LINE OPERATORS IN $N = 2$ THEORIES

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MARCH 30, 2010

WORK IN PROGRESS WITH

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$1003.5657 \leq ? \leq 2003.5657$
I. MOTIVATION & SUMMARY

- WE HAVE SEEN IN ANDY'S TALK THAT $S^1$-COMPACTIFICATION OF $d=4, N=2$ THEORIES PRODUCES INTERESTING FUNCTIONS $\{ Y_\theta \}$ WHICH SATISFY A TBA-LIKE EQUATION AND CAN BE USED TO CONSTRUCT HK MANIFOLDS.

- MOREOVER, FOR $T_{g,n}[A,?]$ THEORIES THESE FUNCTIONS CAN BE CONSTRUCTED FROM FOCK-GONCHAROV COORDINATES ON MODULI OF FLAT $SL(2,\mathbb{C})$ CONNECTIONS.
• We wanted to find a direct physical interpretation of \( \{ y_\gamma \} \) — and the answer is that they are VEV's of IR line operators.

• We are aiming for a formula like

\[
\langle L \rangle = \sum_\gamma \overline{\Omega}(L,\gamma) y_\gamma
\]

\[\uparrow\]

UV line op

\[\uparrow\]

E-M charges

\[\uparrow\]

Expansion coeffs.

Turn out to be indices \( \in \mathbb{Z} \)
While unraveling this we found some other results of independent interest:

1. $\langle L \rangle$ can be viewed as a "classical limit" of

$$F(L) = \sum_{\gamma} \Omega(L, \gamma; y) \ X_{\gamma}$$

$$X_{\gamma}X_{\gamma'} = y^{\langle \gamma, \gamma' \rangle} X_{\gamma + \gamma'}$$

$$\Omega(L, \gamma; y) = \text{character of a rep. of } \text{SU}(2)$$

$$= \sum_{n \geq 0} a_n [n]_y \quad a_n \in \mathbb{Z}_+$$
2. NEW PHYSICAL PROOF OF THE "MOTIVIC KSWCF"

3. THERE ARE CONCRETE ALGORITHMS FOR COMPUTING \( \langle L \rangle \) \( \in \) \( F(L) \) IN THE \( T_{g,n} [A_i] \) THEORIES

4. THE \( F(L) \) ARE RELATED TO QUANTUM GEODESIC OPERATORS IN TEICHMULLER THEORY
OUTLINE

II. SUSY LINE OPERATORS

III. FRAMED BPS STATES

IV. HALOS

V. WALL-CROSSING

VI. DEFORMED PRODUCT

VII. FORMAL LINE OPERATORS

VIII. DARBOUX COORDINATE EXPANSION

IX. COMPUTING $\langle L_5 \rangle$ IN $A_1$-THEORIES

X. SUMMARY & OPEN PROBLEMS
11. SUSY LINE OPERATORS: $d = 4, N = 2$

- $S = \text{RG fixed point with } SU(2,2|2) \text{ symmetry}$

- LINE OPERATOR = BOUNDARY COND. ON S ON CONFORMAL NBD. $\text{AdS}_2 \times S^2$ [KAPUSTIN]

- $L$ IS AN "OPERATOR" IN THE SENSE THAT WE CAN INSERT IN PATH INTEGRAL

- IN THE PRESENCE OF $L$ THERE IS A HILBERT SPACE $\mathcal{H}_L$.

- ACTUALLY $L$ IS AN OBJECT IN A MONOIDAL CATEGORY
We want to restrict attention to a class of line op's preserving 4 susy's.

Let \( f \in U(1) \) and define a superalgebra

\[
\text{osp}(4^*|2)_f \subset \text{su}(2,2|2)
\]
\( \mathfrak{osp}(4*12)_5 \) := FIXED SUBALGEBRA OF AN INvolution COMBINING \( P \) WITH \( U(1)_R \) ROTATION BY \( 5 \)

\[
\mathfrak{osp}(4*12)_5^{\text{even}} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(3) \oplus \mathfrak{su}(2)
\]

\[
\mathfrak{osp}(4*12)_5^{\text{odd}} \text{ has Poincaré susy:}
\]

\[
R^A_a \sim Q^A_a + 5 \sigma_0^a \dot{Q}^{\dot{A}}_b
\]

