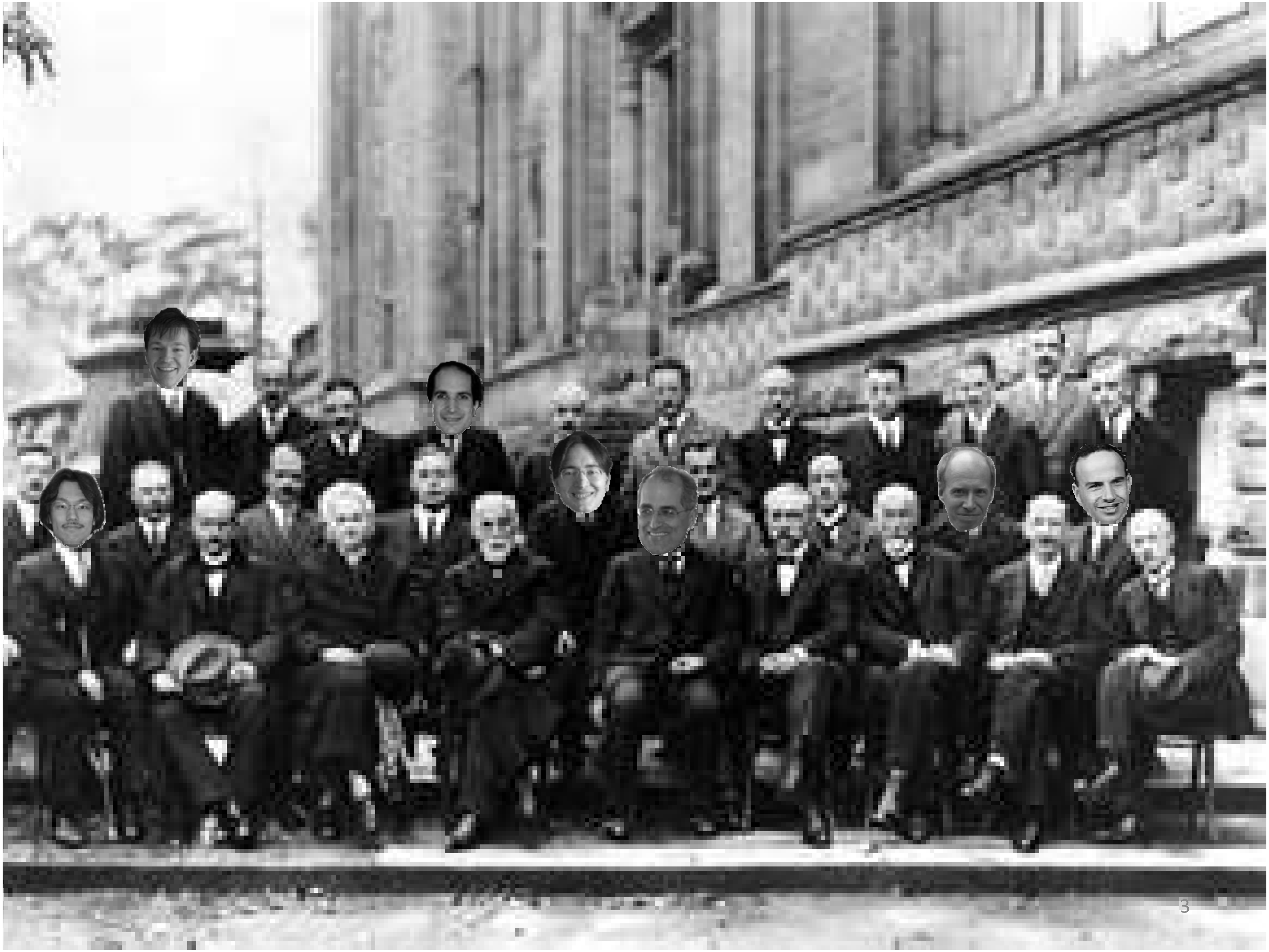
Preface

``… Would you be interested in giving the review ***5-hour*** talk on the applications of six-dimensional (2,0) theories to Physical Mathematics ? …''

``… O.K. I'll give a 1-hour talk, but I'm going to make some heterodox choices …''

``*Nobody goes there anymore; it's too crowded.*'' -- Yogi Berra

My presentation is strongly influenced by many discussions with a number of great physicists. Particular thanks go to



Superconformal Algebras

Nahm's Theorem (1977): Classifies superconformal algebras.
Existence relies on special isomorphisms of Lie algebras related to Clifford algebras, and in particular they only exist in dimensions D ≤ 6.

$$(0,2): \quad osp(6,2|4) \supset so(6,2) \oplus so(5)$$

Poincare subalgebra:

$$\{Q^i_\alpha, Q^j_\beta\} = J^{ij} P_{[\alpha\beta]} + Z^{[ij]}_{[\alpha\beta]} + Z^{(ij)}_{(\alpha\beta)}$$

BPS strings

BPS codimension two objects

Howe, Lambert, West (1997)

The Abelian Tensormultiplet

Fields: H, X^I, ψ^a $I = 1, \ldots, 5,$
$a = 1, \ldots, 4$

$$\partial \cdot \partial X^I = 0 \qquad \gamma \cdot \partial \psi^a = 0$$

$$H \in \Omega^3(M_6) \quad dH = 0 \quad H = *H$$

There is a generalization to theories of many tensormultiplets: $H \in \Omega^3(M_6;V)$

Subtleties

But, already these free field theories are subtle when put on arbitrary manifolds:

Action principle

Partition functions

Charge lattice & Dirac quantization

Hilbert space & Hamiltonian formulation

Formulating these properly is nontrivial and even points to the need to generalize the standard notion of ``field theory.''

Some key papers

1. E. Witten, The five-brane partition function 1996; Duality Relations 1999; Geometric Langlands from Six Dimensions, 2009

2. M. Hopkins and I. Singer, Quadratic Functions in Geometry, Topology and M-Theory, 2002

3. Dolan and Nappi, 1998

4. M. Henningson, et. al. 2000-2010

5. E. Diaconescu, D. Freed, G. Moore 2003; G. Moore, Strings 2004

6. D. Freed, G. Moore, and G. Segal; D. Belov & G. Moore 2006

7. S. Monnier, 2010

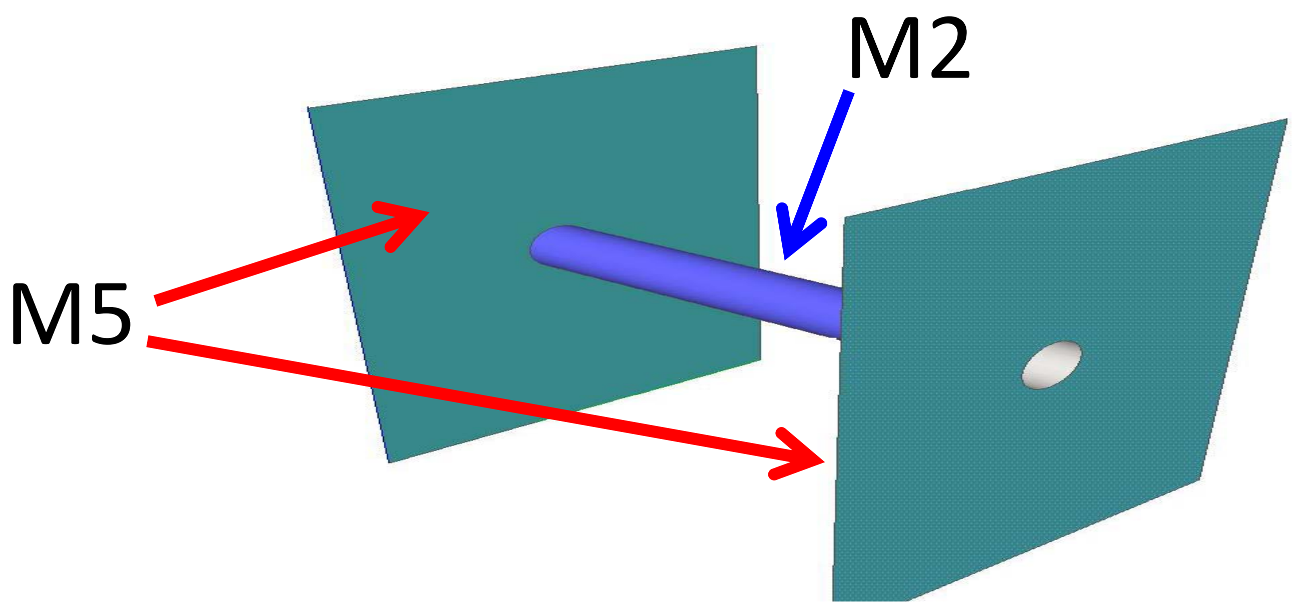
8. N. Seiberg and W. Taylor, 2011

9. Applications to Alexandrov, Persson, Pioline, Vandoren on Sugra HMs.

String/M-theory constructions of interacting theories

Witten, 1995, ``Some comments on string dynamics,'' IIB theory on a hyperkahler ADE singularity with a decoupling limit.

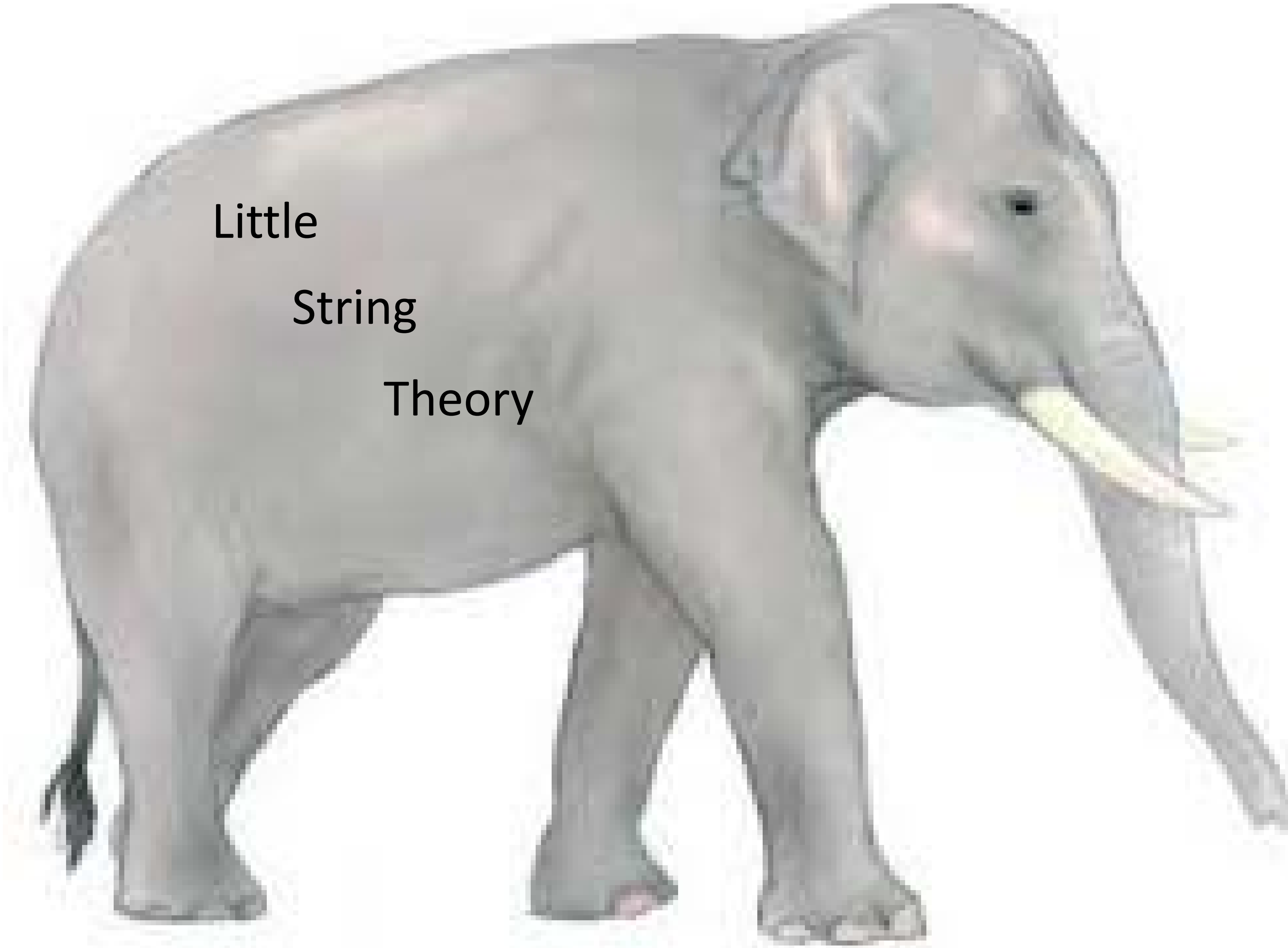
Strominger, 1995: ``Open p-branes.''



