Surface Defects and the BPS Spectrum of 4d $N=2$ Theories

Solvay Conference, May 19, 2011

Gregory Moore, Rutgers University

Davide Gaiotto, G.M., Andy Neitzke

Wall-crossing in Coupled 2d-4d Systems: 1103.2598

Framed BPS States: 1006.0146

Wall-crossing, Hitchin Systems, and the WKB Approximation: 0907.3987

Four-dimensional wall-crossing via three-dimensional Field theory: 0807.4723
A Motivating Question

Given an arbitrary four-dimensional field theory with N=2 supersymmetry, is there an algorithm for computing its BPS spectrum?

Who cares?

A good litmus test to see how well we understand these theories…
``Getting there is half the fun!''
Goal For Today

We describe techniques which should lead to such an algorithm for

``A_k theories of class S``

Some isolated examples of BPS spectra are known:

1. Bilal & Ferrari: SU(2) \( N_f = 0,1,2,3 \)
2. Ferrari: SU(2), \( N = 2^* \), SU(2) \( N_f=4 \)
3. GMN: \( A_1 \) theories of class S
Outline

1. Review of some N=2,d=4 theory

2. Theories of Class S
   a. 6d (2,0) and cod 2 defects
   b. Compactified on C
   c. Related Hitchin systems
   d. BPS States and finite WKB networks

3. Line defects and framed BPS States

4. Surface defects
   a. UV and IR
   b.Canonical surface defects in class S
   c. 2d4d BPS + Framed BPS degeneracies

5. 2d/4d WCF

6. Algorithm for theories of class S

7. Overview of results on hyperkahler geometry
$N=2, \, d=4$ Field Theory

Coulomb branch: $\mathcal{B}$,

generic point $u \in \mathcal{B}$

Local system of charges, with integral antisymmetric form:

$$0 \to \Gamma_f \to \Gamma \to \Gamma_g \to 0,$$

$\Gamma_f$: Charges of unbroken flavor symmetries

$\Gamma_g$: Symplectic lattice of (elec,mag) charges of IR abelian gauge theory
Self-dual IR abelian gauge field

\[ V = \Gamma_g \otimes \mathbb{R} \quad F \in \Omega^2(M_4) \otimes V \]

\[ dF = 0 \quad F = *F \]

\[ F = dA = e_I F^I + e^I G_I \]

\[ S = \int \text{Im} \tau_{IJ} F^I \ast F^J + \text{Re} \tau_{IJ} F^I F^J \]
Central charge: $Z \in \text{Hom}(\Gamma, \mathbb{C})$

$$\mathcal{H} = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma \quad E \geq |Z_\gamma|$$

$$\mathcal{H}^{\text{BPS}}_\gamma = \{ \psi : E\psi = |Z_\gamma|\psi \}$$

$$\Omega(\gamma; u; y) := \text{Tr}_{h_\gamma} (-1)^{2J_3} (-y)^{2J_3 + 2I_3}$$

$$\Omega(\gamma; u) := \text{Tr}_{h_\gamma} (-1)^{2J_3}$$
Seiberg-Witten Moduli Space

\[
\emptyset \subset \mathcal{M} = \Gamma_g^* \otimes \mathbb{R}/(2\pi\mathbb{Z})
\]

Hyperkahler target space of 3d sigma model from compactification on \( \mathbb{R}^3 \times S^1 \)

Seiberg & Witten
Theories of Class S

Consider nonabelian (2,0) theory $T[g]$ for "gauge algebra" $g$

The theory has half-BPS codimension two defects $D(m)$

Compactify on a Riemann surface $C$ with $D(m_a)$ inserted at punctures $z_a$

$$so(5)_R \rightarrow so(3)_R \oplus so(2)_R$$

Twist to preserve $d=4,N=2$

$T[g,C,m]$
Seiberg-Witten = Hitchin

\[ T[g, C, m] \]

\[ S^1 \]

\[ C \]

\[ 5D \ g \ SYM \]

\[ F + R^2[\varphi, \bar{\varphi}] = 0 \]

\[ \bar{\partial}_A \varphi = 0 \]

\[ \mathbb{R}^{1,2} \rightarrow \mathcal{M} \]

\[ \sigma \text{-Model:} \]
Digression:  **Puncture Zoo**

**Regular singular points:**

\[
\varphi \sim \frac{dz}{z-z_a} r + \text{reg}
\]