- A LINE OP. PRESERVING THIS SYMMETRY WILL BE DENOTED \( \mathcal{L}_5(\ldots) \)
- WE WILL CONTINUE TO \( 5 \in \mathbb{C}^* \)
1) \(G = \text{CPT, SIMPLE LIE GROUP}
R = \text{REP}
\]
\[
L_5(G,R) = \int_R \left( \text{Pexp} \int_{\mathbb{R} \times \mathbb{R}} \left( \frac{\varphi}{25} - iA - \frac{\xi}{2} \overline{\varphi} \right) \right)
\]

2) \(\text{COCHARACTER } \nu \in \text{Hom}(U(1), T) \]
\(\text{DEFINES } \,'	ext{t Hooft OPERATOR}\)

3) \(\text{WILSON-'}\text{t Hooft LABELED BY CERTAIN COLLECTIONS (KAPUSTIN)}\)
\[
\mathcal{L} / \mathcal{W}
\]
\[
\Lambda^*_w \times \Lambda^*_r \subset \mathcal{L} \subset \Lambda^*_r \times \Lambda^*_w
\]

\(\uparrow\) \text{'t Hooft}
\(\uparrow\) \text{WILSON}
• **WE CAN DEFINE THE** **SUM** **OF**
  **LINE OPERATORS BY SUMMING PATH INTEGRALS**

• $\mathcal{L}_{L+L'} := \mathcal{L}_L + \mathcal{L}_{L'}$

**DEF:** $L$ is **simple** if $L \neq L' + L''$
  **for two nontrivial** $L', L''$
WE CAN DEFINE THE PRODUCT OF LINE OP'S BY DOUBLE INSERTION

\[ \langle L_1 L_2(x) \rangle := \lim_{x_2 \to x_2} \langle L_1(x_1) L_2(x_2) \rangle \]

- **Commutative** (to be deformed)

C.f. Kapustin, Witten, Kapustin, Saulina \( W = 4 \)
**MUTUAL LOCALITY**

Def: Let $\mathbb{U}(1)_{\tilde{x}_1\tilde{x}_2} \subset \mathbb{SU}(2)_{\text{rot}}$.

If $\mathcal{H}_{L_1(x_1) L_2(x_2)}$ is a rep of $\mathbb{U}(1)_{\tilde{x}_1\tilde{x}_2}$ (and not just $\widetilde{\mathbb{U}(1)_{\tilde{x}_1\tilde{x}_2}}$)

Then $L_1 \& L_2$ are mutually local.

**Example** $G = \text{SpL}e$ group

$\Lambda_{cr} \times \Lambda_{r} \subset L \subset \Lambda_{\text{mw}} \times \Lambda_{\text{w}}$

$L = \text{set of simple mutually local w/ th loops if}$

$\langle p_1, q_2 \rangle \langle p_2, q_1 \rangle \in \mathbb{Z}$

For all $(p, q_1) \& (p_2, q_2) \in L$. 

III. FRAMED BPS STATES \& PSC

$N=2, d=4$ theory has a local system defined by charge lattice

$\Gamma \rightarrow B = \text{VM moduli}$

$\Phi \sim \mathbb{U}$

If $L$ is simple then

$\mathcal{H}_L = \bigoplus_{\delta \in \Gamma_L} \mathcal{H}_{L,\delta}$

$\Gamma_L = \Gamma + \mathbb{Z}_L$ torsor for $\Gamma$

Example: Even in pure SU(2) we can introduce Wilson line in the fundamental.
\( \{ R^A, R^B \} = 4 \left( E + \text{Re}(Z_{8/5}) \right) \epsilon_{\alpha \beta} \epsilon^{AB} \)

\implies E \geq -\text{Re}(Z_{8/5}) \text{ on } \mathcal{H}_{L_{8/5}} \)
THE PROTECTED SPIN CHARACTER

\[ \mathcal{H}_{L_5, \chi}^{\text{BPS}} \] is a rep of \( \mathfrak{so}(3)_2 \oplus \mathfrak{su}(2)_1 \)

\[
\Omega(L_5, \chi; y) := \text{Tr} \mathcal{H}_{L_5, \chi}^{\text{BPS}} y^{2J_3} (-y)^{2I_3}
\]

- Vanishes on massive reps.

