Say `Halo!' to New Indices & New Walls

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Gregory Moore

Rutgers University
OUTLINE

Introduction & Review

Part I:  How BPS statespaces change with moduli in N=2 sugra

Part II:  Line Operators in N=2 Field Theories

  Framed BPS states & Protected Spin Characters

  New derivation of the ``motivic KSWCF''

  Exact results for line operator vevs in $T_{g,n}[A_1]$ theories
Old Question!

Strings2007: Introduced the semi-primitive WCF.

Final Riddle: Why did the BPS state cross the wall?

We need to understand not just the index but how the space of BPS states change as moduli are varied.

New walls
Strings2008: Moduli space $\mathcal{M}$ of an $\mathcal{N} = 2$ theory on $\mathbb{R}^3 \times S^1$

Darboux/Twistor coordinates $X_\gamma$ construct a HK metric

Final Promise:
These have an interpretation in terms of line operators

Part II: Make good on that promise
N=2: Basic Definitions

Moduli of vacua: $B_{vm} \times \mathcal{M}_{hm}$

Local system: $\Gamma \rightarrow B_{vm}$

$\mathcal{H} = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma \supset \bigoplus_{\gamma} \mathcal{H}_\gamma^{BPS}$

$Z : \Gamma \rightarrow \mathbb{C}$ \quad $E \geq |Z(\gamma; u)|$

$\langle \cdot, \cdot \rangle : \Gamma \rightarrow \mathbb{Z}$
Old Indices & Old Walls

\[ \mathcal{H}^{BPS}_\gamma \] Finite dimensional representation of \( SU(2)_{\text{space}} \)

As such: Completely determined by their spin character:

\[ s(\gamma, y; m) := \text{Tr} \mathcal{H}^{BPS}_\gamma y^2 J_3 \]

This is not an index: It depends on \( m \in \mathcal{B}_{vm} \times \mathcal{M}_{hm} \)

Better:

\[ \Omega(\gamma; u) := s|_{y=-1} \]

Piecewise constant but can change across:

\[ M_S(\gamma_1, \gamma_2) := \{ u | Z_1 \parallel Z_2 \} \]
Old Boundstates

Boundstate radius (Denef)

\[ R_{12} = \langle \gamma_1, \gamma_2 \rangle \frac{|Z_1 + Z_2|}{2 \text{Im}(Z_1 Z_2^*)} \]

So the moduli space of vacua is divided into two regions:

\[ \langle \gamma_1, \gamma_2 \rangle \text{Im}(Z_1 Z_2^*) > 0 \quad \text{OR} \quad \langle \gamma_1, \gamma_2 \rangle \text{Im}(Z_1 Z_2^*) < 0 \]
Primitive Wall-Crossing

\[ R_{12} = \langle \gamma_1, \gamma_2 \rangle \frac{|Z_1 + Z_2|}{2 \text{Im}(Z_1 Z_2^*)} \]

Crossing the wall: \( \text{Im}(Z_1 Z_2^*) \to 0 \)

\[ \Delta \mathcal{H} = (J_{12}) \otimes \mathcal{H}(\gamma_1) \otimes \mathcal{H}(\gamma_2) \]

\[ 2J_{12} + 1 = |\langle \gamma_1, \gamma_2 \rangle| \]
Old Halos

\[
\mathcal{H}_{\text{halo}}^{BPS} := \mathcal{H}_{\gamma_2}^{BPS} \otimes \ell \geq 1 \ F \left[ (J_{\gamma_2, \ell \gamma_1}) \otimes \mathcal{H}(\ell \gamma_1) \right]
\]

\[
R_{\text{halo}} \rightarrow \infty \quad \text{across} \quad MS(\gamma_1, \gamma_2)
\]
Part I

Systematic description of how BPS state-spaces change

arXiv:10??.