**r:**

- \( \text{Diag}\{m_1, m_2, \ldots, m_k\} \)  ``Full puncture''
- \( \text{Diag}\{m, \ldots, m, -(k-1)m\} \)  ``Simple puncture''

**Irregular singular points:**

\[
\varphi \sim \frac{dz}{(z-z_a)^\ell} r + \text{reg}
\]
Seiberg-Witten Curve

\[ \Sigma : \det(\lambda - \varphi(z, \bar{z})) = 0 \subset T^*C \]

\[ \lambda = p dq \quad \lambda |_{\Sigma} \quad \text{SW differential} \]

For \( g = \text{su}(k) \quad \pi : \Sigma \to C \]

is a k-fold branch cover

\[ \lambda^k + \lambda^{k-2} \phi_2(z) + \cdots + \phi_k(z) = 0 \]
Local System of Charges

\[
\ker \pi_* : \text{Jac}(\Sigma) \rightarrow \text{Jac}(C)
\]

determines \( \Gamma \subset H_1(\Sigma; \mathbb{Z}) \)

A local system over a torsor for spaces of holomorphic differentials…
BPS States: Geometrical Picture

BPS states come from open M2 branes stretching between sheets \( i \) and \( j \). Here \( i, j, =1,\ldots, k \). This leads to a nice geometrical picture with string networks:

Klemm, Lerche, Mayr, Vafa, Warner; Mikhailov; Mikhailov, Nekrasov, Sethi,

**Def:** A WKB path on \( C \) is an integral path

\[
\langle \lambda_i - \lambda_j, \partial_t \rangle = e^{iv\theta}
\]

Generic WKB paths have both ends on singular points \( z_a \)
Finite WKB Networks - A

But at critical values of $\vartheta = \vartheta^*$ "finite WKB networks appear":

$\vartheta < \vartheta_c$  $\quad \vartheta = \vartheta_c$  $\quad \vartheta > \vartheta_c$

$\rightarrow$  

Hypermultiplet
Finite WKB Networks - B

Closed WKB path

$\vartheta < \vartheta_c$

$\vartheta = \vartheta_c$

$\vartheta > \vartheta_c$

Vectormultiplet
At higher rank, we get string networks at critical values of $\vartheta$:

A "finite WKB network" is a union of WKB paths with endpoints on branchpoints or such junctions.

These networks lift to closed cycles $\gamma$ in $\Sigma$ and represent BPS states with

$$Z_\gamma = \oint_\gamma \lambda = e^{i\vartheta} |Z_\gamma|$$
Line Defects & Framed BPS States

A **line defect** (say along $\mathbb{R}_t \times \{0\}$) is **of type $\zeta$** if it preserves the susys:

$$Q^A_{\alpha} + \zeta \sigma^0_{\alpha \beta} \bar{Q}^\beta A$$

Example:

$$L_\zeta = \exp \int_{\mathbb{R}_t \times \bar{0}} \left( \frac{\varphi}{2\zeta} + A + \frac{\zeta \bar{\varphi}}{2} \right)$$

$$\mathcal{H}_L = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_{L, \gamma}$$

$$E \geq -\text{Re}(Z_{\gamma}/\zeta)$$
Framed BPS States saturate this bound, and have framed protected spin character:

$$\overline{\Omega} := \text{Tr}_\mathcal{H}_L^{bps} (-1)^2 J_3 (-y)^2 J_3 + 2 I_3$$

$$\overline{\Omega}(L, \gamma; y; \zeta; u)$$

Piecewise constant in $\zeta$ and $u$, but has wall-crossing across ``BPS walls'' (for $\Omega(\gamma) \neq 0$):

$$W_\gamma := \left\{ (u, \zeta) : \frac{Z_\gamma(u)}{\zeta} \in \mathbb{R}_- \right\}$$

Particle of charge $\gamma$ binds to the line defect:

Similar to Denef's halo picture
Wall-Crossing for $\overline{\Omega}$

$$F(L) = \sum_\gamma \overline{\Omega}(L, \gamma; y) X_\gamma$$

$$X_{\gamma_1} X_{\gamma_2} = y^{\langle \gamma_1, \gamma_2 \rangle} X_{\gamma_1 + \gamma_2}$$