- PSC for vanilla BPS states

\[
\mathcal{H}_{y}^{\text{BPS}} = \left[ (\frac{1}{2}; 0) \oplus (0; \frac{1}{2}) \right] \otimes \mathcal{H}_{y}^{\text{BPS}}
\]

\[
\Omega(y; y) = \text{Tr} \mathcal{H}_{y}^{\text{BPS}} y^{2J_3} (-y)^{2I_3}
\]
• The PSC (suggested by J.M.) is an improvement on the spin character, which is not a priori an index.

• However examples (Diaconescu \& M., Dimofte \& Gukov, GMN) thus far examined suggest the

**Strong Positivity Conjecture:**

For $d=4$, $\mathcal{N}=2$ field theory

$$\text{PSC} = \text{Spin Character}$$

⇒ Nontrivial statements about $\mathcal{D}$ on monopole moduli spaces…
IV. HALOS & HALO FOCK SPACES

How does the PSC depend on $u \in \mathbb{B}$ and $s \in U(1)$?

It is piecewise constant but can jump when $E_{\text{gap}} \to 0$.

\[\begin{align*}
\text{VANILLA BOUND} \quad |Zy| \\
-\Re(Zy/5) \quad \downarrow E_{\text{gap}} \\
\end{align*}\]
IR MODEL FOR THE BOUND STATES

STATES IN $\mathcal{H}_{L, \chi}$:

1.) $\bullet \leftrightarrow \infty$-HEAVY BPS DYON
    OF CHARGE $\chi$

    \[ E = -\text{Re}\left( \frac{Z_\chi}{\sigma} \right) \]

2.) $\chi = \chi_c + \chi_h$

    \[ \chi_h \rightarrow \bullet \rightarrow \chi_c \]

    HEAVY BPS DYON

    OF CHARGE $\chi_c$

    \[ E = -\text{Re}\left( \frac{Z_{\chi_c}}{\sigma} \right) + |Z_{\chi_h}| + \frac{1}{2}|Z_{\chi_h}|v^2 + \cdots \]
3.) \[ \text{A FRAMED BPS STATE} \]

FROM \( \int m \, ds + \langle \gamma_h, \mathcal{A} \rangle \) GET BOUNDSTATE

\[
\gamma_{\text{HALO}} = \frac{\langle \gamma_h, \gamma_c \rangle}{2 \text{Im}(Z_{\gamma_h}(u)/\sqrt{s})}
\]

\[
E = -\text{Re}(Z_{\gamma_c}/\sqrt{s}) - \text{Re}(Z_{\gamma_h}/\sqrt{s}) = -\text{Re}(Z_{\gamma}/\sqrt{s})
\]

N.B. THIS DESCRIPTION OF FRAMED BPS STATES IS ONLY VALID FOR \( \gamma_{\text{HALO}} \gg 1 \).
\[ E_{\text{gap}} = |Z_{\gamma_h}| + \text{Re}(Z_{\gamma_h}/\xi) \]

**GAP CLOSES ACROSS WALLS**

\[ \hat{\mathcal{W}}(\gamma_h) := \{(u,\xi) \mid Z_{\gamma_h}(u)/\xi < 0 \}\]

\[ < \mathbb{B} \times \mathbb{C}^* \]
LOSING BOUNDSTATES

NOTE THAT ACROSS WALLS

\[ \hat{W}(\gamma_h) := \{ (u, s) \mid Z_{\gamma_h}(u)/s < 0 \} \subset \mathbb{B} \times \mathbb{C}^* \]

\[ r_{\text{HALO}} \rightarrow \infty \]

\[ r_{\text{HALO}} = \frac{\langle \gamma_h, \gamma_c \rangle}{2 \text{ Im}(Z_{\gamma_h}(u)/s)} \]
**HALO Fock Spaces - Part I**

Since $n_i \mathcal{Y}_h$ - particles are mutually BPS:

Models a framed BPS state for

$$\gamma = \left( \sum_i n_i \right) \mathcal{Y}_h + \mathcal{Y}_c$$

States build up a Fock space; we can "add" particles with charge parallel to $\mathcal{Y}_h$.
STABILITY CONDITION FOR FOCK SPACES

\[ \langle \gamma_h, \gamma_c \rangle \text{Im} \left( \frac{Z_{\gamma_h}(\omega)}{z} \right) > 0 \]

WALL-CROSSING OF THE PSC OCCURS WHEN WE GAIN/LOSE THESE FOCK SPACES:

\[ r_{\text{HALO}} \rightarrow \infty \]

HALO

\[ r_H > 0 \]