But is it *field theory*?

Following some important works on probe-brane theories (Seiberg; Seiberg & Witten), and three-dimensional mirror symmetry (Intriligator-Seiberg), Seiberg stressed in 1996 that the decoupled theories should be viewed as *local quantum field theories*.

Note this is not obvious, given the elephant in the room.


Little
String
Theory

Summary of Section 1

So we conclude that

Already the free abelian theories are very nontrivial on general backgrounds.

There are hypothetical interacting theories $S[\mathfrak{g}]$, for simply laced $\mathfrak{g}$ -- we are going to try to learn something about their dynamics.

Outline

- Introduction: Abelian & Nonabelian (2,0) Theories
- ~~Extended Topological Field Theory~~
- Characteristic properties of S[$\mathfrak{g}$]
- Theories of class S
- BPS States
- Line defects and framed BPS States
- Surface defects
- Hyperkahler geometry
- N=2, d=4 Geography
- Egregious Omissions

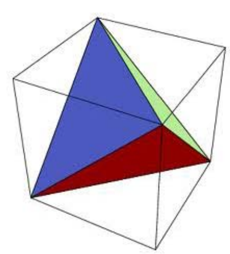
2

Extended Field Theories

A key idea of the Atiyah-Segal definition of TFT is to encode the most basic aspects of locality in QFT.

Axiomatics encodes some aspects of QFT locality:
Gluing = composition of ``morphsims''.

Unsatisfactory: In a truly local description we should be able to build up the theory from a simplicial decomposition.



``If you don't go to other peoples' funerals, they won't go to yours.'' -- Yogi Berra

What is the axiomatic structure that would describe such a completely local decomposition?

D. Freed; D. Kazhdan; N. Reshetikhin; V. Turaev; L. Crane; Yetter; M. Kapranov; Voevodsky; R. Lawrence; J. Baez + J. Dolan ; G. Segal; M. Hopkins, J. Lurie, C. Teleman,L. Rozansky, K. Walker, A. Kapustin, N. Saulina,…

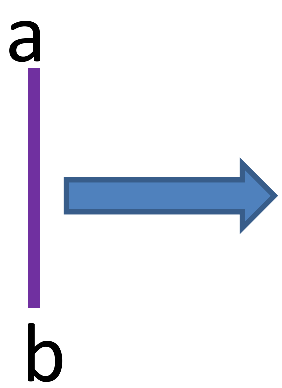
Example: 2-1-0 TFT:

$$F(M_2) \in \mathbb{C}$$ Partition Function

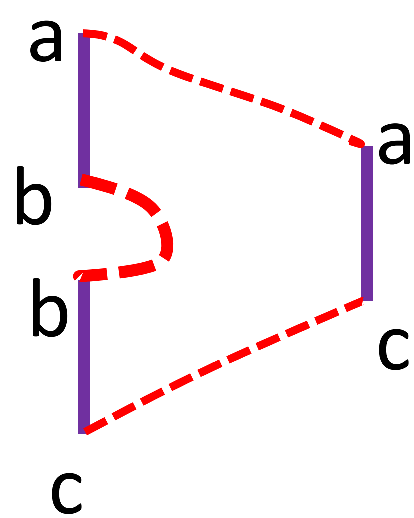
$$F(M_1) \in VECT$$ Hilbert Space

$$F(M_0) \in \mathrm{CAT}$$ Boundary conditions

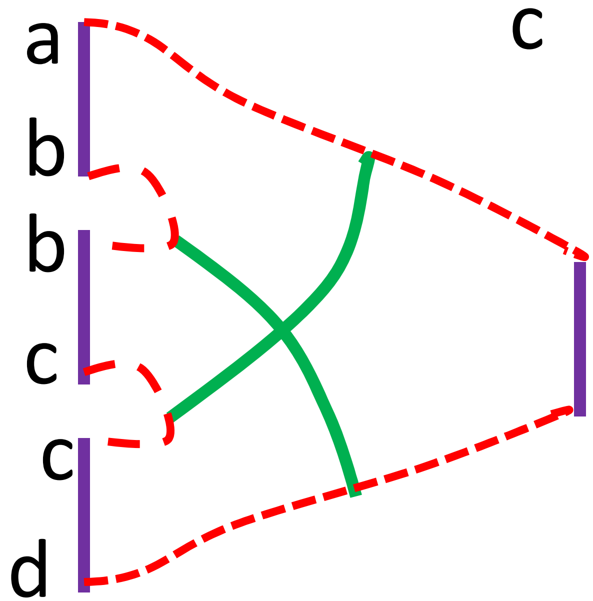
Why are boundary conditions objects in a category?



$\mathcal{O}_{ab}$



$$\mathcal{O}_{ab} \times \mathcal{O}_{bc} \to \mathcal{O}_{ac}$$



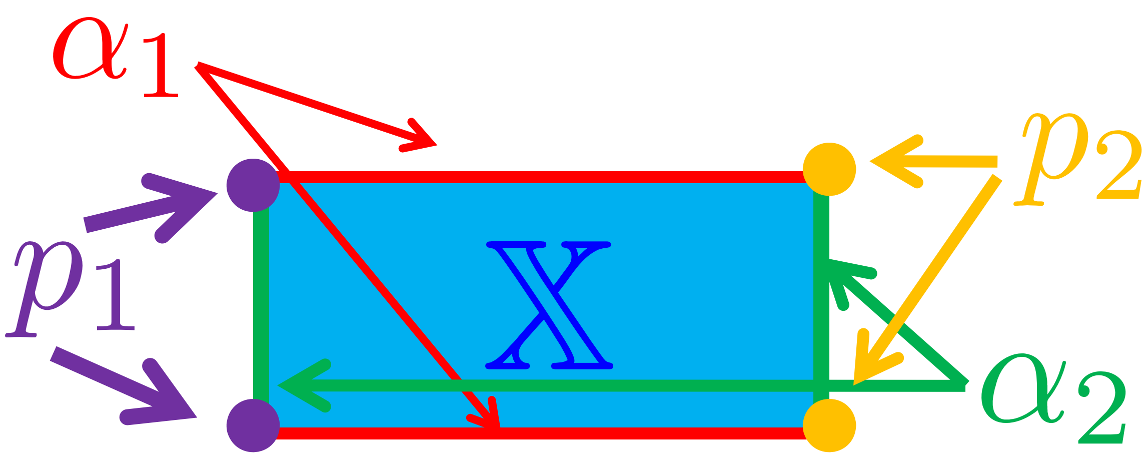
associative

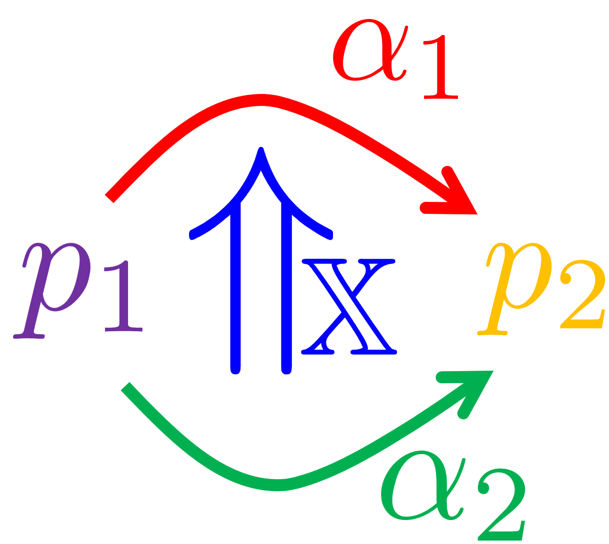
Clay volume: Branes & Mirror Symmetry

N-Categories

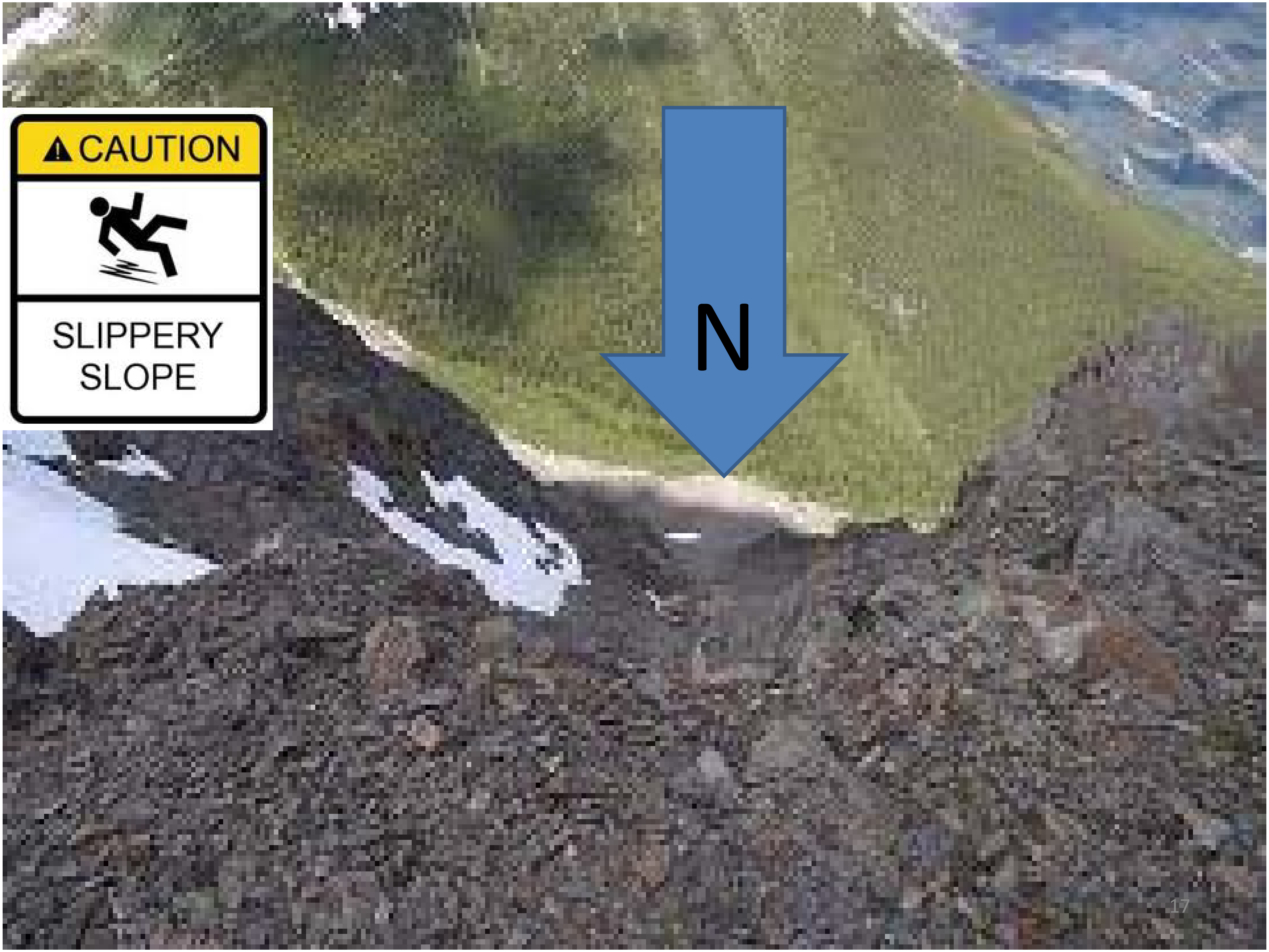
Definition: An N-category is a category C whose morphism spaces are N-1 categories.