Across $W(\gamma_h)$ Denef’s halo picture leads to:

$$F^+(L) = \Phi(X_{\gamma_h}) F^-(L) \Phi(X_{\gamma_h})^{-1}$$

$\Phi(X_{\gamma_h})$ constructed from $\Omega(\gamma_h; y)$
Wall-Crossing for $\Omega$

Consistency of wall crossing of framed BPS states implies the Kontsevich-Soibelman ``motivic WCF'' for

$$\Omega(\gamma; y)$$

This strategy also works well in supergravity to prove the KSWCF for BPS states of Type II/Calabi-Yau
(but is only physically justified for $y=-1$)

Andriyash, Denef, Jafferis, Moore
Line defects in $T[g,C,m]$

6D theory $T[g]$ has supersymmetric surface defects $S(R, \sigma)$

For $T[g,C,m]$ consider

\[ \sigma = \mathbb{R} \times \{\vec{0}\} \times \mathcal{O} \]

$C$ \quad \text{Line defect in 4d labeled by isotopy class of a closed path $\mathcal{O}$ and $R$}

$k=2$:

Drukker, Morrison, Okuda
Complex Flat Connections

\((A, \phi)\) solve Hitchin equations iff

\[ A = R\zeta^{-1} \varphi + A + R\zeta \overline{\varphi} \]

is a complex flat connection on \(\mathbb{C}\) \(\forall \zeta \in \mathbb{C}^*\)

On \(\mathbb{R}^3 \times S^1\) line defects become local operators in the 3d sigma model:

\[ \langle L_\zeta (\mathcal{R}, \varphi) \rangle = \text{Tr}_\mathcal{R} \text{Hol}(A, \varphi) \]
Surface defects

$S \text{ at } x^1 = x^2 = 0$

Preserves $d=2$ (2,2) supersymmetry subalgebra

Twisted chiral multiplet: $\Upsilon = \varphi + \cdots$

Finite set of vacua $i \in \mathcal{V}$

$S_{IR} = \int d^4x d^4\theta \mathcal{F}(a) + \int d^2x d^2\theta \mathcal{W}^{eff}(\Upsilon)$
Effective Solenoid

\( \int A = \gamma_i \in V = \Gamma_g \otimes \mathbb{R} \)

\( \gamma_i = \eta^I e^I + \alpha^I e_I \)

\( \eta + \tau \cdot \alpha = \frac{\partial W_{eff}^{eff}}{\partial a} \)

\( \alpha, \eta \) ARE NOT QUANTIZED
Torsor of Effective Superpotentials

A choice of superpotential =
a choice of gauge =
a choice of flux $\gamma_i$

$$Z_{\gamma_i} := \mathcal{W}^{eff}$$

$$Z_{\gamma_i + \gamma} := \mathcal{W}^{eff} + Z_{\gamma}$$

Extends the central charge to a $\Gamma$ - torsor $\Gamma_i$
Canonical Surface Defect in $T[\mathfrak{g}, C, m]$

For $z \in C$ we have a **canonical surface defect** $S_z$

It can be obtained from an M2-brane ending at $x^1 = x^2 = 0$ in $\mathbb{R}^4$ and $z$ in $C$.

In the IR the different vacua for this M2-brane are the different sheets in the fiber of the SW curve over $z$.

Therefore the chiral ring of the 2d theory should be the same as the equation for the SW curve!

$$\lambda^k + \lambda^{k-2} \phi_2(z) + \cdots + \phi_k(z) = 0$$

Alday, Gaiotto, Gukov, Tachikawa, Verlinde; Gaiotto
Example of SU(2) SW theory

\[ \lambda^2 = \left( \frac{1}{z} + \frac{2u}{z^2} + \frac{\Lambda^2}{z^3} \right) (dz)^2 \]

\[ \lambda = xdz \quad z = e^t \]

\[ x^2 = e^t + 2u + \frac{\Lambda^2}{e^t} \]

Chiral ring of the $\mathbb{CP}^1$ sigma model.

