The u-Plane Integral As A Tool In The Theory Of Four-Manifolds

Gregory Moore
Rutgers University

SCGP, April 26, 2017
Introduction

1. Lightning Summary: The Physical Derivation Of Witten’s Conjecture Relating Donaldson & SW
2. Brief Overview Of The World Of N=2 Theories
3. Donaldson-Witten Partition Function For General Compact Simple Lie Group
4. Theories With Matter & Superconformal Simple Type
5. Two Possible Future Directions
Introduction

Most of this talk reviews work done around 1997-1998:

Moore & Witten  Marino & Moore  Marino, Moore, & Peradze

Overlapping work: Losev, Nekrasov, & Shatashvili

Central Question: Given the successful application of N=2 SYM for SU(2) to the theory of 4-manifold invariants, are there interesting applications of OTHER N=2 field theories?

Recently re-visited with Iurii Nidaiev
1 Introduction

2 Lightning Summary: The Physical Derivation Of Witten’s Conjecture Relating Donaldson & SW

3 Brief Overview Of The World Of N=2 Theories

4 Donaldson-Witten Partition Function For General Compact Simple Lie Group

5 Theories With Matter & Superconformal Simple Type

6 Two Possible Future Directions
Review: Derivation Of Witten Conjecture From SU(2) SYM

\(X\): Smooth, compact, \(\partial X = \emptyset\), oriented, \((\pi_1(X) = 0)\)

Twisted N=2 SYM on \(X\) for simple Lie group \(G\): Sum over connections \(A \in \mathcal{A}(P)\) on all \(G_{adj}\) bundles \(P \to X\) with fixed 't Hooft flux \(\xi \in H^2\left(X; \pi_1(G_{adj})\right)\) together with various fields valued in \(\text{ad } P\):

\[
\phi \in \Omega^0(\text{ad}P \otimes \mathbb{C}) \quad \chi \in \Pi\Omega^{2+}(\text{ad}P) \quad \eta \in \Pi\Omega^0(\text{ad}P) \quad \psi \in \Pi\Omega^1(\text{ad}P)
\]

Formally: Correlation functions of Q-invariant operators localize to integrals over the finite-dimensional moduli spaces of G-ASD conn’s.

Witten’s proposal: For \(G = SU(2)\) correlation functions of Q-invariant operators are the Donaldson polynomials.
Local Observables

\[ U \in \text{Inv}(g) \Rightarrow U(\phi) \]

\[ H_*(X, \mathbb{Z}) \xrightarrow{\text{Descent formalism}} H^*(\text{Fieldspace}; Q) \]

\[ g = \mathfrak{su}(2) \quad U = \text{Tr}_2 \left( \frac{\phi^2}{8\pi^2} \right) \quad U(S) \sim \int_S \text{Tr}(\phi F + \psi^2) \]
Donaldson-Witten Partition Function

\[ Z_{DW}^\xi(p, s) = \langle e^{2p \, U + s^a U(S_a)} \rangle_{\Lambda} \]

\[ = \Lambda^{-\frac{3}{4}(\chi + \sigma)} \sum \frac{(\Lambda^2 p)^\ell (\Lambda s)^r}{\ell! \, r!} \vartheta_D(p t^\ell S^r) \]

Mathai-Quillen & Atiyah-Jeffrey:
Path integral formally localizes:

\[ \mathcal{M} \leftrightarrow \mathcal{A}/\mathcal{G} \]

Strategy: Evaluate in LEET:
Integrate over vacua on \( \mathbb{R}^4 \)
Spontaneous Symmetry Breaking

\( SU(2) \rightarrow U(1) \) by vev of adjoint Higgs field \( \phi \):

Order parameter:

\[ u = \langle U(\phi) \rangle \]

Coulomb branch: \( \mathcal{B} = t \otimes \mathbb{C}/\mathcal{W} \cong \mathbb{C} \)

\( adP \rightarrow L^2 \oplus \mathcal{O} \oplus L^{-2} \)

Photon: Connection \( A \) on \( L \)

\( U(1) \) VM: \( (a^q, A, \chi, \psi, \eta) \)

\( a^q \): complex scalar field on \( \mathbb{R}^4 \):

Do path integral of quantum fluctuations around

\[ a^q(x) = a + \delta a(x) \]

What is the relation of \( a = \langle a^q(x) \rangle \) to \( u \)?