Bord_n: Objects = Points; 1-Morphisms = 1-manifolds; 2-Morphisms = 2-manifolds (with corners); …





Definition: An N-extended field theory is a ``homomorphism'' from Bord_n to a symmetric monoidal N-category.


CAUTION
SLIPPERY SLOPE
N

Defects

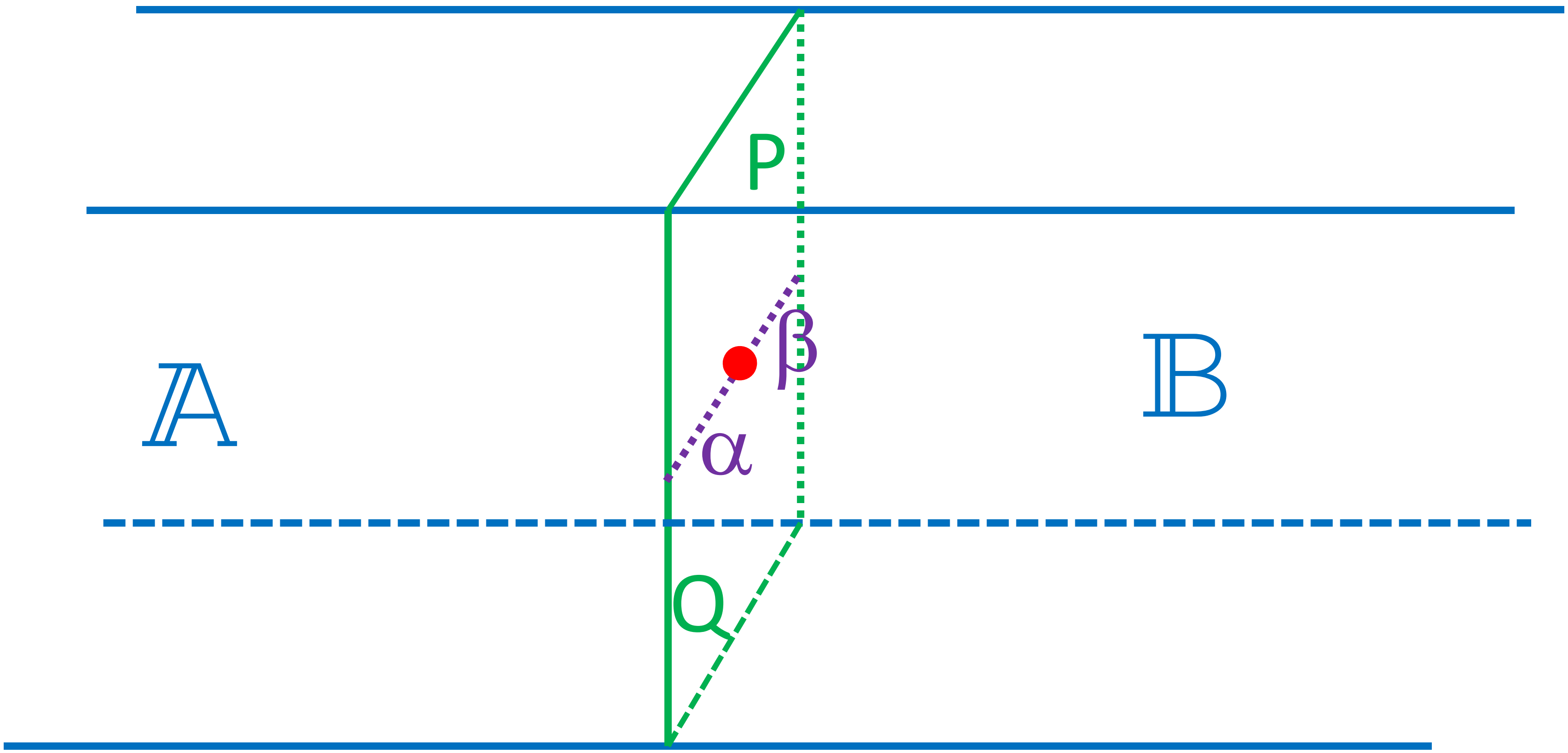
A nice physical way to approach this topic is to study defects in QFT – i.e. local disturbances of positive codimension.

Kapustin's ICM 2010 talk argues that $F(M_{n-k})$ is the category of (k-1)-dimensional defects .

Categories of boundary conditions have been very important in homological mirror symmetry (Kontsevich 1994), whose physical significance was elucidated by Douglas and collaborators around 2001-2002.

Subsequently, these categories have been important in the physical approach to the Geometric Langlands Program in the work of Kapustin-Witten and Gukov-Witten.

Defects Within Defects



Conclusion: Spatial boundary conditions in an n-dimensional ETFT are objects in an (n-1)-category:

k-morphism = (n-k-1)-dimensional defect in the boundary.

The Cobordism Hypothesis

$F(M_n) \in \mathbb{C}$ Partition Function

$F(M_{n-1}) \in VECT$ Hilbert Space

$F(M_{n-2}) \in CAT$ Boundary conditions

$$F(M_{n-k}) \in k - CAT$$

Cobordism Hypothesis of Baez & Dolan: An n-extended TFT is entirely determined by the n-category attached to a point.

For TFTs satisfying a certain finiteness condition this was proved by Jacob Lurie. Expository article. Extensive books.

Generalization: Theories valued in field theories

DEFINITION: An m-dimensional theory $\mathfrak{H}$ valued in an n-dimensional field theory F, where n= m+1, is one such that

$$\mathfrak{H}(N_j) \in F(N_j) \qquad j = 0,1,\ldots, m$$

The ``**partition function**'' of $\mathfrak{H}$ on N_m is a ***vector in a vector space***, and not a complex number . The Hilbert space…

1. The chiral half of a RCFT.
2. The abelian tensormultiplet theories

3 Important characteristics of the six-dimensional (2,0) theories S[$\mathfrak{g}$]

These theories have not been constructed – even by physical standards - but some characteristic properties of these hypothetical theories can be deduced from their relation to string theory and M-theory.

Three key properties + Four important ingredients

These properties will be treated as axiomatic.
Later they should be theorems.

1: What sort of theory is it?

It should be defined on 6-manifolds with certain topological data:

- Orientation
- Spin structure
- Quadratic functions on various cohomology theories

But we require a generalization of the notion of field theory.
(Witten 1998; Witten 2009)

The theory of singletons suggests that it is a ``six-dimensional field theory valued in a 7-dimensional topological field theory. "

For S[u(N)] it is 7-dimensional Chern-Simons field theory

$$N \int C dC$$

C is a 3-form gauge potential in $\Omega^3(M_7)$

2: Why is it interacting?

When S[$\mathfrak{g}$] is KK reduced on $\mathbb{R}^{1,4} \times S^1$,
where S^1 has radius R, with nonbounding, or Ramond spin structure, the long distance dynamics is governed by a maximally supersymmetric five-dimensional Yang-Mills theory with a gauge Lie algebra $\mathfrak{g}$

$$\frac{1}{R}\int_{M_5} \mathrm{Tr}\left(F * F + DX^I * DX^I + \cdots\right)$$

$$g^2_{YM} = R$$

3: Low Energy Physics

The theory in Minkowski space has a moduli space of vacua given by $\mathcal{M} = (\mathbb{R}^5 \otimes \mathfrak{t})/W$

$$\mathcal{M}(u(N); \mathbb{R}^{1,5}) = (\mathbb{R}^5 \otimes \mathbb{R}^N)/S_N$$

Low energy dynamics is described by N free tensormultiplets and we view the space of vacua as parametrizing:

$$\langle X^{I,i} \rangle \quad I = 1, \ldots, 5 \quad i = 1, \ldots, N$$

Important Ingredients: Charged Strings

There are dynamical string-like excitations around generic vacua which are simultaneously electric and magnetic sources for the free tensormultiplets H^i:

$$dH^i = q^i \delta(W_2 \subset \mathbb{R}^6)$$

Important Ingredients: Surface Defects

There are surface defects $\mathbb{S}[\mathcal{R},\Sigma]$ associated to representations $\mathcal{R}$ of u(N). Far out on the moduli space they are well approximated by

$$\mathbb{S}[\mathcal{R},\Sigma] \sim \sum_w \exp[2\pi i \int_\Sigma w \cdot B + \cdots]$$

Important Ingredients: Chiral Operators

- Study short representations of osp(6,2|4).
- Chiral operators are labeled by Casimirs of degree d.
- Operators of lowest conformal dimension =2d transform in irreps of so(5)

In 5D SYM:

$$\mathcal{O}^{I_1,\dots,I_j} = \mathrm{Tr} X^{(I_1} \dots X^{I_j)}$$

Aharony, Berkooz, Seiberg (1997)

J.Bhattacharya, S.Bhattacharyya, S.Minwalla and S.Raju (2008)

Important Ingredients: Codimension 2 Defects

There are codimension two supersymmetric defects preserving half the susy.

There is an important class D(ρ,m) determined by

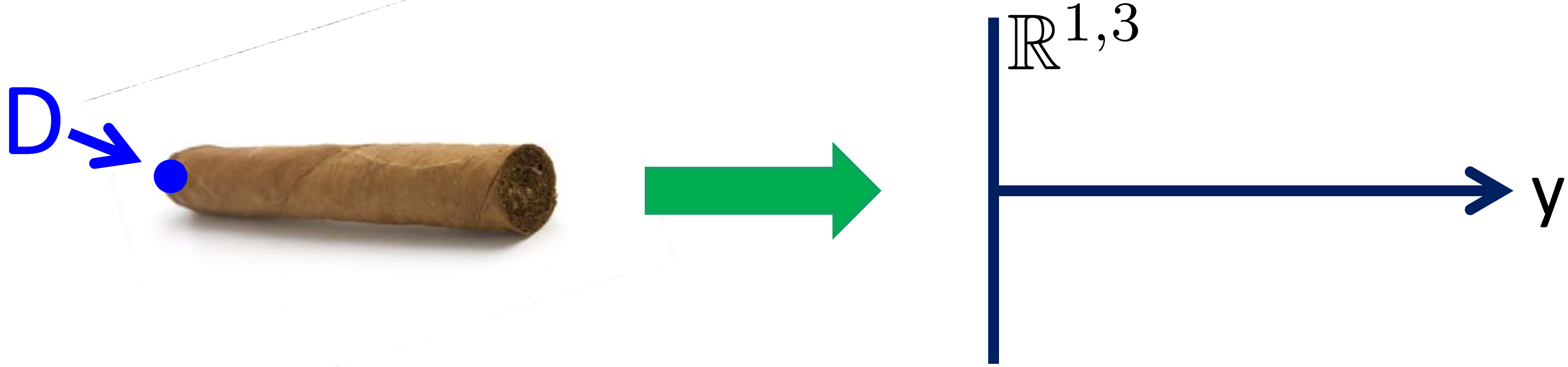
$$\rho : sl(2,\mathbb{C}) \rightarrow \mathfrak{g}_c$$

$$m \in Z(\mathrm{Im}(\rho))$$

Characterizing D(ρ,m)