Twisted mass

2d-4d instanton effects

Gaiotto
Superpotential for $S_z$ in $T[g,C,m]$

\[ Z \gamma_i - Z \gamma_j = \int_{\gamma_{ij}} \lambda \]

\[ \gamma_{ij} \text{ Homology of an open path on } \Sigma \text{ joining } x_i \text{ to } x_j \text{ in the fiber over } S_z \]

\[ \gamma_{ij} \in \Gamma_{ij} \subset H_1(\Sigma, \{x_i, x_j\}; \mathbb{Z}) \]
New BPS Degeneracies: $\mu$

$\mu(\gamma_{ij})$ 2D soliton degeneracies.

\[
 Flux: \gamma_{ij} \in \Gamma_{ij}
\]

For $S_z$ in $T[su(k),C,m]$, $\mu$ is a signed sum of open finite BPS networks ending at $z$. 
New BPS Degeneracies: $\omega$

Degeneracy: $\omega(\gamma; \gamma_i)$

$\omega(\gamma; \gamma_i + \gamma') = \omega(\gamma; \gamma_i) + \Omega(\gamma)\langle\gamma, \gamma'\rangle$

Flux: $\gamma \in \Gamma$
Supersymmetric Interfaces - A

Flux: $\gamma_{ij'} \in \Gamma_{ij'}$
Supersymmetric Interfaces -B

\[ \mathcal{H}_{SSLs'} = \bigoplus_{\gamma_{ij}, \in \Gamma_{ij}, \mathcal{H}_{SSLs'}, \gamma_{ij}} \]

Our interfaces preserve two susy’s of type \( \zeta \) and hence we can define framed BPS states and form:

\[ F(L) = \sum_{\Gamma_{ij}} \bar{\Omega}(L, \gamma_{ij}) X_{\gamma_{ij}} \]

\[ X_{\gamma_{ij}} X_{\gamma_{j'\prime\prime\prime\prime}} = \begin{cases} \pm X_{\gamma_{ij}} + \gamma_{j'\prime\prime\prime}, & j' = k'' \\ 0, & \text{else} \end{cases} \]
Susy interfaces for $T[g,C,m] - A$

Interfaces between $S_z$ and $S_{z'}$ are labeled by open paths $\varnothing$ on $C$.

So: framed BPS states are graded by open paths $\gamma_{ij'}$ on $\Sigma$ with endpoints over $z$ and $z'$.

$$\Gamma_{ij'} \subset H_1(\Sigma, \{x_i, x_{j'}\}; \mathbb{Z})$$
Susy interfaces for $T[\mathcal{g},\mathcal{C},m] - B$

Wrapping the interface on a circle in $\mathbb{R}^3 \times S^1$ compactification:

$$\langle L_\zeta(\mathcal{R}, \phi) \rangle = \rho_\mathcal{R} Hol(A, \phi)$$

$$A = R^{-1}_\zeta \phi + A + R \zeta \overline{\phi}$$
Framed BPS Wall-Crossing

Across BPS $W_{\gamma}$ walls the framed BPS degeneracies undergo wall-crossing.

Now there are also 2d halos which form across walls

$$W_{\gamma_{ik}} := \{ (u, \zeta) : Z_{\gamma_{ik}}(u)/\zeta \in \mathbb{R}_- \}$$

$$F(L) \rightarrow SF(L)S^{-1} \quad S = 1 + \mu(\gamma_{ik})X_{\gamma_{ik}}$$

As before, consistency of the wall-crossing for the framed BPS degeneracies implies a general wall-crossing formula for unframed degeneracies $\mu$ and $\omega$. 
Framed Wall-Crossing for $T[g,C,m]$

The separating WKB paths of phase $\zeta$ on $C$ are the BPS walls for

$$\Omega(L_\zeta, \mathcal{P}(z,z'), \gamma ij')$$
Formal Statement of 2d/4d WCF

1. Four pieces of data
2. Three definitions
3. Statement of the WCF
4. Relation to general KSWCF
5. Four basic examples
A. Groupoid of vacua, $\mathcal{V}$: Objects = vacua of $\mathcal{S}$: $i = 1, \ldots, k$ & one distinguished object 0. Morphism spaces are torsors for $\Gamma$, and the automorphism group of any object is isomorphic to $\Gamma$:
2d-4d WCF: Data

B. Central charge $Z \in \text{Hom}(\mathcal{V}, \mathbb{C})$:

$$Z(a + b) = Z(a) + Z(b)$$

Here $a, b$ are morphisms $\gamma, \gamma_i, \gamma_{ij}$; valid when the composition of morphisms $a$ and $b$, denoted $a+b$, is defined.