E. Andriyash, F. Denef, D. Jafferis,
An Old Puzzle

$R_{12} > 0$

$MS(\gamma_1, \gamma_2)$

$Z_1 = 0$

$R_{12} < 0$

$AMS(\gamma_1, \gamma_2)$

$\mathcal{P}$ crosses no other walls of MS

Boundstate $(\gamma_1, \gamma_2)$ exists near $MS(\gamma_1, \gamma_2)$

Boundstate $(\gamma_1, \gamma_2)$ cannot exist near $AMS(\gamma_1, \gamma_2)$!
What happened?

Just got married… VM + HM pair up

\[ \Delta \Omega_{vm} + \Delta \Omega_{hm} = 0 \]  
(Note: \( \Delta s \neq 0 \))

Not the whole story: \((\gamma_1, \gamma_2)\) bound states contribute to \(\Omega \neq 0\)
$R_{12} > 0$

$MS(\gamma_1, \gamma_2)$

$Z_1 = 0$

$R_{12} < 0$

$AMS(\gamma_1, \gamma_2)$

$\mathcal{H}(\gamma_1) = 0$ near $Z_1 = 0$  

Recombination wall

$\text{Menage a trois}$

$\gamma_1$

$\gamma_2$

$\gamma_3$

$\gamma_2 + \gamma_4$
$R_{12} > 0$

$MS(\gamma_1, \gamma_2)$

$Z_1 = 0$

$AMS(\gamma_1, \gamma_2)$

$R_{12} < 0$

$H(\gamma_1) \neq 0$

New wall:

\[ \{ u | (\gamma_1 + \gamma_2) \text{ attractor flow crashes on } Z_1 = 0 \} \]

$A_{\text{probe}} = \int (m ds + \langle \gamma, A \rangle) \propto |Z(\gamma_1, u(\vec{x}))|$
``... without any fuss, the stars were going out." – Arthur C. Clarke
Monodromy

\[ R_{12} > 0 \]
\[ MS(\gamma_1, \gamma_2) \]
\[ Z_1 = 0 \]
\[ AMS(\gamma_1, \gamma_2) \]
\[ R_{12} < 0 \]

\[ \gamma_2 \rightarrow \gamma_2 + I \gamma_1 \]

\[ I = \langle \gamma_2, \gamma_1 \rangle \sum_{\ell=1}^{\infty} \ell^2 \Omega(\ell \gamma_1) \]
Further Predictions

The halo picture makes some further predictions about the spectrum of light states near a singular point of moduli space.

\[ \sum_{\ell=1}^{\infty} \ell^2 \Omega(\ell \gamma_1) > 0 \]

\[ \prod_{\ell>0} (1 - (-1)^\ell |\langle \gamma_2, \gamma_1 \rangle| q^\ell) \ell |\langle \gamma_2, \gamma_1 \rangle| \Omega(\ell \gamma_1) \]

These appear to contradict some of the literature on geometric engineering and extremal transitions.

We’re trying to sort it out.
Part II: N=2 Field Theories

Davide Gaiotto & Andy Neitzke

arXiv:0807.4723 – Hyperkahler metrics and Darboux coordinates

Now focus on $d=4$ $N=2$ field theory defined by some $\text{su}(2,2|2)$ superconformal fixed point $S$.

Line operator = boundary condition for $S$ on $\text{AdS}_2 \times S^2$ [Kapustin]
Unbroken Susy

Restrict attention to line op’s preserving $osp(4^*|2)\zeta$

Fixed points of an involution of $su(2,2|2)$

$\vec{x} \rightarrow -\vec{x}$ & $U(1)_R$ rotation by $\zeta$

$osp(4^*|2)_{\zeta}^{\text{even}} = sl(2,\mathbb{R}) \oplus so(3) \oplus su(2)$

$R^A_{\alpha} \sim Q^A_{\alpha} + \zeta \sigma^0_{\alpha\dot{\beta}} \bar{Q}^{\dot{\beta}} A$

Spatial rotation & R-symmetry

Line operator $L$ of type $\zeta$, $L_{\zeta}(\cdots)$
Choose a line operator $L$ preserving $osp(4^*|2)_\zeta$