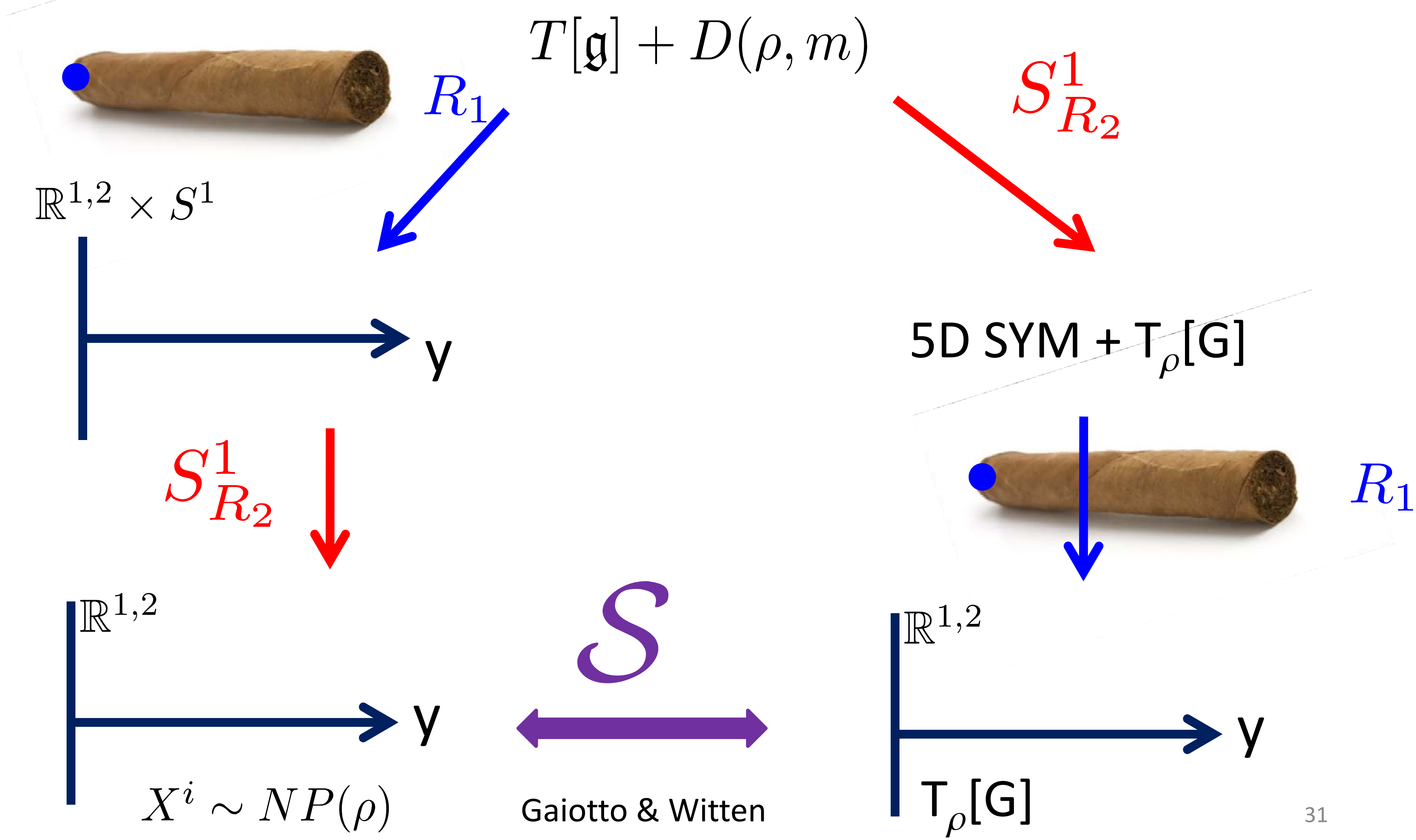
Reduction on longitudinal circle $\rightarrow$ 5D SYM + 3D defect.
5D SYM weak in IR $\rightarrow$ 3D defect = $T_\rho[G]$ theory of Gaiotto & Witten.

Reduction on linking circle: (compactification on cigar):
5DSYM + Boundary condition:



$$\mathcal{B}(D): \quad X^i \sim \frac{\rho(t^i)}{y} \qquad X^{4+i5} = m$$

Defects and S-Duality



Taxonomy of Defects

$$su(2) \xrightarrow{\rho} su(N)$$

$$n_1 + \cdots n_s \leftarrow N$$

Name	Partition of N	ρ	Global Symmetry
Full Defect	$[1^N]$	$\rho = 0$	SU(N)
Simple Defect	[N-1,1]	``Sub-regular''	U(1)
Trivial Defect	[N]	``Regular'' ``Principal''	1

Correspondences

Compactify and Compare

2d/4d, or, 6=4+2

Nakajima Geometric Langlands (Kapustin & Witten)

Alday-Gaiotto-Tachikawa; Gaiotto-Moore-Neitzke; Nekrasov-Shatashvili

Cecotti, Neitzke, & Vafa

6= 3 + 3

Domain walls, three-dimensional compactification & non-compact Chern-Simons theory: Drukker, Gaiotto, Gomis; Hosomichi, Lee, Park; Dimofte & Gukov et. al. ; Terashima & Yamazaki

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4

Theories of Class S

Consider nonabelian (2,0) theory S[$\mathfrak{g}$] for ``gauge algebra'' $\mathfrak{g}$

The theory has half-BPS codimension two defects D

Compactify on a Riemann surface C with D_a inserted at punctures z_a

$$so(5)_R \to so(3)_R \oplus \underbrace{so(2)_R}$$

Twist to preserve d=4,N=2

Witten, 1997
GMN, 2009
Gaiotto, 2009

$$S[\mathfrak{g}, C, D]$$

Type II duals via
``geometric engineering''
KLMVW 1996

(1) (2) (3) S-Duality

Higgs branches

(4) (5) (6) (7) (8) Coulomb branch & Hitchin moduli

(9) (10) (11) BPS states & wall-crossing

(12) (13) (14) Line & Surface defects

(15) (16) HK geometry

(17) (18) Cluster algebras

(19) (20) Holographic duals

(21) N=4 scattering

(22) (23) (24) Ω-backgrounds, Nekrasov partition functions, Pestun localization.

(25) (26) $Z(S^3 \times S^1)$ Scfml indx

(27) (28) Aut($\mathfrak{g}$) twists

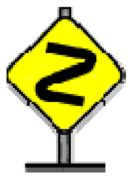
(29) AGT & Toda CFT

(30) (31) (32) (33) (34) Quantum Integrable

(35) Domain walls & 3D Chern-Simons

Generalized Conformal Field Theory

Twisting → S[$\mathfrak{g}$,C,D] only depends on the conformal structure of C.



For some C, D there are subtleties in the 4d limit.

``Conformal field theory valued in d=4 N=2 field theories''

(AGT → Non-rational CFT!)

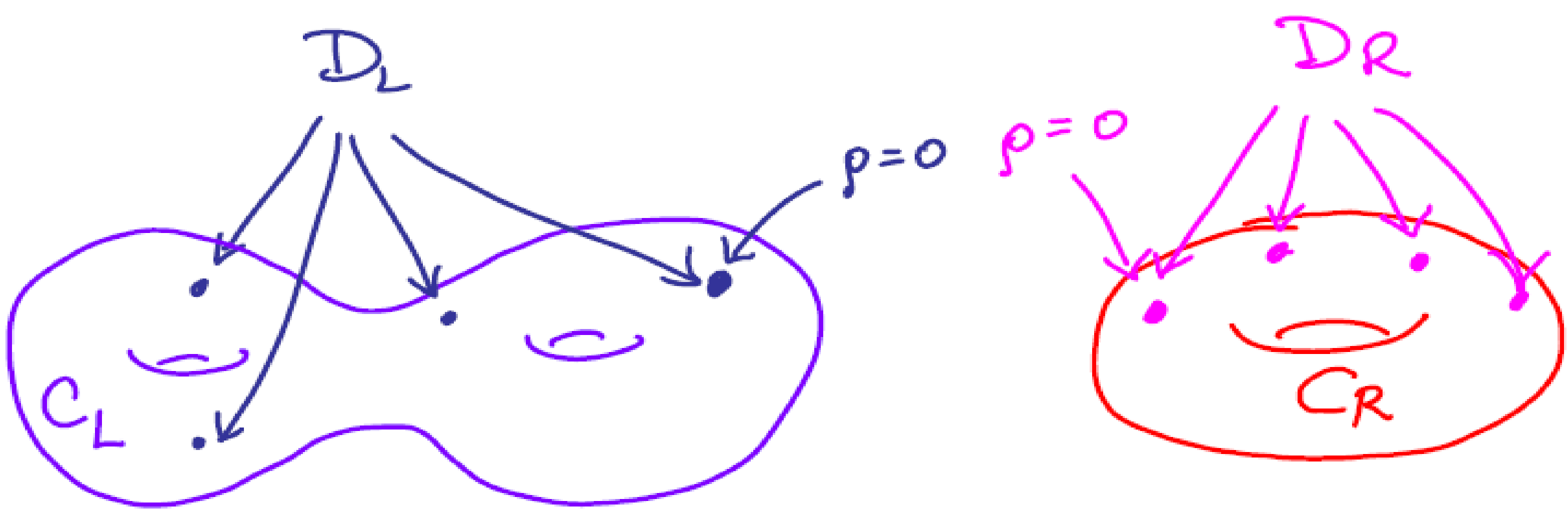
Space of coupling constants = $\mathcal{M}_{g,n}$

Go to the boundaries of moduli space....

Gaiotto Gluing Conjecture -A

D. Gaiotto, ``N=2 Dualities''

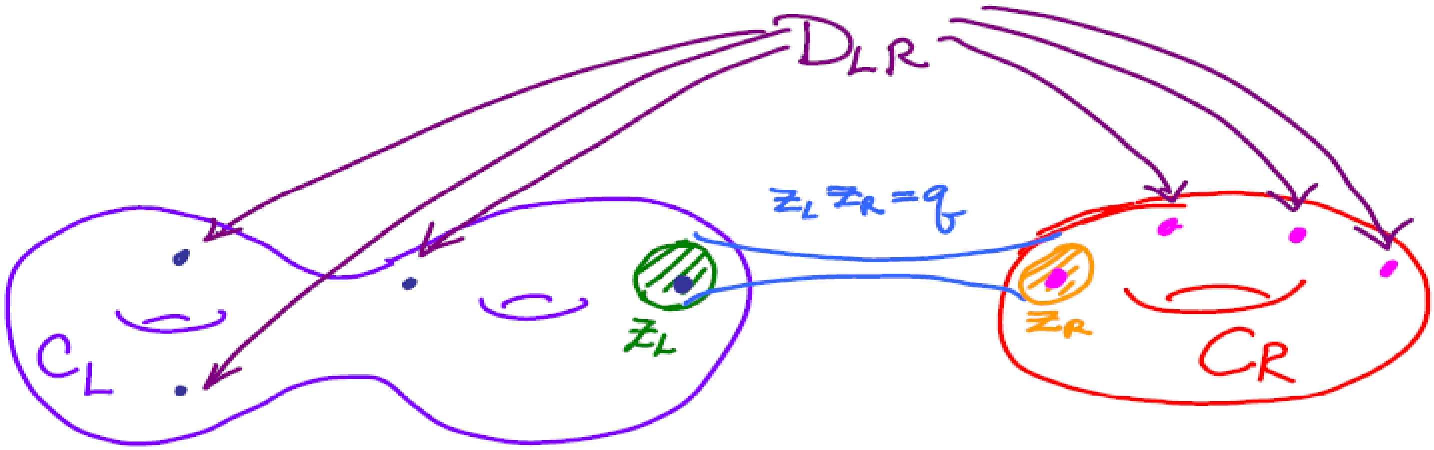
Slogan: Gauging = Gluing



Gauge the diagonal $G \subset G_L \times G_R$ symmetry with $q = e^{2\pi i\tau}$:

$$S[\mathfrak{g}, C_L, D_L] \times_{G,q} S[\mathfrak{g}, C_R, D_R]$$

Gaiotto Gluing Conjecture - B



Glued surface: $z_L z_R = q$ ➡ $C_L \times_q C_R$

$$S[\mathfrak{g}, C_L \times_q C_R, D_{LR}] = S_L \times_{G,q} S_R$$

Nevertheless, there are situations where one gauges just a subgroup – the physics here could be better understood. (Gaiotto; Chacaltana & Distler; Cecotti & Vafa)

S-Duality - A

Cut C into pants = 3-hole spheres = trinions.