C. BPS Data:

$$\mu(\gamma_{ij}) \in \mathbb{Z} \quad \& \quad \omega(\gamma, a) \in \mathbb{Z}$$

$$\omega(\gamma; a + \gamma') = \omega(\gamma; a) + \Omega(\gamma) \langle \gamma, \gamma' \rangle$$

D. Twisting function:

$$\sigma(a, b) \in \mathbb{Z}_2 \text{ when } a+b \text{ is defined}$$
A. A **BPS ray** is a ray in the complex plane:

\[ \ell_\gamma = Z(\gamma)\mathbb{R}_- \quad \text{IF} \quad \omega(\gamma, \cdot) \neq 0 \]
\[ \ell_{\gamma_{ij}} = Z(\gamma_{ij})\mathbb{R}_- \quad \text{IF} \quad \mu(\gamma_{ij}) \neq 0 \]

B. The **twisted groupoid algebra** \( \mathbb{C}[\mathcal{V}] \):

\[ X_a X_b = \begin{cases} 
\sigma(a, b)X_{a+b} & a + b \text{ composable} \\
0 & \text{else}
\end{cases} \]
2d-4d WCF: 3 Definitions

C. Two automorphisms of $\mathbb{C}[V]$:

CV-like: $S_{\gamma i j}^\mu$:

$$X_a \rightarrow (1 - \mu(\gamma_{ij})X_{\gamma_{ij}})X_a(1 + \mu(\gamma_{ij})X_{\gamma_{ij}})$$

KS-like: $\mathcal{K}_{\gamma}^\omega$

$$X_a \rightarrow (1 - X_\gamma)^{-\omega(\gamma;a)}X_a$$
2d-4d WCF: Statement

Fix a convex sector: \( \angle \)

\[ A(\angle) =: \prod S^{\mu}_{\gamma_{i,j}} K_{\gamma}^{\omega} : \]

The product is over the BPS rays in the sector, ordered by the phase of \( Z \)

WCF:

\( A(\angle) \) is constant as a function of \( Z \), so long as no BPS line enters or leaves the sector
Kontsevich & Soibelman stated a general WCF attached to any graded Lie algebra $g$ with suitable stability data.

The 2d-4d WCF (with $y= -1$ ) is a special case for the following Lie algebra

\[ T = \Gamma^* \otimes \mathbb{C}^* \]

Generated by

\[ \mathcal{V}_\gamma \quad \mathcal{V}_\gamma \mathcal{V}_{\tilde{\gamma}} = \sigma(\gamma, \tilde{\gamma})\mathcal{V}_{\gamma + \tilde{\gamma}} \]

\[ g = M_k(\mathcal{A}) \oplus \text{SympVect}(T) \]
Four “`types” of 2d-4d WCF-A

A. Two 2d – central charges sweep past each other:

\[
Z(\gamma_{ij}) \quad Z(\gamma_{ik}) \quad Z(\gamma_{jk}) \quad Z(\gamma_{kj})
\]

\[
S^\mu_{\gamma_{ij}} S^\mu_{\gamma_{il}} S^\mu_{\gamma_{jl}} = S^{\mu'}_{\gamma_{jl}} S^{\mu'}_{\gamma_{il}} S^{\mu'}_{\gamma_{ij}}
\]

Cecotti-Vafa
Four ```types``` of 2d-4d WCF - B

B. Two 4d – central charges sweep past each other:

\[
\prod_{\frac{n}{m}} \mathcal{K}^\omega_{n\gamma^1 + m\gamma^2} = \prod_{\frac{n}{m}} \mathcal{K}^{\omega'}_{n\gamma^1 + m\gamma^2}
\]
Four ``types” of 2d-4d WCF - C

C. A 2d and 4d central charge sweep past each other:

\[ Z(\gamma_{ij}) \rightarrow Z(\gamma_{ij} + n\gamma) \rightarrow Z(\gamma) \rightarrow Z(\gamma_{ij} + n\gamma) \rightarrow Z(\gamma_{ij}) \]

\[ \mathcal{K}^\omega_\gamma \prod_n S^\mu_{\gamma_{ij} + n\gamma} = \prod_n S^\mu'_{\gamma_{ij} + n\gamma} \mathcal{K}^\omega'_{\gamma} \]
Four "types" of 2d-4d WCF - D