What are the couplings in the LEET for the U(1) VM?
LEET: Constraints of N=2 SUSY

General result on N=2 abelian gauge theory with Lie algebra $\mathfrak{t} \cong \mathfrak{u}(1) \oplus \cdots \oplus \mathfrak{u}(1)$: Action determined by a family of Abelian varieties and an "N=2 central charge function":

$$\mathfrak{A} \rightarrow \mathfrak{t} \otimes \mathbb{C} - \mathcal{D} \quad \Gamma := H_1(\mathfrak{A}; \mathbb{Z})$$

$$Z: \Gamma \rightarrow \mathbb{C} \quad \langle dZ, dZ \rangle = 0$$

Duality Frame: $\Gamma \cong \Gamma^{\text{electric}} \oplus \Gamma^{\text{magnetic}}$

$$a^I = Z(\alpha^I) \quad a_{D,I} := Z(\beta_I) = \left( \frac{\partial F}{\partial a^I} \right) \quad \tau_{IJ} := \frac{\partial a_{D,I}}{\partial a^J}$$

Action $\sim \int_X \bar{\tau}(F^+)^2 + \tau(F^-)^2 + \text{d}a^q \ast (\text{Im} \, \tau) d\bar{a}^q + \cdots$
Seiberg-Witten Theory:

For $G=SU(2)$ SYM, $\mathcal{A}$ is a family of elliptic curves:

$$E_u: \quad y^2 = x^2(x-u) + \frac{\Lambda^4}{4} x \quad u \in \mathbb{C}$$

$$Z(\gamma) = \oint_y \lambda \quad \lambda = \frac{dx}{y} (x-u)$$

$u \to \infty$: Invariant cycle $A$:

$$a(u) = \oint_A \lambda$$

Choose B-cycle: $\Rightarrow \tau(a) \Rightarrow$ Action for LEET

LEET breaks down at $u = \pm \Lambda^2$ where $Im(\tau) \to 0$
LEET breaks down because there are new massless fields associated to BPS states

\[ u = -\Lambda^2 \quad u = \Lambda^2 \]

\[ U(1)_D \text{ VM: } (a_D, A_D, \chi_D, \psi_D, \eta_D) \]

Near \( \mathcal{U}_{\Lambda^2} \):

\[ + \]

Charge 1 HM: \( (M = q \oplus \tilde{q}^*, \cdots) \)

\[ Z_{DW}^{\xi}(p, s) = Z_u + Z_{\Lambda^2} + Z_{-\Lambda^2} \]
u-Plane Integral $Z_u$

Can be computed explicitly from QFT of LEET

Vanishes if $b_2^+ > 1$

$$Z_u = \int da \ d\bar{a} \ \left( \frac{du}{da} \right)^{\frac{\chi}{2}} \Delta_8^{\sigma} \ e^{2pu + S^2 T(u)}$$

$$\Delta = (u - \Lambda^2)(u + \Lambda^2)$$

Contact term: $T(u) = \left( \frac{du}{da} \right)^2 E_2(\tau) - 8 \ u$

$\Theta$: Sum over line bundles for the U(1) photon.
Photon Theta Function

\[ \Theta = e^{y^{-1} \left( \frac{du}{da} \right)^2 S_+^2} \sum_{\lambda = \lambda_0 + H^2(X, \mathbb{Z})} y^{-\frac{1}{2}} e^{-i \pi \lambda^2 - i \pi \lambda_0^2} \]

\[ (-1)^{w_2(X) \cdot (\lambda - \lambda_0)} e^{-i \left( \frac{du}{da} \right) S \cdot \lambda} \left( \frac{d\tau}{d\bar{\alpha}} \right) (\lambda_+ + \frac{1}{4\pi y} S_+ \left( \frac{du}{da} \right)) \]

\[ \tau = x + i \ y \]

\[ 2\lambda_0 \text{ is an integral lift of } \xi = w_2(P) \]

Metric dependent! \[ \lambda = \lambda_+ + \lambda_- \]
Contributions From $\mathcal{U}_\Lambda^2$

Path integral for $U(1)_D$ VM + HM:

General considerations imply:

$$\sum_{\lambda \in \frac{1}{2} w_2(X)+H^2(X,\mathbb{Z})} SW(\lambda) e^{2\pi i \lambda \cdot \lambda_0} R_\lambda(p,S)$$

$$R_\lambda(p,S) = \text{Res} \left[ \left( \frac{da_D}{1+\frac{d(\lambda)}{2}} \right)^{2pu+S^2 T(u)+i \left( \frac{du}{da_D} \right)^{S \cdot \lambda}} C(u)^{\lambda^2} P(u)^\sigma E(u)^\chi \right]$$

$$d(\lambda) = \frac{(2\lambda)^2 - c_1^2}{4} \quad u = \Lambda^2 + \text{Series } a_D$$

$$c_1^2 = 2\chi + 3 \sigma$$

Deriving $C, P, E$ From Wall-Crossing

$$\frac{d}{dg_{\mu\nu}} Z_u = \int \text{Tot deriv} = \int_{-\Lambda^2} du(....) + \int_{\Lambda^2} du(....) + \int_{\infty} du(....)$$

$Z_u$ piecewise constant: Discontinuous jumps across walls:

$$\Delta_{\infty} Z_u: \quad W(\lambda): \quad \lambda_+ = 0 \quad \lambda = \lambda_0 + H^2(X, \mathbb{Z})$$

Precisely matches formula of Göttsche!

$$\Delta_{\pm \Lambda^2} Z_u: \quad W(\lambda): \quad \lambda_+ = 0 \quad \lambda = \frac{1}{2} w_2(X) + H^2(X, \mathbb{Z})$$

$$\Delta_{\Lambda^2} Z_u + \Delta Z_{\Lambda^2} = 0 \Rightarrow \quad C(u), P(u), E(u)$$
\[ Z_{DW} = \langle e^{pO+I(S)} \rangle_{\text{micro}} = Z_{\text{Coulomb}} + Z_{\text{Higgs}} = Z_{u} + Z_{SW} \]

Donaldson polynomials do not jump at SW walls \(\Rightarrow\)

\[ 0 = \delta Z_{DW} = \delta Z_{\text{Coulomb}} + \delta Z_{\text{Higgs}} \]

\[ \mathcal{M}_{SW}(\lambda) = \{(A^D_{\mu}, M_\alpha) : F_+(A^D) = \bar{M}M, \bar{\nabla}M = 0\} \]
Witten Conjecture

Now, with $C, P, E$ known one takes $b_2^+(X) > 1$ and SWST to recover the Witten conjecture:

$$Z_{DW}^\xi(p, s) = 2^{c^2 - \chi_h} \left( e^{\frac{1}{2}S^2 + 2p} \sum_\lambda SW(\lambda)e^{2\pi i \lambda \cdot \lambda_0} e^{2S \cdot \lambda} + e^{-\frac{1}{2}S^2 - 2p} \sum_\lambda SW(\lambda)e^{2\pi i \lambda \cdot \lambda_0} e^{-2iS \cdot \lambda} \right)$$

$$\chi_h = \frac{\chi + \sigma}{4} \quad c^2 = 2\chi + 3\sigma$$
1. Introduction

2. Lightning Summary: The Physical Derivation Of Witten’s Conjecture Relating Donaldson & SW

3. Brief Overview Of The World Of N=2 Theories

4. Donaldson-Witten Partition Function For General Compact Simple Lie Group

5. Theories With Matter & Superconformal Simple Type

6. Two Possible Future Directions
N=2 Theories

Lagrangian theories: Compact Lie group $G$, quaternionic representation $\mathcal{R}$ with $G$-invariant metric,

$$\tau_0 \in \prod_{\text{simple factors}} \mathcal{H} \quad m \in \text{Lie}(G_f) \quad G_f = Z(G) \subset O(\mathcal{R})$$

Class S: Theories associated to Hitchin systems on Riemann surfaces.

Superconformal theories

Couple to N=2 supergravity
1. Introduction

2. Lightning Summary: The Physical Derivation Of Witten’s Conjecture Relating Donaldson & SW

3. Brief Overview Of The World Of N=2 Theories

4. Donaldson-Witten Partition Function For General Compact Simple Lie Group

5. Theories With Matter & Superconformal Simple Type

6. Two Possible Future Directions
G-Donaldson Invariants

Pure VM theory for $G$ a compact simple Lie group of rank $r$

$0,2$ observables derived from independent invariant polynomials $U, V \in Inv(g)$

$$V(S) \sim \int_S V_{ab} \left( \phi^a F^b + \psi^a \psi^b \right)$$

$$Z_{DW}^{G, \xi}(U, V(S)) = \langle e^{U+V(S)} \rangle_{\Lambda}$$

Formally the path integral localizes to $G$-ASD moduli space

Generating function of generalization of Donaldson polynomials for any $G$. 