(More precisely, consider embedded trivalent graphs.)

Presentation of S[$\mathfrak{g}$,C,D] as a gauging of ``trinion theories.''

Trivalent graphs label asymptotic regions of $\mathrm{Teich}_{g,n}$[C], hence correspond to weak-coupling limits.

S-Duality - B

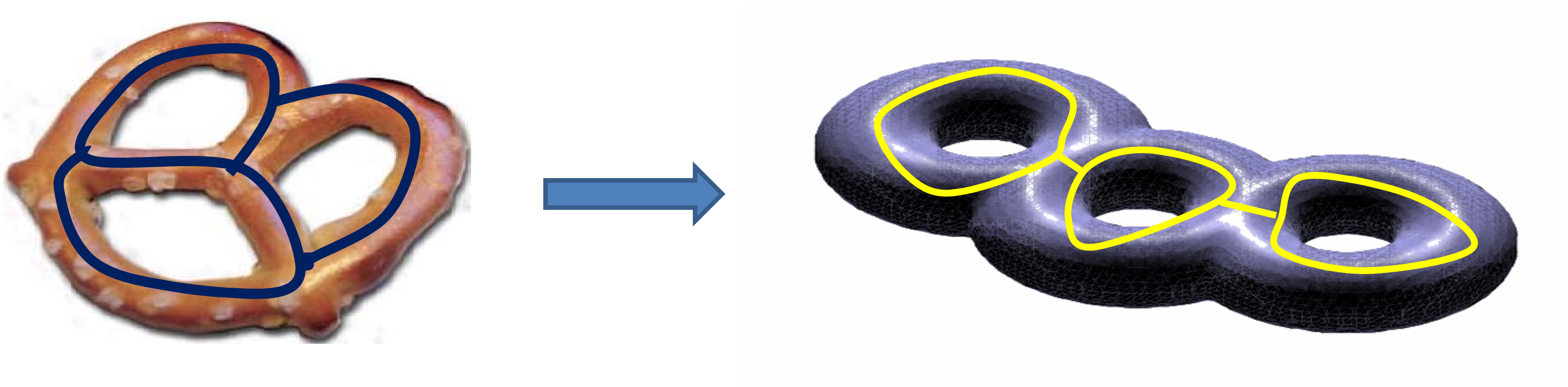
Two different points in $\mathrm{Teich}_{g,n}$ can project to the ***same*** point in $\mathcal{M}_{g,n}$

When we have two different presentations of the same theory we call it a duality.

This is an S-duality.

S-Duality - C

More generally, in conformal field theory it is useful to introduce a ``duality groupoid'':



Braiding, Fusing and S – moves generate the morphisms.

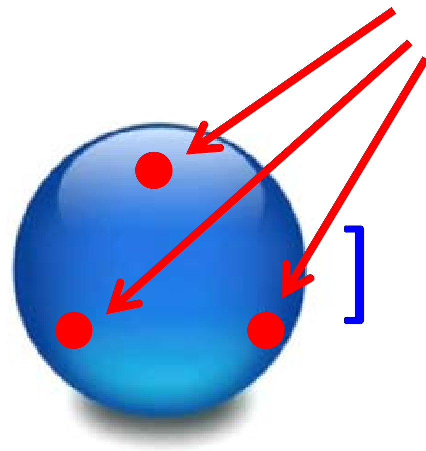
In this context there is therefore an ``S-duality groupoid.''
It was usefully applied to the AGT correspondence in:

Alday, Gaiotto, Gukov, Tachikawa; Drukker, Gomis, Okuda, Teschner

Trinion Theories

Gaiotto (2009)

$\rho = 0$

TRIN(N) := S[su(N), ]

N=2: Free ``half'' hypermultiplet in (2,2,2) of SU(2)^3

N=3: E_6 SCFT of Minahan-Nemeschansky

⇒ Geometrical description of Argyres-Seiberg duality, and a generalization to SU(N).

Moduli of Vacua

$$\mathcal{M} = \amalg_\alpha (\mathcal{SK}_\alpha \times \mathcal{H}_\alpha)/\sim$$

Higgs branches of theories with Lagrangian presentation are linear hyperkahler quotients

But Higgs branches of the non-Lagrangian TRIN(N) theories are unknown and interesting.

Gaiotto, Neitzke, Tachikawa 2009

Benini, Benvenuti, Tachikawa, 2009

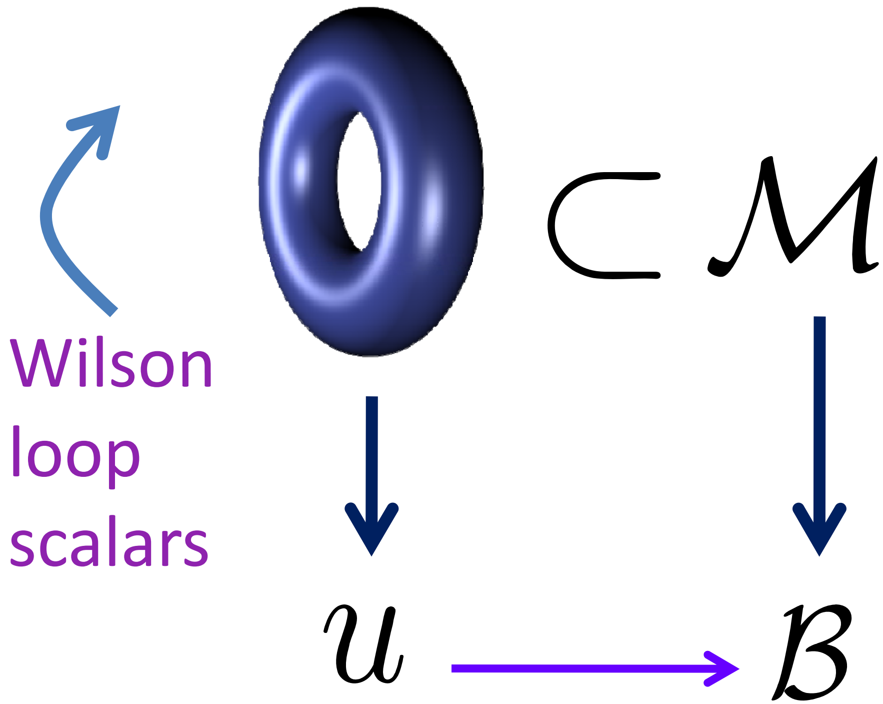
Benini, Tachikawa, Xie 2010

Hanany & Mekareeya, 2010

Tachikawa: http://www.math.upenn.edu/StringMath2011/notespt.html

Moore & Tachikawa, short summary

Seiberg-Witten Moduli Space



Compactify $\mathbb{R}^3 \times S^1$

Get a 3d sigma model with hyperkahler target space: A torus fibration over the Coulomb branch.

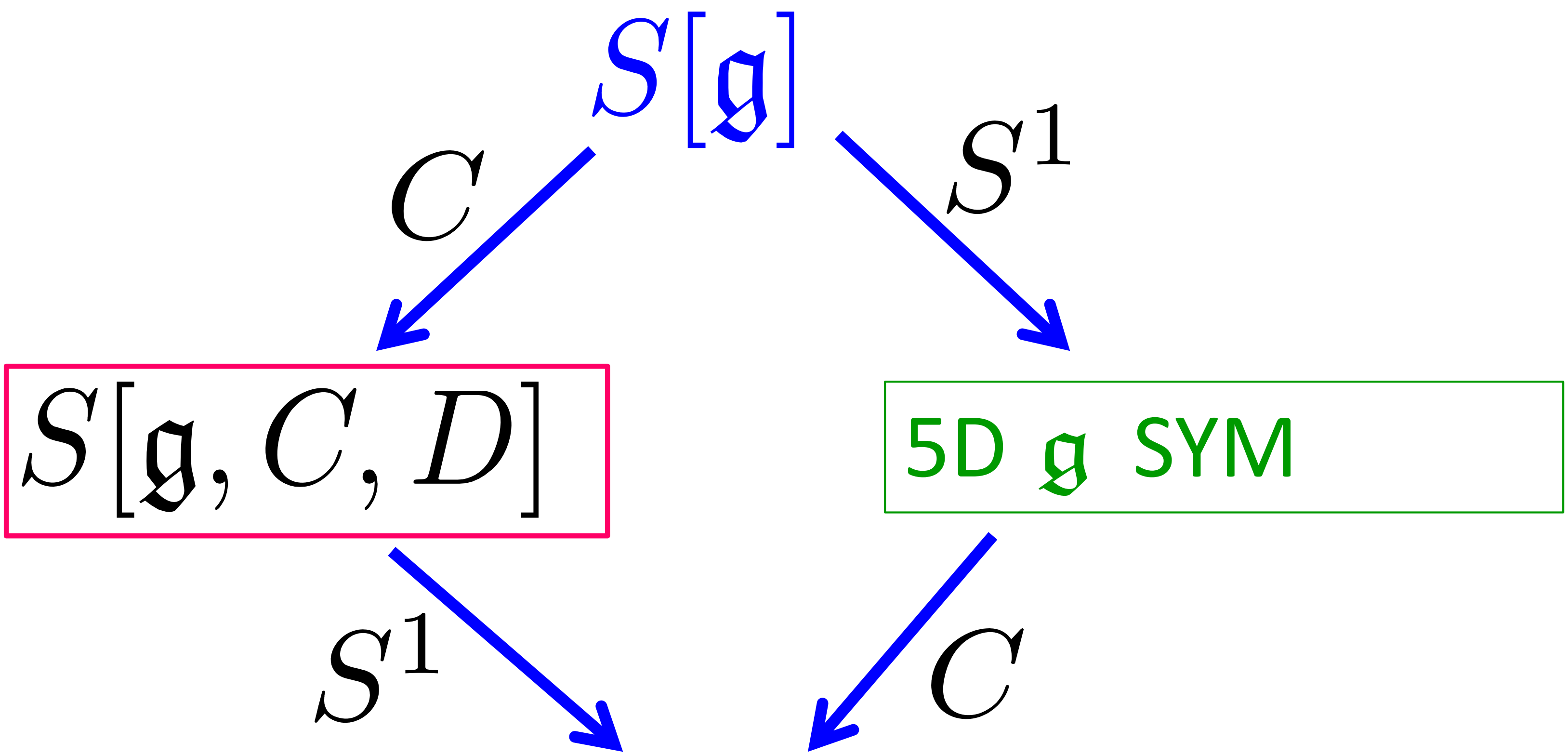
Seiberg & Witten (1996)

The relation to integrable systems goes back to:

Gorsky, Krichever, Marshakov, Mironov, Morozov; Martinec & Warner; Donagi & Witten (1995)

Which one?