GAIN

\[ \hat{W}(\gamma_h) \]

HALO

\[ r_H < 0 \]

LOSE
HALO Fock SPACE - PART II

“ADDING PARTICLES”: CREATION OP’S

\[ V_{\gamma_c, \gamma_h} = (J_{\gamma_c, \gamma_h}) \otimes \mathfrak{h}^{\text{BPS}}_{\gamma_h} \]

\[ (J_{\gamma_c, \gamma_h}) = \text{so(3) IRREP, DIM} = \left| \langle \gamma_c, \gamma_h \rangle \right| \]

\[ \mathcal{F}_{\text{HALO}} = \mathfrak{h}_{L, \gamma_c} \otimes \mathcal{F} \left[ V_{\gamma_c, \gamma_h} \right] \]

\[ \mathbb{Z}_2\text{-GRADED SPACE OF CREATION OP’S} \]

HYPERMULT’S = FERMIONS

VECTORMULT’S = BOSONS

\[ \text{Tr}_{\mathfrak{g}^{\text{BPS}}_{\gamma_h}} (-y)^{2s_3} y^{2\bar{s}_3} = \sum_{-M_h}^{M_h} a_{m, \gamma_h} y^m \]
Taking into account particles with charge parallel to $\gamma_h$:

$$F_{\text{HALO}} = \mathcal{H}_{L,\gamma_c} \bigotimes_{l=1}^{\infty} F [V_{\gamma_c, l \gamma_h}]$$
V. Wall-Crossing Formula

(Classical) Generating Function

\[ F^\text{cl}(L) := \sum_y \Omega(L; \gamma; y) x_\gamma \]

If \( x_\gamma x_{\gamma'} = x_{\gamma + \gamma'} \), then Gain/Loss of Halos is expressed by

\[ x_{\gamma c} \rightarrow x_{\gamma c} \prod (1 + (-1)^{m_m + m_m'})^{a_{m, \gamma} y_h} \]

\[ -2J_{\gamma c, \gamma h} \leq m' \leq 2J_{\gamma c, \gamma h} \]

\[ -M_h \leq m \leq M_h \]

\[ \text{Tr}_{\g} \left( -y \right)^{2J_3} y^{2I_3} = \sum_{-M_h}^{M_h} a_{m, \gamma} y^{m} \]
THIS FORMULA IS AWKWARD.

IT TURNS OUT THERE IS A BETTER WAY:

INTRODUCE QUANTUM $X_\gamma$:

$$X_\gamma X_{\gamma'} = \gamma^{\gamma \gamma'} X_{\gamma + \gamma'}$$

$$F(L) = \sum_\gamma \Omega(L; \gamma; \gamma') X_\gamma$$

$\hat{W}(y_n)$ FOR $\Omega(y_n; u) \neq 0$

DIVIDE $\mathcal{B} \times \mathbb{C}^*$ INTO CHAMBERS

$$F(L; c) = \text{VALUE OF } F(L) \text{ IN CHAMBER } c$$
\[ F(L, c^+) = S_{\gamma_h} \quad F(L, c^-) S_{\gamma_h}^{-1} \]

\[ S_{\gamma_h} = \prod_{m_h} \prod_{m_h} (-1)^{m_h} y^{m_h} X_{\gamma_h} a_{m_h, \gamma_h} \]

\[ \psi(X) = \prod_{k=1}^{\alpha} (1 + y^{2k-1} X)^{-1} \]

= \text{QUANTUM Dilog}
• Derive the "motivic KSWCF": Two different paths between a fixed pair of chambers must induce the same transformation!

• Consistent with Dimofte/Gukov/Strominger

• Classical limit: \( y = -1 \)

\[
\hat{x}_y \hat{x}_{y'} = (-1)^{\langle y, y' \rangle} \hat{x}_{y+y'},
\]

Twisted group law on commutative variables.

\[
\hat{x}_y \rightarrow k_{y, (\gamma_h)}^{-\Omega(\gamma_h)} (\hat{x}_y)
\]

\[
k_{y, (\gamma_h)}(\hat{x}_y) = \hat{x}_y \left( 1 - \hat{x}_{\gamma_h} \right)^{\langle y, \gamma_h \rangle}
\]

Cluster transformation
VI. DEFORMED PRODUCT

WE CAN DEFINE A $y$-PRODUCT SUCH THAT

$$F(L_1 \circ_y L_2) = F(L_1) F(L_2)$$

$$(\mathcal{H}_{L_1(x,1), L_2(x,2)})_{y_0}$$

$$= \bigoplus_{y + y' = y_0} \mathcal{H}_{L_1, y} \otimes \mathcal{H}_{L_2, y'} \otimes N_{y, y'}$$

$$\text{Tr}_{N_{y_0, y_0}} y^2 J_3 = y^{<y, y'>}$$
WE HAVE NOW REVIEWED THE BASICS OF LINE OP'S.