$\mathcal{H}_L$ Hilbert space in presence of $L$

$\mathcal{H}_L = \bigoplus_\gamma \mathcal{H}_{L,\gamma}$

$\{ R^A_\alpha, R^B_\beta \} = 4\epsilon_{\alpha\beta}\epsilon^{AB}(E + \text{Re}(Z_\gamma/\zeta))$

$E \geq -\text{Re}(Z_\gamma/\zeta)$
Framed BPS States

\[ E = -\text{Re}(Z_\gamma / \zeta) \]

\[ E = |Z_\gamma| \]

Vanilla BPS bound

Continuum

\[ E_{\text{gap}} > 0 \]

Framed BPS states

\[ \mathcal{H}_{L,\gamma}^{\text{BPS}} \]
Protected Spin Character

Framed PSC for framed BPS states:

\[
\bar{\Omega}(L, \gamma; y) := \text{Tr}_{\mathcal{H}^{BPS}_{L,\gamma}} y^{2J_3} (-y)^{2I_3}
\]

(Thanks to Juan Maldacena for an important suggestion.)

This is an index!

Vanilla PSC for vanilla BPS states:

\[
\Omega(\gamma; y) := \text{Tr}_{\mathcal{H}^{BPS}_{\gamma}} y^{2J_3} (-y)^{2I_3}
\]

Also an index.
Closing the Gap

Gap can close when

\[ Z_{\gamma_h}/\zeta \in \mathbb{R}_- \]

for some BPS charge

DEF: BPS ray:

\[ \ell_{\gamma_h,u} = \{ \zeta | Z_{\gamma_h}/\zeta \in \mathbb{R}_- \} \]
IR Description of Framed BPS States: Say halo!

A good description of some states in $\mathcal{H}_{L,\gamma}^{BPS}$ is near $l_{\gamma_h,u}$.

$$\zeta$$

$$r_{\text{halo}} = \frac{\langle \gamma_c,\gamma_h \rangle}{2\text{Im}(Z_{\gamma_h}/\zeta)}$$
Wall-crossing is elegantly summarized by introducing a generating function:

\[ F = \sum_{\gamma} \overline{\Omega}(L, \gamma; y) X_\gamma \]

How is \( F^+ \) related to \( F^- \)?

We gain & lose halo Fock-spaces, exactly as in the derivation of the semi-primitive WCF!
Wall-crossing: Noncommutativity

\[ X_{\gamma_1} X_{\gamma_2} = y^{\langle \gamma_1, \gamma_2 \rangle} X_{\gamma_1 + \gamma_2} \]

\[ F^+ = S_{\gamma h} F^- S_{\gamma h}^{-1} \]

\[ S_{\gamma h} \text{ Product of quantum dilogs } \Phi(X_{\gamma h}) \]
Wall-Crossing II

\[ \Phi(X) = \prod_{k=1}^{\infty} (1 + y^{2k-1}X) \]

\[ \Omega(\gamma_h; -y) = \sum_m a_m^{\gamma_h} y^m \]

\[ S_{\gamma_h} = \prod_m \Phi(( -y)^m X_{\gamma_h}) a_m^{\gamma_h} \]
Wall-Crossing III

This result implies the `motivic wall crossing formula' of Kontsevich & Soibelman

Our discussion is consistent with the form of the result as discussed in

Dimofte & Gukov; Cecotti & Vafa; Dimofte, Gukov & Soibelman
Relation to "Darboux" coordinates on Seiberg-Witten Moduli Spaces

Now we explain how line operators are related to an interesting collection of functions

\[ \chi \]

Lightning review of

Gaiotto, Moore, Neitzke, arXiv:0807.4723
Circle Compactification of $N=2$

$N=2$ Supersymmetric theory on $\mathbb{R}^3 \times S^1_R$

Low energies: Sigma model

$\mathbb{R}^3 \rightarrow \mathcal{M}$

Torus of (elec, mag) Wilson lines

Susy $\rightarrow \mathcal{M}$ hyperkahler

Coulomb branch
``Darboux coordinates``

Giving a HK metric is equivalent to giving holomorphic symplectic structure on $\mathcal{M}^\zeta \quad \zeta \in \mathbb{P}^1$