Coulomb branch & Hitchin moduli



σ-Model: $\mathbb{R}^{1,2} \to \mathcal{M}$

$$F + R^2[\varphi, \bar{\varphi}] = 0$$
$$\bar{\partial}_A \varphi = 0$$

Effects of Defects

<u>Regular singular punctures</u>:

$$\varphi \sim \frac{dz}{z - z_a}\mathfrak{r} + \mathrm{reg}$$

Nature of the puncture is determined by the complex orbit of residue $\mathfrak{r}$.

<u>Irregular singular punctures</u>:

$$\varphi \sim \frac{dz}{(z - z_a)^{\ell}}\mathfrak{r} + \cdots \qquad \ell > 1$$

UV Curve

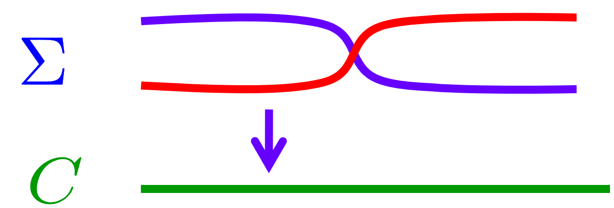
Seiberg-Witten Curve

$$\Sigma : \det(\lambda - \varphi(z,\bar{z})) = 0 \subset T^*C$$

$\lambda = p\,dq$ $\qquad \lambda|_\Sigma$ $\qquad$ SW differential

For $\mathfrak{g}$ = su(N) $\qquad \pi : \Sigma \to C$

is an N-fold branch cover



$$\lambda^N + \lambda^{N-2}\phi_2(z) + \cdots + \phi_N(z) = 0$$

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5 BPS States and Indices (in general)

Γ Lattice of flavor + electromagnetic charges

$$\text{Central charge: } Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$$

$$\mathcal{H} = \oplus_{\gamma\in\Gamma}\mathcal{H}_\gamma \qquad E \geq |Z_\gamma|$$

$$\mathcal{H}_\gamma^{\mathrm{BPS}} = \{\psi : E\psi = |Z_\gamma|\psi\}$$

$$\Omega(\gamma; u) := \mathrm{Tr}_{h_\gamma}(-1)^{2J_3}$$

$$\Omega(\gamma; u; y) := \mathrm{Tr}_{h_\gamma}(-1)^{2J_3}(-y)^{2J_3+2I_3}$$

Wall-Crossing (in general)

Ω jumps across the walls of marginal stability:

$$W(\gamma_1, \gamma_2) := \{u | Z(\gamma_1) \parallel Z(\gamma_2)\}$$

BPS states can form BPS boundstates: CFIV, CV, SW

Cecotti & Vafa (1992): Computed analogous jumps for BPS solitons in 2d (2,2) models.

Denef & Moore (2007) , Kontsevich & Soibelman (2007) , Joyce & Song (2007)

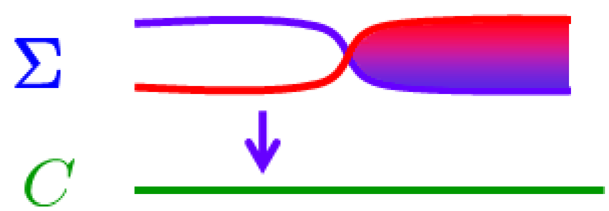
Most complete and conceptually simple formulation: KSWCF, but we will see that the halo configurations of F. Denef capture the essential physics.

Nice recent development: Manschot, Pioline, and Sen: Apply localization to Denef's multi-centered solution moduli space. See Pioline review.

BPS States for S[$\mathfrak{g}$,C,D]:

BPS states come from finite open M2 branes ending on Σ. Here This leads to a nice geometrical picture with string networks:

Klemm, Lerche, Mayr, Vafa, Warner; Mikhailov; Henningson & Yi; Mikhailov, Nekrasov, Sethi,



Combining wall crossing techniques with line and surface defects leads to an algorithm for determining the BPS spectrum. (GMN 2009; to appear).

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Line Defects & Framed BPS States (in general)

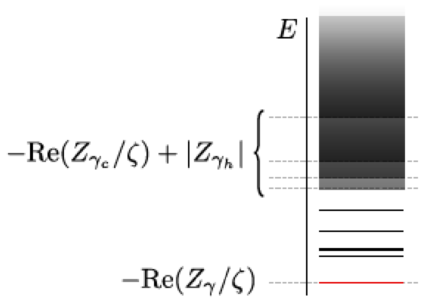
A ***line defect*** L (say along $\mathbb{R}_t \times \{0\}$) is ***of type ζ*** if it preserves the susys:

$$Q_{\alpha}^{A} + \zeta \sigma^{0}_{\alpha\dot{\beta}} \bar{Q}^{\dot{\beta} A}$$

Example:

$$L_{\zeta} = \exp \int_{\mathbb{R}_t \times \vec{0}} \left(\frac{\varphi}{2\zeta} + A + \frac{\zeta}{2}\overline{\varphi} \right)$$

$$\mathcal{H}_L = \oplus_{\gamma \in \Gamma} \mathcal{H}_{L,\gamma}$$



$$E \geq -\mathrm{Re}(Z_{\gamma}/\zeta)$$

Framed BPS States saturate this bound, and have framed protected spin character:

$$\overline{\underline{\Omega}} := \mathrm{Tr}_{\mathcal{H}^{bps}_{L,\gamma}} (-1)^{2J_3} (-y)^{2J_3+2I_3}$$

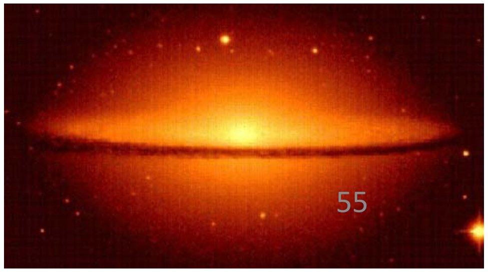
$$\overline{\underline{\Omega}}(L, \gamma; y; \zeta; u)$$

Piecewise constant in ζ and u, but has wall-crossing across ``BPS walls'' (for $\Omega(\gamma) \neq 0$):

$$W_\gamma := \{(u, \zeta) : Z_\gamma(u)/\zeta \in \mathbb{R}_-\}$$

Particle of charge γ binds to the line defect:

→ Similar to Denef's halo picture



Wall-Crossing for $\underline{\overline{\Omega}}$

$$F(L) = \sum_\gamma \underline{\overline{\Omega}}(L,\gamma;y) X_\gamma$$

$$X_{\gamma_1} X_{\gamma_2} = y^{\langle \gamma_1, \gamma_2 \rangle} X_{\gamma_1+\gamma_2}$$

Across W(γ_h) Denef's halo picture leads to:

$$F^+(L) = \Phi(X_{\gamma_h}) F^-(L) \Phi(X_{\gamma_h})^{-1}$$

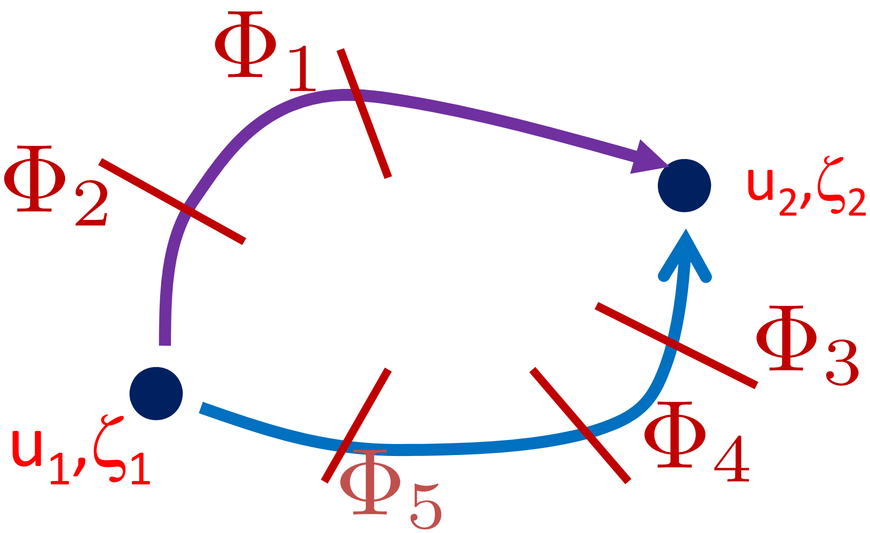
$\Phi(X_{\gamma_h})$ constructed from $\Omega(\gamma_h; y)$

Wall-Crossing for Ω

Consistency of wall crossing of framed BPS states implies the Kontsevich-Soibelman WCF for ***unframed*** states

$$\Omega(\gamma; y)$$

We simply compare the wall-crossing associated to two different paths relating F(L)(u_1,ζ_1) and F(L)(u_2,ζ_2)



$$\Phi_1 \Phi_2 = \Phi_3 \Phi_4 \Phi_5$$

References

1. GMN, ``Framed BPS States,’’ 2010

2. Andriyash, Denef, Jafferis, Moore, 2010: Supergravity

3. Dimofte & Gukov; Dimofte, Gukov & Soibelman 2009: ``motivic’’  vs. ``refined’’

4. Cecotti & Vafa, 2009; Cecotti, Neitzke, & Vafa 2010: Use topological strings in non-compact Calabi-Yau

R-symmetry generator as Hamiltonian

Appearance of Heisenberg algebra is natural and physical.

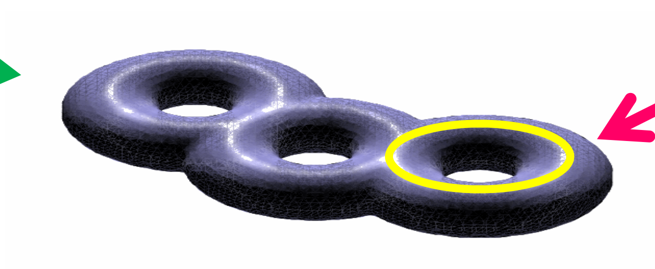
Line defects in S[$\mathfrak{g}$,C,m]

6D theory S[$\mathfrak{g}$] has supersymmetric surface defects $\mathbb{S}(\mathcal{R}, \sigma)$

For S[$\mathfrak{g}$,C,D] consider

$$\sigma = \mathbb{R} \times \{\vec{0}\} \times \wp$$

C



$$L_\zeta(\mathcal{R}, \wp)$$

Line defect in 4d *labeled* by isotopy class of a ***<u>closed</u>*** path $\wp$ and $\mathcal{R}$

Classifying Line Defects

For $\mathfrak{g} = su(2)$ and R = fundamental , the Dehn-Thurston classification of isotopy classes of closed curves matches nicely with the classification of simple line operators as Wilson-'t Hooft operators: Drukker, Morrison & Okuda.

The generalization of the Drukker-Morrison-Okuda result to higher rank has not been done, and would be good to fill this gap.