REMAINDER OF THE TALK IS DEVOTED TO 3 RESULTS MENTIONED IN THE INTRO.

A. CONSEQUENCES OF THE STRONG POSITIVITY CONJECTURE

B. DARBOUX–COORDINATE EXPANSION

C. COMPUTATION OF $\langle L_5 \rangle$ ON $\mathbb{R}^3 \times S^1$ FOR $T_{g,n}[A_1]$ THEORIES
VIII. FORMAL LINE OP'S

WE NOW FORMALIZE THE NOTION OF LINE OPERATOR.

THIS IS A PURELY MATHEMATICAL CONSTRUCTION.

DATA

1. LOCAL SYSTEM $\Gamma \rightarrow \mathbb{B}$
2. $\langle \cdot , \cdot \rangle : \wedge^2 \Gamma \rightarrow \mathbb{Z}$
3. $\mathbb{Z} \in \text{Hom}(\Gamma, \mathbb{C})$
4. $\Omega(x;u;y)$ SATISFYING THE KSWCF
DIVIDE $\mathbb{B} \times \mathbb{C}^*$ INTO CHAMBERS 
BY WALLS $\hat{W}(Y)$ WITH $\Omega(Y; u) \neq 0$.

**DEF:** A STRONGLY POSITIVE FORMAL LINE OPERATOR IS A COLLECTION $F(c)$ ON CHAMBERS

- $F(c) = \sum_y P_y^c \cdot X_y$, \textit{finite sum},

  $P_y^c = $ TRUE SPIN CHARACTER

  \[ = \sum_{n \geq 0} a_n \left( \frac{y^n - y^{-n}}{y - y^{-1}} \right) \]

  \[ = \sum_{n \geq 0} a_n \left[ n \right]_y \quad a_n \in \mathbb{Z}_+ \]

- $F(c^+) = S_{\gamma_h} F(c^-) S_{\gamma_h}^{-1}$
Strong positivity is a nontrivial constraint:

\[ X_{\gamma_c} \rightarrow \overset{\uparrow}{\Theta}(X_{\gamma_h}) X_{\gamma_c} \overset{\uparrow}{\Theta}(X_{\gamma_h})^{-1} \]

\[
= \begin{cases} 
\sum_{j=0}^{N} c^{j}(N, \rho_N) X_{\gamma_c + j\gamma_h} & \langle \gamma_c, \gamma_h \rangle = -N < 0 \\
\sum_{j=0}^{\infty} (-1)^j c^{j}(S, \rho_N) X_{\gamma_c + j\gamma_h} & \langle \gamma_c, \gamma_h \rangle = +N > 0
\end{cases}
\]

Thus you can't just start with some arbitrary \( F(c) \) in some chamber and generate the rest.
EXAMPLE: $U(1)$ THEORY
WITH SINGLE HYPERMULTIPLE

$\mathcal{B} = \mathbb{C} - \{0\}, \quad \Gamma = \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2, \quad \langle \delta_1, \delta_2 \rangle = 1$

$$\Omega(\gamma) = \begin{cases} 
1 & \gamma = \pm \delta_2 \\
0 & \text{ELSE} 
\end{cases}$$

$Z_{\delta_2}(u) = u$

$\hat{W}(-\delta_2)$

$C_{-3}$ $C_{-2}$ $C_{-1}$ $C_0$ $C_1$ $C_2$ $C_3$

$-2\pi$ $-\pi$ $0$ $\pi$ $2\pi$
TRY TO DEFINE \( F_{p,q} \) BY

\[
F_{p,q}(c_0) = X_{p,q}
\]