$\omega_\zeta = \zeta^{-1}\omega_+ + \omega_3 + \zeta\omega_-\$

$\omega_\zeta$ is determined from a collection of functions

$\chi_\gamma : \mathcal{M}^\zeta \times \mathbb{C}^* \to \mathbb{C} \quad \gamma \in \Gamma$

$\chi_\gamma \chi_{\gamma'} = \chi_{\gamma + \gamma'}$

$\omega_\zeta = \langle d \log \chi_\gamma, d \log \chi_\gamma \rangle$
Constructing $\mathcal{X}_\gamma$

$\mathcal{X}^{\text{sf}}_\gamma := \exp \left[ \frac{\pi R}{\zeta} Z_\gamma + i \gamma \cdot \theta + \pi R \zeta \bar{Z}_\gamma \right]$

(Neitzke, Pioline, & Vandoren)

Solve a TBA-like integral equation

$\log \mathcal{X}_\gamma = \log \mathcal{X}^{\text{sf}}_\gamma +$

$+ \sum_{\gamma'} \Omega(\gamma') K_{\gamma, \gamma'} \ast \log(1 + \mathcal{X}_{\gamma'})$
IR Line Operator Expansion

\[ L_\zeta \text{ wraps circle in } \mathbb{R}^3 \times S^1_R \]

\[
\langle L_\zeta \rangle = \sum_\gamma \overline{\Omega}(L_\zeta, \gamma) \chi_\gamma
\]

Holomorphic on \( \mathcal{M}_\zeta \)

\[
\langle L_\zeta \rangle = \text{Tr}_{\mathcal{H}_{L_\zeta}} (-1)^F e^{-2\pi R H} e^{i\theta \cdot Q}
\]

\[
R \rightarrow \infty \quad \sum_\gamma \overline{\Omega}(L_\zeta, \gamma) e^{2\pi R \text{Re}(Z_\gamma/\zeta)} + i \gamma \cdot \theta
\]

\[
\langle L_\zeta \rangle \quad \text{Has no wall-crossing!}
\]
Generalizing a construction of Witten 98, GMN studied these theories in order to construct Darboux coordinates, and found an algorithm for computing the BPS spectrum for $T_{g,n}[A_1]$ theories.

They have attracted a lot of attention following the discovery by Gaiotto, arXiv:0904.2715, that they can be described as generalized quiver gauge theories.
6D to 3D

C: Genus g surface with n punctures

\[ d = 6 \ (2,0) \ g\text{-theory} [\mathbb{R}^3 \times S^1_R \times C] \]

- \( d = 6 \): 6D (2,0) theory
- \( d = 4 \): 4D theory, \( T_{g,n} [\mathfrak{g}] \)
- \( d = 5 \): \( g\)-SYM
- \( d = 3 \): \( \mathbb{R}^3 \rightarrow \mathcal{M} \) Sigma model
Hitchin = Seiberg-Witten

\[ \mathcal{M} : \begin{cases} F + R^2 [\varphi, \bar{\varphi}] = 0 \\ \bar{\partial}_A \varphi = 0 \end{cases} \] on \( \Sigma \)

Hitchin system

Spectral curve in \( T^*C \) \( \Sigma : \det(\lambda - \varphi) = 0 \)

\( \Sigma \) Seiberg-Witten curve

\( \lambda \) Seiberg-Witten differential
Flat Connections

For $\zeta \neq 0, \infty$

$\mathcal{M}^\zeta = \text{Moduli of flat } \mathcal{G}_C \text{ connections}$

$A^\zeta = R^\zeta^{-1} \phi + A + R^\zeta \bar{\phi}$

For $\phi \subset C$:

$\text{Hol}(\mathcal{R}, \phi) := \text{Tr}_\mathcal{R} P \exp \int_\phi A^\zeta$

will be an important holomorphic function for us.
Surface to Line Operators

\[ \langle S(\mathcal{R}, S^1_R \times \varphi) \rangle \]

\[ C \quad \quad \quad S^1_R \]

\[ \langle L_\zeta(\mathcal{R}, \varphi) \rangle = \text{Tr}_\mathcal{R} P \exp \oint_{\varphi} A_\zeta \]