D. Two 2d central charges sweep through a 4d charge:

\[ \Pi_n \rightarrow S^\mu_{\gamma_{ij} + n\gamma} \prod_{m=1}^{\infty} K^\omega_{m\gamma} \prod_n \downarrow S^\mu_{\gamma_{ji} + n\gamma} = \]

\[ \Pi_n \rightarrow S^\mu'_{\gamma_{ji} + n\gamma} \prod_{m=1}^{\infty} K'^{\omega'}_{m\gamma} \prod_n \downarrow S^\mu'_{\gamma_{ij} + n\gamma} \]
Fix a phase \( \vartheta \). On the UV curve \( C \) draw the separating WKB paths of phase \( \vartheta \): These begin at the branch points but end at the singular points (for generic \( \vartheta \)):

Massive Nemeschansky-Minahan E\(_6\) theory, realized as a trinion theory a la Gaiotto.
Label the walls with the appropriate $S^\mu$ factors – these are easily deduced from wall-crossing.

Now, when a $ij$-line intersects a $jk$-line, new lines are created. This is just the CV wall-crossing formula $SSS = SSS$. 
C: Iterate this process.

Conjecture: It will terminate after a finite number of steps (given a canonical structure near punctures).

Call the resulting structure a ``minimal S-wall network” (MSWN)

D: Now vary the phase $\theta$.

This determines the entire 2d spectrum
\[ \mu(\gamma_{ij}) \text{ for all } S_z, i, j \]
The MSWN will change isotopy class precisely when an S-wall sweeps past a K-wall in the $\zeta$-plane. Equivalently, when an (ij) S-wall collides with an (ij) branch point:
Finally, use the 2d/4d WCF to determine the 4d BPS spectrum:

\[
Z_{\gamma_{ij}} \parallel Z_{\gamma}
\]

\[
(1 - X_{\gamma}) \omega(\gamma, \gamma_{ij}) = \Sigma'_{ij} / \Sigma_{ij}
\]

\[
\Sigma_{ij} := \sum_{n=0}^{\infty} \mu(\gamma_{ij} + n\gamma) X_{\gamma}^n
\]
\[ Z_{\gamma_{ij}} \parallel Z_{\gamma} \parallel Z_{\tilde{\gamma}_{ji}} \]

\[ \Pi_{ij} = \frac{\Sigma_{ij}'}{\Sigma_{ij}} - \sigma(\gamma_{ij}, \gamma_{ji}) \Sigma_{ij} \Sigma_{ji} X_{\gamma} \]

\[ \Pi_{ij} = \prod_{n=1}^{\infty} \frac{1}{1 - X_{\gamma}^n} \omega(n\gamma, \gamma_{ij}) \]
Concluding slogan for this talk

The 2D spectrum controls the 4D spectrum.
Spectrum Generator?

Can we work with just one $\zeta$?

Perhaps yes, using the notion of a "spectrum generator" and "omnipop"

This worked very well for $T[\text{su}(2),C,m]$ to give an algorithm for computing the BPS spectrum of these theories.

Stay tuned....
Hyperkahler geometry: A system of holomorphic Darboux coordinates for SW moduli spaces can be constructed from a TBA-like integral equation, given $\Omega$.

1. From these coordinates we can construct the HK metric on $M$.

3. $\langle L_\zeta, \phi \rangle = \sum_\gamma \bar{\Omega}(L, \gamma) \mathcal{V}_\gamma$
Hyperkahler Summary - B

4. For $T[\text{su}(2),C,m]$, $\mathcal{Y}_\gamma$ turn out to be closely related to Fock-Goncharov coordinates

5. We are currently exploring how the coordinates for $T[\text{su}(k),C,m]$ are related to the "higher Teichmuller theory" of Fock & Goncharov
Hyperkahler Summary - C

6. For $T[\mathfrak{su}(2), \mathbb{C}, m]$ the analogous functions: $\mathcal{V}_{\gamma_{ij}'}$, associated to

$$\left\langle L_\zeta, \mathcal{P}(z, z') \right\rangle$$

are sections of the universal bundle over $\mathcal{M}$, and allow us moreover to construct hyper-holomorphic connections on this bundle.

7. Explicit solutions to Hitchin systems (a generalization of the inverse scattering method)
That's all Folks!