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- N=2, d=4 Geography
- Egregious Omissions

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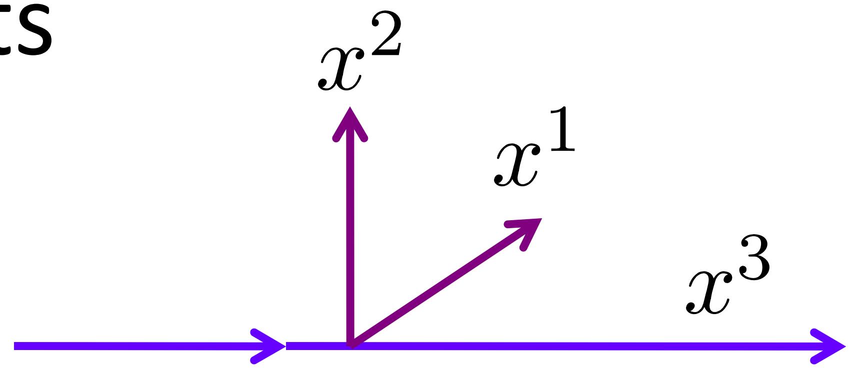
Surface defects (in general)

$\mathbb{S}$ at $x^1 = x^2 = 0$



UV Definition:

Preserves d=2 (2,2) supersymmetry subalgebra

Twisted chiral multiplet : $\Upsilon = \varphi + \cdots$

IR Description: ***Coupled 2d/4d system***

$$S_{IR} = \int d^4x d^4\theta \, \mathcal{F}^{eff}(a)$$

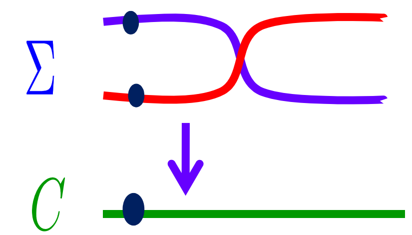
$$+ \int d^2x d^2\theta \, \mathcal{W}^{eff}(\Upsilon)$$

Canonical Surface Defect in S[$\mathfrak{g}$,C,m]

Alday, Gaiotto, Gukov, Tachikawa, Verlinde (2009); Gaiotto (2009)

For z $\in$ C we have a ***canonical surface defect*** $\mathbb{S}_z$

It can be obtained from an M2-brane ending at $x^1=x^2=0$ in $\mathbb{R}^4$ and z in the UV curve C



In the IR the different vacua for this M2-brane are the different sheets in the fiber of the SW curve Σ over z.

Therefore the chiral ring of the 2d theory should be the same as the equation for the SW curve!

$$\lambda^N + \lambda^{N-2}\phi_2(z) + \cdots + \phi_N(z) = 0$$

Example of SU(2) SW theory

$$\lambda^2 = \left(\frac{1}{z} + \frac{2u}{z^2} + \frac{\Lambda^2}{z^3}\right)(dz)^2$$

$$\lambda = x\,dz \qquad z = e^t$$

$$\underbrace{x^2 = e^t}_{} + 2u + \frac{\Lambda^2}{e^t}$$

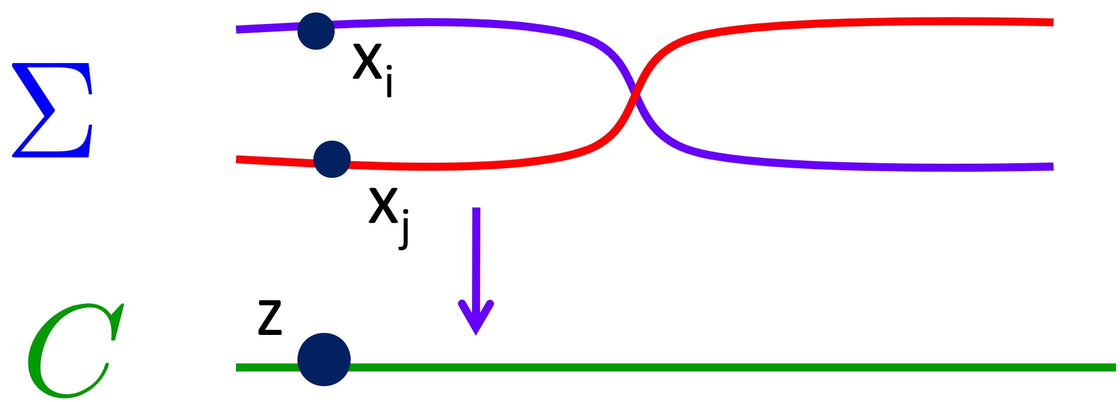
Chiral ring of the $\mathbb{C}$P[1] sigma model.

Twisted mass

2d-4d instanton effects

Gaiotto

Superpotential for $\mathbb{S}_z$ in S[$\mathfrak{g}$,C,m]

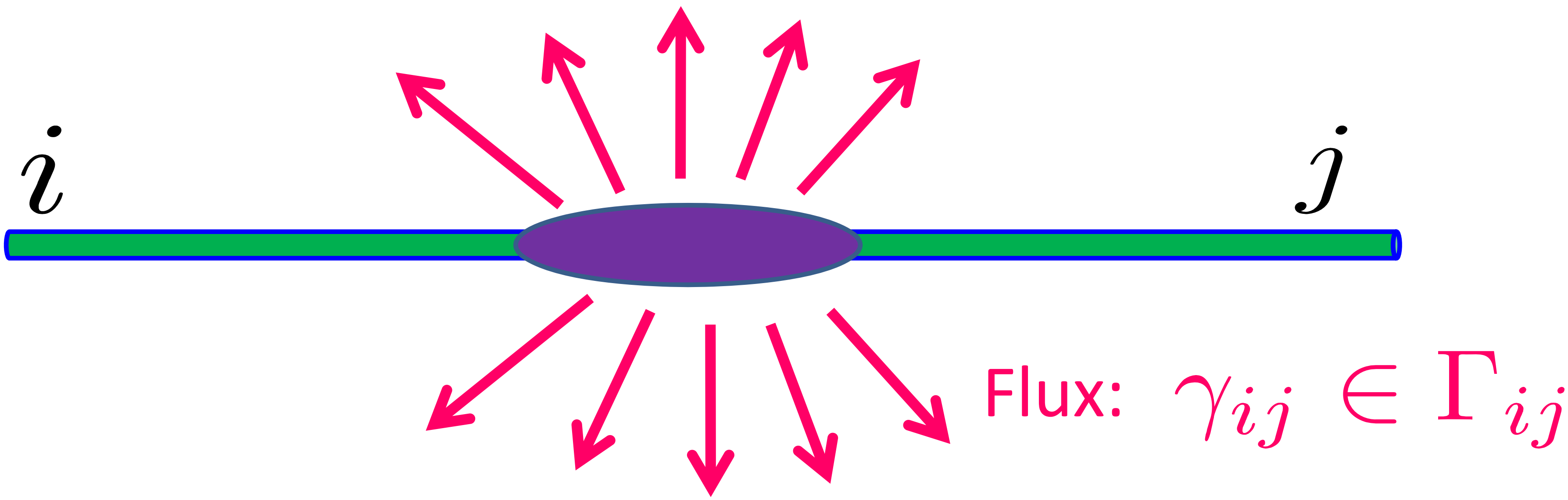


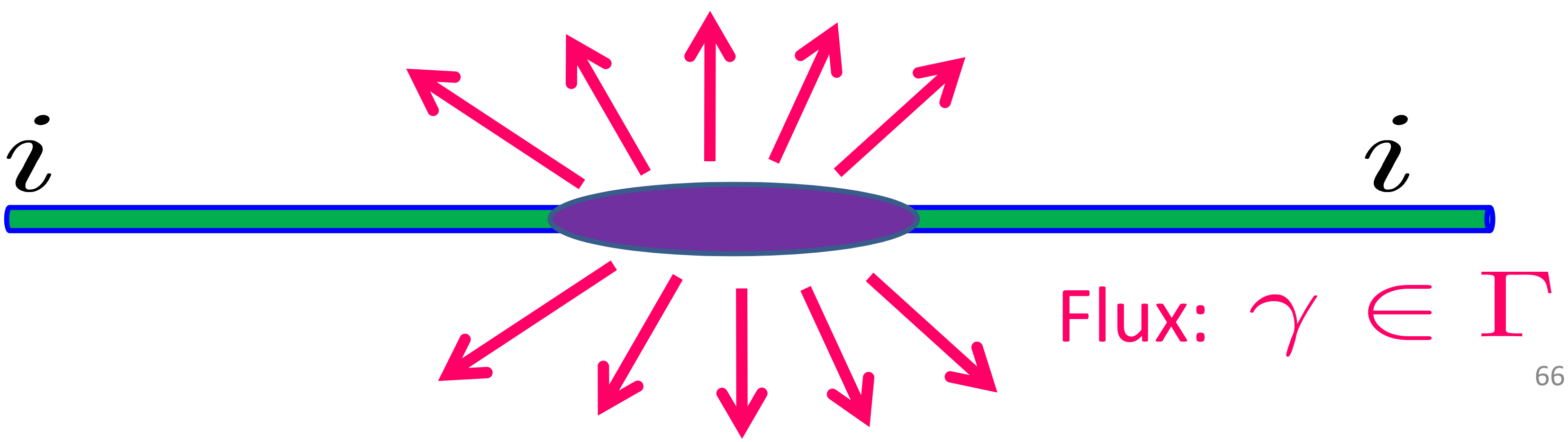
$$\mathcal{W}(x_i) - \mathcal{W}(x_j) = \int_{\gamma_{ij}} \lambda$$

γ_{ij} Homology of an ***open*** path ***on Σ*** joining x_i to x_j in the fiber over $\mathbb{S}_z$

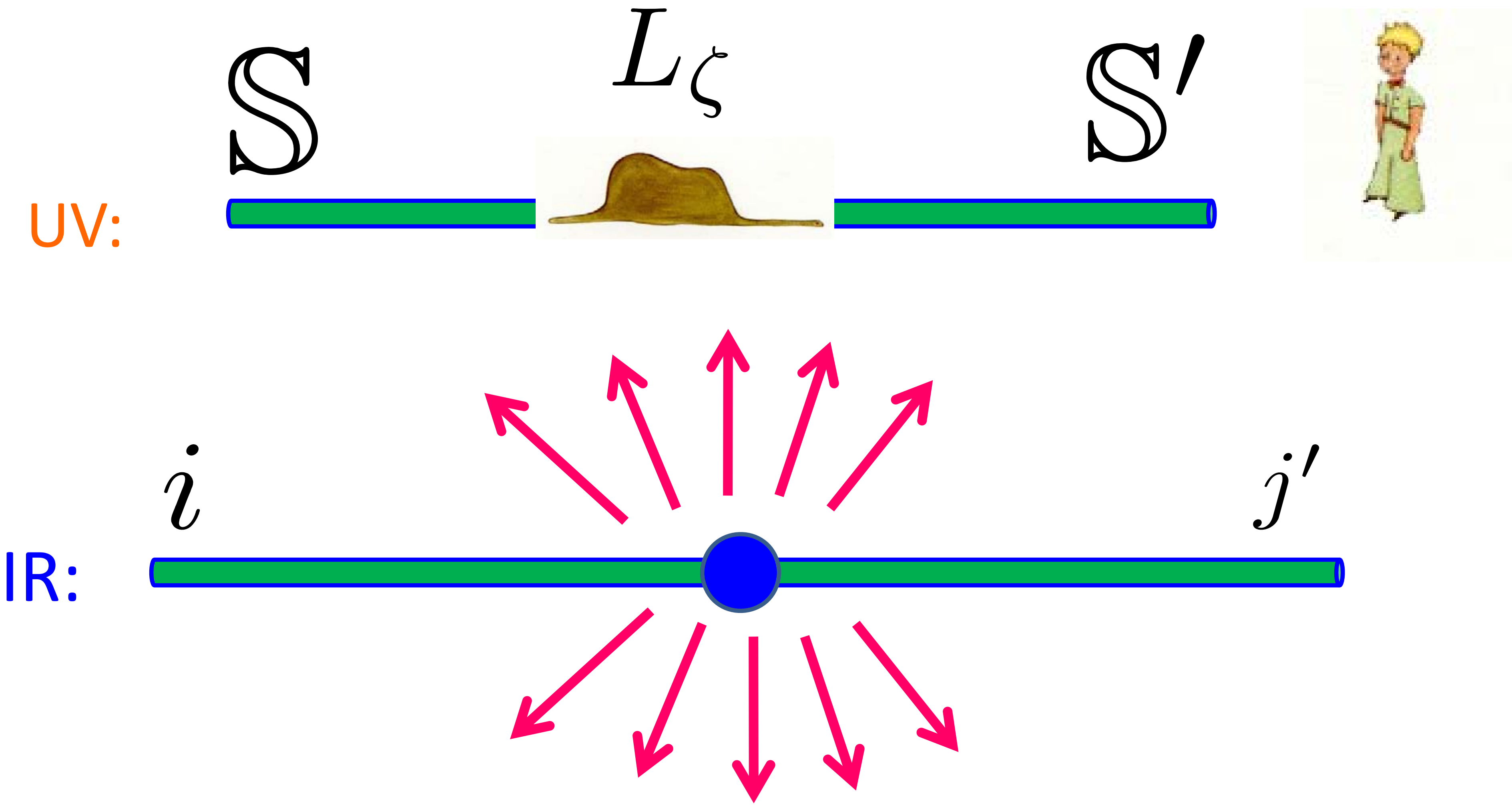
$$\gamma_{ij} \in \Gamma_{ij} \subset H_1(\Sigma, \{x_i, x_j\}; \mathbb{Z})$$