FOR \( p > 0 \)

\[
F_{p,q}(c_1) = \sum_{j=0}^{p} \text{ch} \Lambda_{p,q}^j \cdot X_{p,q+j} = \sum_{j=0}^{p} \left( \begin{array}{c} p \\ j \end{array} \right) X_{p,q+j}
\]
But for \( p < 0 \)

\[
\hat{\Omega}(-\delta_2) \xrightarrow{\text{HALO}} \hat{\Omega}(\delta_2) \xrightarrow{\text{NO HALO}} \hat{\Omega}(-\delta_2)
\]

\[ X_{p,q} \]

\[
F_{p,q}(c_1) = \sum_{j=0}^{\infty} (-1)^j \chi_h(S^j \rho_p) X_{p,q+j} = X_{p,q} - [\Phi] X_{p,q+1, \pm} \ldots
\]

**Strong Positivity Fails**
WE CAN DETERMINE THE FULL ALGEBRA OF LINE OPERATORS

SIMPLE LINE OP's:

\[ F_{p_{1q}} = \begin{cases} 
X_{p_{1q}} & \text{in } C_0, \quad p \geq 0 \\
X_{p_{1q}} & \text{in } C_1, \quad p \leq 0 
\end{cases} \]

\[
F_{p_{1q}} F_{r_{1s}} = \begin{cases} 
\sum_{P,r} y^{ps-qr} F_{p+r_{1q}+s} & P,r \text{ SAME SIGN} \\
\sum_{P,r} y^{ps-qr} F_{p+r_{1q}+s} + \cdots & \text{ELSE} 
\end{cases}
\]

\[ F_{1_{10}} F_{-1_{10}} = 1 + y F_{0_{11}} \]
SIMPLE GENERALIZATION

\[ \Omega(\gamma) = \begin{cases} 1 & \gamma = \pm Q \gamma_2, \ Q > 1 \\ 0 & \text{else} \end{cases} \]

\[ F_{1,0} F_{-1,0} = \sum_{j=0}^{Q} \left[ \begin{array}{c} Q \\ j \end{array} \right] y^{Q_j} F_{0, Q_j} \]

\exists \text{ similar examples for AD } \frac{1}{3} {\text{SU(2)}} \text{ theories...}

CONJECTURE:

THE ALGEBRA OF FORMAL LINE OPERATORS IS CANONICALLY ISOMORPHIC TO THE ALGEBRA OF TRUE LINE OPERATORS
On $\mathbb{R}^3 \times S^1_R$ we have a loop operator and

$$\langle L_5 \rangle = \text{function on moduli of } d=3 \text{ vacua } \mu \rightarrow \mathcal{B}$$

$$\langle L_5 \rangle = \sum_{\gamma} \Omega(L_5, \gamma) \chi_{\gamma}$$

2: $\exists$ several moduli spaces $\mathcal{M}_x$ with isogenous fibers.
"Proof"

1. SUSY $\Rightarrow$ $\langle L_5 \rangle$ Holomorphic Function on $M^5$

2. $R \to \infty$ Asymptotics:

\[
\langle L_5 \rangle = \text{Tr} \left( (-1)^F e^{-2\pi RH + i\Theta \cdot Q} \right) \mathcal{H}_{L_5} \sigma(Q)
\]

$\Theta \sim$ Boundary cond's on (elec, mag)
Wilson lines $\in \text{Hom}(T_1, \mathbb{R}/2\pi \mathbb{Z})$

$Q \sim$ Charge operator

$\sigma \sim$ Q.R. of $(-1)^{\langle \gamma, \gamma' \rangle}$ due to self-duality

\[
\left( e^{i\hat{\Theta} \cdot \gamma} = e^{i\Theta \cdot \gamma} \right) \sigma(\gamma) \text{ is canonical, but twisted}
\]
$R \to \infty$ projects to framed BPS states

$$\langle L_5 \rangle \sim \sum_{\gamma} \overline{\Omega} (L_5, \gamma) \gamma_{\gamma}^{sf}$$

$$\gamma_{\gamma}^{sf} = e^{\pi R S' Z_\gamma + i \theta \cdot \gamma + \pi R S \overline{Z}_\gamma} \sigma(\gamma)$$

So we make the ansatz

$$\langle L_5 \rangle = \sum_{\gamma} \overline{\Omega} (L_5, \gamma) \tilde{\gamma}_{\gamma}$$

$\tilde{\gamma}_{\gamma}$ holomorphic with

$$\tilde{\gamma}_{\gamma} \sim \gamma_{\gamma}^{sf} \quad R \to \infty$$

(also $S \to 0, \infty$)
3. WALL-CROSSING

\[ \overline{\Omega} (L_s, \gamma) \] HAS WALL-CROSSING:

\[ F_{\text{cl}} = \sum \overline{\Omega} (L_s, \gamma) \hat{x}_\gamma \]