Rigorous setup: Kronheimer & Mrowka
LEET On Coulomb Branch

Coulomb branch: $\mathcal{B} = \mathfrak{t} \otimes \mathbb{C} \sslash \mathcal{W}$

SSB: $G \rightarrow T \Rightarrow$ Abelian VM’s valued in $T$

Example of SW Geometry: $G=SU(N)$

$\Sigma_u$: $y^2 = P(x)^2 - \Lambda^{2N}$ \hspace{1cm} $P(x) = x^N + u_2 x^{N-2} + \cdots + u_N$

$\mathfrak{A}_u = \text{Jac}(\Sigma_u)$ \hspace{1cm} $\Gamma_u = H_1(\Sigma_u; \mathbb{Z})$

$Z(\gamma) = \oint_{\gamma} \lambda$ \hspace{1cm} $\lambda = x \, d \log \frac{y + P}{y - P}$
u-Plane Integral

Can compute u-plane integral explicitly from QFT:

\[ \text{∃ almost canonical duality frame in weak-coupling region at } \alpha \]
\[ \implies t - \text{valued VM: } (a, A, \chi, \eta, \psi) \]

\[ Z_u = \int_{t_c}^{\infty} [da] A^\chi B^\sigma e^{U + S^2TV} \Theta \]

\[ A = \alpha \left( \text{Det} \left( \frac{\partial u^I}{\partial a^J} \right) \right)^{\frac{1}{2}} \]
\[ B = \beta \Delta^\frac{1}{8} \]

\( \Delta: \) Holomorphic function vanishing along ``discriminant locus’’ \( \mathcal{D} \)

\( T_V: \) Contact term. General theory Losev-Nekrasov-Shatashvili; Edelstein, Gomez-Reino, Marino

For quadratic Casimir: \( T_{u_2} \sim 2 \ u_2 \sim a^I \frac{\partial u_2}{\partial a^I} \)
Theta Function

$\Theta$: Theta function for abelian gauge fields remaining after SSB

Reduction of structure group $G_{adj} \rightarrow T$

Classes for fluxes in a torsor for $H^2(X; \Lambda_{wt}(g)) \cong \Lambda_{wt}(g) \otimes H^2(X; \mathbb{Z})$

$$[F] = 4 \pi \lambda \quad \lambda \in \Lambda_{wt}(g) \otimes H^2(X; \mathbb{Z}) + \lambda_0 := \Lambda$$

$$\Theta \sim \sum_{\lambda \in \Lambda} e^{-i \pi \lambda_+ \bar{\tau} \lambda_+ - i \pi \lambda_- \bar{\tau} \lambda_-} e^{i \pi (\lambda - \lambda_0) \cdot \rho \otimes w_2(X)} e^{i \frac{\partial V}{\partial a^I} \lambda_+^I \cdot S}$$

Metric dependent $\Rightarrow$ Possible wall-crossing
Discriminant Locus

Just as in rank 1, the integrand is singular along a ``discriminant locus'' $\mathcal{D}$ where BPS states become massless and some $U(1) \subset T$ becomes strongly coupled.

$$\mathcal{D} = \bigcup_i \mathcal{D}_i$$

$\mathcal{D}_i$ generalizes $u = \pm \Lambda^2$

Higher rank: complicated intersections where multiple BPS states become massless, i.e. multiple periods of the curve vanish.

Integral $Z_u$ must be regularized by cutting out tubular regions around $\mathcal{D}_i$
General Form Of $Z_{DW}$

$$Z_{DW}^{G, \xi}(p, s) = Z_u + \sum_i Z_{D_i} + \sum_{i \neq j} Z_{D_{ij}} + \cdots + Z_{mx}$$

Cancelling wall-crossing inductively determines $Z_{D_i}$ from $Z_u$ and $Z_{D_{ij}}$ from $Z_{D_i}$, etc.

All $Z_{D_{i_1 \cdots i_k}}$ vanish for $b_2^+ > 1$ EXCEPT $Z_{mx}$

So for $b_2^+ > 1$ the answer is given entirely by $Z_{mx}$
The ``N=1 Vacua''

\( \mathcal{D}_{mx} \) contains \( h = h^\vee(G) \) isolated points \( \nu_\alpha \) permutated by spontan. broken \( \mathbb{Z}/h\mathbb{Z} \) R-symmetry

\[ b_2^+ > 1 \quad Z_{DW}^{G,\xi}(U, V) = \sum_{\nu_\alpha} Z_{DW}^{G,\xi}(U, V; \nu_\alpha) \]

In principle, other maximal degenerations – corresponding to superconformal points- might have contributed.