Coupled 2d/4d: New BPS States





Supersymmetric Interfaces

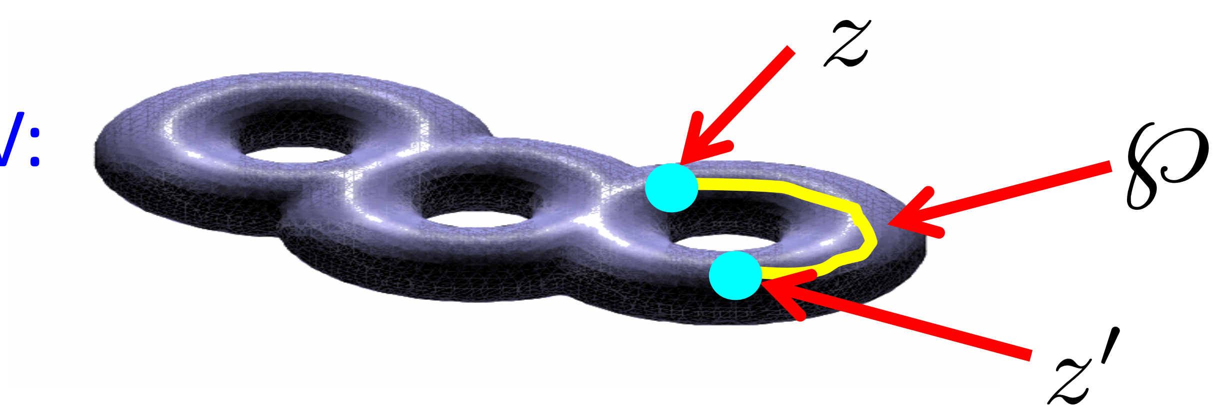


Flux: $\gamma_{ij'} \in \Gamma_{ij'}$

Susy interfaces for S[$\mathfrak{g}$,C,m]

Interfaces between $\mathbb{S}_z$ and $\mathbb{S}_{z'}$ are labeled by open paths $\wp$ on C between z and z':

UV:



$\Rightarrow L_{\zeta,\mathcal{P}(z,z')}$

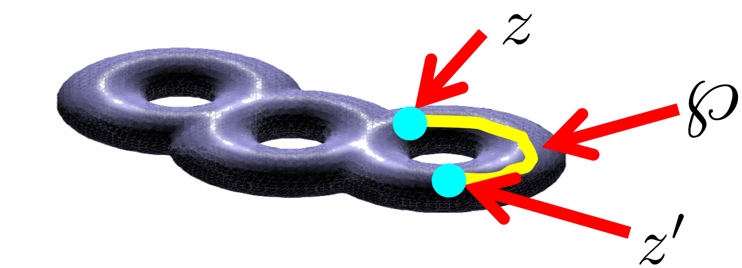
IR: *Framed* BPS states are graded by open paths $\gamma_{ij'}$ on Σ with endpoints over z and z'

$$\Gamma_{ij'} \subset H_1(\Sigma, \{x_i, x_{j'}\}; \mathbb{Z})$$

2D/4D Wall-Crossing

Consistency of WC for framed BPS indices implies a generalization of the KSWCF: ``2d/4d WCF'' (GMN 2011).

Studying WC as functions of z, and interpreting $\mathcal{P}(z,z')$ as a Janus configuration makes contact with the new work of Gaiotto & Witten, and gives some nice new perspectives on old work of Hori, Iqbal, & Vafa.



Finally, very recently, GMN used the study of framed BPS states to give an algorithm for computing the BPS spectrum of A_N theories of class S in a certain region of the Coulomb branch.

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8 Coulomb branch Hyperkahler Geometry - A

1. $\langle L_{\zeta,\wp} \rangle_{m \in \mathcal{M}} = \sum_{\gamma} \overline{\underline{\Omega}}(L_{\zeta,\wp}, \gamma) \mathcal{Y}_{\gamma}(m)$

2. $\mathcal{Y}_{\gamma}$ define a system of holomorphic Darboux coordinates for SW moduli spaces. They can be constructed from a TBA-like integral equation.

3. From these coordinates we can construct the HK metric on $\mathcal{M}$.

4. 1-instanton confirmed via direct computation: Chen, Dorey, & Petunin 2010;2011; Multi-instanton?

Coulomb branch HK Geometry - B

5. For S[su(2),C,m], $\mathcal{Y}_\gamma$ turn out to be closely related to the Fock-Goncharov coordinates which appear in the F&G work on quantum Teichmuller theory.

6. For S[su(2),C,m] the analogous functions: $\mathcal{Y}_{\gamma_{ij'}}$

$$\langle L_{\zeta,\mathcal{P}(z,z')}\rangle = \sum_{\gamma_{ij'}} \underline{\overline{\Omega}}(L,\gamma_{ij'})\mathcal{Y}_{\gamma_{ij'}}$$

are sections of a bundle over $\mathcal{M}$, and allow us moreover to construct hyper-holomorphic connections (BBB branes).

``Hyper-holomorphic'': $F^{2,0}=0$ in all complex structures

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9 Geography of d=4,N=2 Theories

Many theories are simply classified by their Lagrangian: G, R, m, ζ

Infinitely many Lagrangian theories are class S .

But infinitely many class S theories are non-Lagrangian.

Some Lagrangian theories appear to be outside class S

$$\begin{array}{cccccccc} & & & & & 3N & & \\ & & & & & | & & \\ N & -2N & -3N & -4N & -5N & -6N & -4N & -2N \end{array}$$

Many N=2 theories can be geometrically engineered by considering type II theories on noncompact Calabi-Yaus with singularities…

> *``You gotta be careful if you don't know where you're going, otherwise you might not get there.''* -- Yogi Berra

Approaches to classification - A

A fast-developing mathematical subject of cluster algebras* and cluster varieties can be usefully applied to the study of class S theories. (GMN 2009; GMN 2010; Cecotti, Neitzke , & Vafa 2010; Cecotti & Vafa 2011)

CNV suggested a classification via a 2d/4d correspondence: Engineer the 4d theory with a noncompact CY and extract a (2,2) massive QFT from the worldsheet of the string. Apply old classification of d=2 (2,2).

C&V then combined the 2d/4d correspondence with math results on cluster algebras to classify a subclass of ``complete N=2 theories.''  The resulting list is ``mostly'' of class S but has exceptional cases which appear to be outside class S.

* http://www.math.lsa.umich.edu/~fomin/cluster.html

Approaches to classification - B

Gaiotto (2009) has suggested a different approach:

Use the geometry associated to a pair (T,$\mathbb{S}$) there is a Seiberg-Witten-like geometry, based on Hitchin-like equations, as an invariant.

Any reasonable theory should have surface defects....

Large N Geography of d=4,N=2 Theories

Large N limits with M-theory duals are described by bubbling geometries of Lin, Lunin, and Maldacena.

Gaiotto & Maldacena showed that large N class S theories, with regular punctures, fit into the LLM bubbling geometry scheme.

A ``classification'' would require a classification of physically sensible boundary conditions of the 3d Toda equation at the heart of the LLM construction.

In particular, what is the large N description of superconformal class S theories with irregular punctures (e.g. Argyres-Douglas theories) ?

What about large N limits with IIB duals?

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Egregious Omissions - A

1. Applications to strong coupling scattering in N=4

For a particular (Argyres-Douglas) theory of class S, the functions $\mathcal{Y}_\gamma$ appearing in the twistor construction of the HK geometry of $\mathcal{M}$ are the cross-ratios solving the Alday-Maldacena minimal area problem in AdS.

→ Past and future Technology Transfer

TBA, Y-systems, small flat sections, …

Alday-Maldacena; Alday, Gaiotto, Maldacena; Alday, Maldacena, Sever, Vieira; Lukyanov & Zamolodchikov; Lukyanov; ….

Egregious Omissions - B

2. ***AGT correspondence***: Nekrasov partition functions = Conformal blocks → Connections to Liouville and Toda theory!

KITP workshop: http://online.itp.ucsb.edu/online/duallang-m10/

3. Witten: Combining work on the analytic continuation of Chern-Simons theory, he uses class S theories to obtain a new gauge-theoretic/Morse-theoretic approach to the ``***categorification of knot polynomials***.''

4. ***2D Quantum Integrability & 4D SYM***: Nekrasov-Shatashvili; Nekrasov-Witten; Nekrasov,Rosly,Shatashvili

SCGP Workshop on Branes and Bethe Ansatz: http://scgp.stonybrook.edu/?p=145

KITP Workshop: http://www.kitp.ucsb.edu/activities/dbdetails?acro=integral11

Un-omittable Omission

But how do we construct the theory $S[\mathfrak{g}]$?

Aharony, Berkooz,Seiberg (1997): Matrix theory & DLCQ
$\rightarrow$ Super QM on instanton moduli space.

Lambert-Papageorgakis (2010): Try to generalize the susy transformations of the abelian tensormultiplet to a nonabelian version (following Bagger & Lambert and Gustavsson).
They were partially successful – Perhaps they just rediscovered the D4 brane, but there is more to understand.

Lambert,Pagageorgakis & Schmidt-Sommerfeld; Douglas:
Try to define the theory by flowing up the RG from 5D SYM and guessing at a UV completion.

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Summary and the Future

``It's tough to make predictions, especially about the future.''
– Yogi Berra

The hypothetical existence and properties of the six-dimensional (2,0) theories leads to many exact results for partition functions, line and surface defect correlators, BPS spectra, etc.

There are several ``2d-4d correspondences'' and other remarkable interrelations in this area of Physical Mathematics.

And these theories even push the envelope, challenging what should be the proper definition of a ``quantum field theory.''

Other Potential applications:

P1: Three dimensional quantum field theory & three dimensional gravity.

P2: Compactification on 4-manifolds: Applications to the MSW (0,4) theories might have applications to the study of supersymmetric black holes.

P3: Supergravity and string compactification.

M1: Relation to the work of Fock & Goncharov, cluster algebras, and geometric Langlands suggests *possible* deep connections to number theory (motives, polylogs, Bloch groups, algebraic K-theory, …)

M2: Knot homology and categorification:

Will physicists beat the mathematicians?

A Central Unanswered Question

Can we construct $S[\mathfrak{g}]$?

MIND THE GAP


NOT
That's all Folks!