\( L_\zeta \) labeled by \( \varphi \subset C \) and \( \mathcal{R} \):

Consistent with Drukker, Morrison, Okuda
How to compute $\Omega$

\[
\langle L_\zeta \rangle = \sum_{\gamma} \Omega(L_\zeta, \gamma) \chi_\gamma
\]

\[
\langle L_\zeta (R, \wp) \rangle = \text{Hol}(R, \wp)
\]

For $T_{g,n}[A_1]$ theories:

1. Expand $\text{Hol}(R, \wp)$ using
   Fock-Goncharov coordinates $\chi_E$

2. Write $\chi_\gamma$ in terms of $\chi_E$
Choose a triangulation of $C$ with vertices at the punctures:

Choose flat sections $s_i$

\[ (d + A_\zeta) s_i = 0 \]

Coordinates $\chi_E$ on $\mathcal{M}_\zeta$

\[ \chi_E := - \frac{(s_1 \wedge s_2)(s_3 \wedge s_4)}{(s_2 \wedge s_3)(s_4 \wedge s_1)} \]
Fock-Goncharov gives Darboux

$$2 : 1 \quad \Sigma \rightarrow C \quad \Gamma = H_1(\Sigma; \mathbb{Z})^-$$

Use canonical triangulation of $C$:

from integral curves of $\lambda$

$$\gamma_E \quad \text{(nice!)} \text{ basis for } \Gamma$$

$$\chi_{\gamma_E} := \chi_E$$
Computing the Holonomy

\[ \text{Hol}(\mathcal{R}, \wp) = \sum_{\gamma} c_{\gamma} \chi_{\gamma} \]

Computable via a simple traffic rule algorithm

\[ R = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \quad M[x] = \frac{1}{\sqrt{x}} \begin{pmatrix} 0 & -1 \\ -x & 0 \end{pmatrix} \]

\[ \overline{\Omega}(\mathcal{L}_\zeta(\mathcal{R}, \wp)) \text{ computable for } T_{g,n}[A_1] \]
Example: SU(2) $N_f=4$

Would be nontrivial to compute from the geometry of monopole moduli spaces!
Asymptotically Free Theories:

$T_{g,n}[A_1]$ are perturbations of conformal field theories

Reach AF theories by decoupling HM’s: Send masses to infinity

Geometrically:

Collide RSP’s to get ISP’s

UV labeling of line operators:
Example: SU(2) \( N_f=0 \)

Wilson loop:
Charge (0,1)

‘t Hooft-Wilson loops:
Charge (1,2n)
N$_f$=0 Framed BPS Degeneracies

$\Gamma = \mathbb{Z}^2$

$\chi_\gamma = X^m Y^n$

$\langle W_2 \rangle = (XY)^{1/2} + (XY)^{-1/2} + \sqrt{\frac{X}{Y}}$

Naive

Many elaborate results for lamination vevs
Quantum Holonomy

Classically, we have interpreted the framed BPS indices as traces of holonomy:

$$\text{Hol}(\mathcal{R}, \varphi) = \sum_{\gamma} \overline{\Omega}(L_\zeta(\varphi), \gamma) X_\gamma$$

What about the PSC?

Combining with results of Teschner on quantization of Teichmuller space shows that

$$\mathcal{O}(\text{Hol}(\mathcal{R}, \varphi)) \sim \sum_{\gamma} \overline{\Omega}(L_\zeta(\varphi), \gamma; y) X_\gamma$$

Satisfy the same operator algebra.
Looking Ahead…

There’s been lots of progress on $\mathcal{N}=2$, and there’s lots more to do: this will surely keep us focused in the near future…

Litmus test: There is no effective algorithm for computing the BPS spectrum of an arbitrary $\mathcal{N}=2$ FT or string cpct.