But detailed analysis shows they do not for \( G=SU(3) \) and it is natural to conjecture that this is the case for all \( G \).
Analog Of Witten Conjecture

In duality frame where max degeneration is $a_D^l = 0$ the $\Theta$ function is a sum over

$$\lambda \in \frac{1}{2} \rho \otimes w_2(X) + \Lambda_{wt}(g) \otimes H^2(X; \mathbb{Z})$$

$r$ independent spin-c structures: $f_I \in \frac{1}{2} w_2(X) + H^2(X; \mathbb{Z})$

$$SW(\lambda) := \prod_I SW(f_I)$$

$$Z_{DW}^{G, \xi}(U, V; \nu) = e^{i \theta_a} \sum_{\lambda} e^{(2 \pi i \lambda \cdot \lambda_0)} SW(\lambda) R_\lambda(U; V)$$

$$R_\lambda(U, V) = \text{Res}[ (da_D^1 \wedge \cdots \wedge da_D^r) / \prod (a_D^l)^{1+\frac{d(f_I)}{2}} \epsilon(a_D^l) e^{Us+S^2Tv+i \frac{\partial V}{\partial a_D^l} \cdot f_I}$$

All computable from the degenerate curve and its first order variation.
Thus we can derive the $SU(N)$ Donaldson invariants.

Corollary: $X$ of simple type, $b_1 = 0, b_2^+ > 1$:

Only the $\mathcal{N} = 1$ points contribute. Local analysis near $\mathcal{N} = 1$ points $\Rightarrow$

$$
\langle e^{U+I_2(S)} \rangle_{SU(N)} = \tilde{\alpha}_N^\chi \tilde{\beta}_N^\sigma \sum_{k=0}^{N-1} \sum_{\lambda^I} \omega^k (N^2-1)^\delta \left( \prod_{I=1}^{N-1} SW(\lambda^I) \right)
$$

$$
\cdot \exp \left[ \sum_{s=1}^{\left[ \frac{N-1}{2} \right]} p_{2s} \omega^{2ks} u_{2s} + 2\omega^{2k} S^2 + 4\omega^k \sum_{I=1}^{N-1} (S, \lambda^I) \sin \frac{\pi I}{N} \right]
$$

$$
\omega = \exp[i\pi/N] \quad \delta = (\chi + \sigma)/4 \quad u_{2s} = 4^s \binom{2s}{s}_N
$$

The sum $\sum_{\lambda^I}$ is over the finite set of SW classes with: $4\lambda^2 = 3\chi + 2\sigma$
1. Introduction

2. Lightning Summary: The Physical Derivation Of Witten’s Conjecture Relating Donaldson & SW

3. Brief Overview Of The World Of N=2 Theories

4. Donaldson-Witten Partition Function For General Compact Simple Lie Group

5. Theories With Matter & Superconformal Simple Type

6. Two Possible Future Directions
Including Matter

Now consider the general Lagrangian theory:

Data: \( G, R, m, \Lambda \) or \( q = e^{2\pi i \tau} \)

Twisted N=2 theory is again of MQ form:

Localize on moduli space of generalized monopole equations.

Q: Differential for \( G_f \)-equivariant cohomology with parameters \( m \in \text{Lie}(G_f) \)

Labastida & Marino; Losev-Nekrasov-Shatashvili

\[
Z(U, V(S)) = \sum_{\ell, r} \frac{1}{\ell! r!} \int_{\mathcal{M}} \omega^\ell_U \omega^r_{V(S)} \text{Eul}(\text{Cok}(\mathbb{F}))
\]
SU(2) With Fundamental Hypers

Moore & Witten

\[ \mathcal{R} = N_{fl}(2 \oplus 2^*) \quad Spin(2N_{fl}) \subset G_f \]

Mass parameters \( m_f \in \mathbb{C}, f = 1, ..., N_{fl} \)

Must take \( \xi = w_2(X) \)

Seiberg-Witten: \( E_u : y^2 = x^3 + a_2 x^2 + a_4 x + a_6 \)

\( a_k \): Polynomials in \( \Lambda, m_f, u \)

\( Z_u \) has exactly the same expression as before but now, e.g. \( da/du \) depends on \( m_f \)