TRANSFORMS BY

\[ \hat{x}_\gamma \rightarrow \hat{x}_\gamma (1 - \hat{x}_{\gamma_{\text{th}}}) \]

\[ = K_{\gamma_{\text{th}}}^{-\Omega(\gamma_{\text{th}})} (\hat{x}_\gamma) \]

BUT \( <L_s> \) SHOULD HAVE NO WALL-CROSSING:

THERE IS NO PHASE TRANSITION IN THE UV THEORY.

\[ \Rightarrow \tilde{y}_\gamma \] TRANSFORM LIKE \( y_\gamma \)

\[ \Rightarrow \tilde{y}_\gamma = y_\gamma \]
• Thus Darboux coord's $y^i$ can be interpreted as "Ir line operator vev's," or chiral ring operators in the 3D $\mathcal{G}$-model $\mathbb{R}^3 \rightarrow M$

• Moreover the n.c. algebra of line operators defines a deformation of the algebra of functions on $M$

? Relation to algebra $\mathfrak{hebb}$ of coisotropic branes of Kapustin-Witten $\&$ Nekrasov-Witten?
EXAMPLE: U(1) THEORY

\[ U = Y_{0,1} = \exp \left( \frac{\pi R}{5} a + i \hat{\Theta}_e + \pi R s \alpha \right) \]

\[ V_+ = Y_{1,0} \quad \text{Im} \frac{Z_{r_2}}{5} > 0 \]

\[ V_- = Y_{-1,0} \quad \text{Im} \frac{Z_{r_2}}{5} < 0 \]

\[ V_+ V_- = 1 - U \]

COMPARE WITH

\[ F_{1,0} F_{-1,0} = 1 + y F_{0,1} \]

@ \ y = -1 \ 
!!
$U(1) \text{ with } Q > 1$

$V_+ = Y_{1,0} \quad V_- = Y_{-1,0}$

$V_+ V_- = (1 + (-u)^Q)^Q$

COMPARE WITH

$F_{1,0} F_{-1,0} = \sum_{j=0}^{Q} \left[ \begin{array}{c} Q \\ j \end{array} \right] y^{Qj} F_{0, Qj}$
IX. Computing $\langle L_5 \rangle$

in $T_{g,n}[A_1]$ theories

$T_{g,n}[\mathfrak{g}] := (2,0)$ $\mathfrak{g}$-theory

with twist on $C \times \mathbb{R}^{1,3}$

There exist SUSY surface operators in 6D: $\mathcal{S}(\mathfrak{g}, \Sigma)$

$\mathcal{R} : \mathfrak{g}$ rep, $\Sigma :$ surface
1. ON THE COULOMB BRANCH ~

\[ \sum_{\nu \in W^+ (R)} \exp \left\{ 2\pi i \left\langle \nu, \int_{\Sigma} B + k n^v Y^v \frac{\text{vol}(\Sigma)}{\Sigma} \right\rangle \right\} \]

2. REDUCED ON $S^1$

\[ \text{Tr}_{\mathcal{R}} P \exp \left\{ 2\pi i \int_{\mathcal{P}} (A + k n^v Y^v) \, ds \right\} \]

\[ \mathcal{P} \subset \mathcal{R}^5 \]
For $T_{g,n} [y]$ theories take

$$\Sigma = \mathbb{R} \times \gamma$$

$\gamma$ = closed, nonintersecting curve on $C$

$\mathcal{S}(\mathbb{R}, \Sigma)$ descends to a line operator of type $\mathcal{S}$

$\gamma \sim \text{angle between } n^x \quad \xi \quad \rho$
IN THIS WAY WE CAN CONSTRUCT SIMPLE LINE OPERATORS

\[ L_5 (\alpha, \beta) \]

\( \sim \) ISOTOPY CLASS OF CLOSED CURVE IN \( \mathbb{C} \)

NOW CONSIDER \( T_{9, m} [e^y] / \mathbb{R}^3 \times S^1 \)

AND THE VEV'S

\[ \langle L_5 \rangle = \sum \Omega (L_5, \gamma) Y_\gamma \]
Recall:

$(2,0) \ 1 \text{-Theory} / \mathbb{R}^3 \times S_L^1 \times C$

$\mathcal{T}_{g,m} [\mathfrak{g}] / \mathbb{R}^3 \times S_R^1$

$5 \text{D } \mathfrak{g}-\text{SYM} / \mathbb{R}^3 \times C$

$\sigma\text{-Model}$

$\mathbb{R}^3 \rightarrow \mathcal{M}$

$\mathcal{M} (\mathbb{R}^3 \times S_R^1 \times C) = \text{Hitchin Moduli Space}$
\( \mathcal{M}^S = \text{moduli of flat connection} \)

\[ A_S = R \frac{\varphi}{S} + A + R S \bar{\varphi} \]

\[ \langle S(R, S_R \times \mathcal{P}) \rangle \]

\[ \text{C} \]

\[ \text{s}^1 \]

\[ \langle L_S(\mathcal{P}, \mathcal{P}) \rangle \]

\[ \text{Tr}_R \text{Pexp} \int A_S \]

\[ \langle L_S(\mathcal{P}, \mathcal{P}) \rangle = \text{Tr}_R \text{Hol}(\varphi, A_S) \]
For \( \gamma = A \), the holonomy is computable in terms of FG coord's via "traffic rules".

\[
\mathbf{R}(E) = \epsilon(E) \begin{pmatrix} \sqrt{y_E} & \sqrt{y_E} \\ 0 & -\frac{i}{\sqrt{y_E}} \end{pmatrix}
\]

\[
\mathbf{L}(E) = \epsilon(E) \begin{pmatrix} -\sqrt{y_E} & 0 \\ \frac{1}{\sqrt{y_E}} & \frac{1}{\sqrt{y_E}} \end{pmatrix}
\]

\[
\epsilon(E) = \pm i
\]
**KEY EXAMPLE: WILSON LOOP**

\[
\langle L_g (z, \varrho) \rangle = \sqrt{y_{\varphi_1} y_{\varphi_2}} + \frac{1}{\sqrt{y_{\varphi_1} y_{\varphi_2}}} + \sqrt{\frac{y_{\varphi_1}}{y_{\varphi_2}}}
\]

\text{NAIVE} \quad \text{SURPRISE}!
\[ \langle L_\sigma(2, \rho) \rangle = \sqrt{y_\sigma_1 y_\sigma_2} + \frac{1}{\sqrt{y_\sigma_1 y_\sigma_2}} + \sqrt{\frac{y_\sigma_1}{y_\sigma_2}} \]

\[ \Rightarrow \]

BY STRONG POSITIVITY THIS CAN BE PROMOTED TO A GENERATOR OF PROTECTED SPIN CHARACTERS!

\[ F(L_\sigma) = x_{\frac{1}{2}(\sigma_1 + \sigma_2)} + x_{-\frac{1}{2}(\sigma_1 + \sigma_2)} + x_{\frac{1}{2}(\sigma_1 - \sigma_2)} \]
ALGORITHM TO COMPUTE $F(L_5(\mathcal{R}, \mathcal{P}))$

- $u, s$ DETERMINES A WKB TRIANGULATION OF $\mathcal{C}$

- NBD. OF $\mathcal{P}$ IS AN ANNULUS. SO CHOOSE A TRIANGULATION OF $\mathcal{C}$ SUCH THAT:

- THEN FLIP USING THE QUANTUM DILOG.
Remark: This algorithm appears to coincide with the prescription by Teschner for computing quantum geodesic lengths in Fock coordinates in the quantization of Teichmüller space.
X. SUMMARY \& OPEN PROBLEMS

S1. We defined line op's \( L_5 \) and their protected spin char's. They satisfy the KSWCF and define a deformation of the algebra of functions on \( M \).

S2. We defined strongly positive formal line operators - conjecturally isomorphic to line op's of \( \mathcal{N}=2 \).

S3. We described concrete algorithms for computing the framed BPS degen's \& PSC's in \( T_{g,n} [A_1] - \text{theories} \).
01. Clarify relation to quantization of Teichmüller and Hitchin moduli

02. Clarify relation to "canonical bases" and "total positivity" (Lusztig, Fomin-Zelevinsky, ....)

03. How to compute at higher rank?

04. Quantum Dilog also appears in noncompact CSW-theory

05. Extension to sugra

06. Geometry of monopole moduli space

07. Applications to surface OP's and 2D/4D W.C.F.