New ingredient: \( \mathcal{D} \) has \( 2 + N_{fl} \) points \( u_i : \Delta = \prod_i (u - u_i) \)
Analog Of Witten Conjecture

\[
 b_2^+ > 1 \quad Z(p; s; m_f) = \sum_{j=1}^{2+N_f l} Z(p, s; m_f; u_j)
\]

\[
 Z(p, s; m_f; u_j) = \bar{\alpha}^\chi \tilde{\beta}^\sigma \sum_{\lambda} SW(\lambda) e^{2\pi i \lambda \cdot \lambda_0} R_j(p, s)
\]

\( X \) is SWST \( \Rightarrow \)

\[
 R_j(p, s) = \kappa_j^{h^+} (\frac{du}{da})^{\chi h + \sigma} \exp \left( 2 p u_j + S^2 T(u_j) - i \left( \frac{du}{da} \right)_j \cdot \lambda \right)
\]

\[
 u = u_j + \kappa_j q_j + O(q_j^2)
\]

Everything computable explicitly as functions of the masses from first order degeneration of the SW curve.
Superconformal Points

Consider $N_{fl} = 1$. At a critical point $m = m_*$ two singularities $u_{\pm}$ collide at $u = u_*$ and the SW curve becomes a cusp: $y^2 = x^3$ [Argyres, Plesser, Seiberg, Witten]

Two mutually nonlocal BPS states have vanishing mass:

\[
\int_{\gamma_1} \lambda \to 0 \quad \int_{\gamma_2} \lambda \to 0 \quad \gamma_1 \cdot \gamma_2 \neq 0
\]

Physically: No local Lagrangian for the LEET:
Signals a nontrivial superconformal field theory.

\[
m = m_* + z
\]
Superconformal Simple Type – 1/2

Analyze contributions at the two colliding points $u_{\pm}$

$$R_j(p, s) = \kappa_j x_n \left( \frac{du}{da} \right)^{x_n + \sigma} \exp \left( 2p u_j + S^2 T(u_j) - i \left( \frac{du}{da} \right)_j S \cdot \lambda \right)$$

$$= \text{const.} \ e^{2pu_* + S^2 T(u_*)} e^{i \theta_{\pm}} \frac{c^2 - x_n}{2} \left( 1 + \text{Series in } z^{\frac{1}{2}} \right)$$

$$\exp \left( e^{i \theta_{\pm}} z^{\frac{1}{4}} \left( 1 + \text{Series in } z^{\frac{1}{2}} \right) S \cdot \lambda \right)$$

$$\frac{c^2 - x_n}{2} = \frac{7 \chi + 11 \sigma}{8} < 0 \quad \text{Perfectly reasonable!}$$

Physics: $\lim_{z \to 0} Z_{DW} \ (p, s; m_* + z) < \infty$
Superconformal Simple Type – 2/2

Physics: \[ \lim_{z \to 0} Z_{DW} (p, s; m_* + z) < \infty \]

No IR divergences on X  No noncompact moduli spaces of vacua

Form of explicit answer implies the only way this can hold for all polynomials in pnt and S is for a series expansion in z with coefficients made from \( SW(\lambda) \) to be regular

Theorem [MMP]: There is no divergence in \( Z_{DW} \) if:

a.) \[ \chi_h - c^2 - 3 \leq 0 \]

b.) \[ \sum_\lambda SW(\lambda) e^{2\pi i \lambda \cdot \lambda_0} \lambda^k = 0 \quad 0 \leq k \leq \chi_h - c^2 - 4 \]

Conditions a,b define SST.

MMP checked that all known (c. 1998) 4-folds with \( b_2^+ > 1 \) are SST.
1. Introduction

2. Lightning Summary: The Physical Derivation Of Witten’s Conjecture Relating Donaldson & SW

3. Brief Overview Of The World Of N=2 Theories

4. Donaldson-Witten Partition Function For General Compact Simple Lie Group

5. Theories With Matter & Superconformal Simple Type

6. Two Possible Future Directions
Two Possible Future Directions

Invariants for families of 4-manifolds

New invariants
(or new facts about old invariants) from superconformal theories??
Families Of Four-Manifolds – 1/5

Donaldson invariants can be generalized to families of four-manifolds: Donaldson, Durham lectures 1989

Naïve attempt at a physical approach:

Couple N=2 field theory to N=2 supergravity:

\[ g_{\mu \nu}, \psi^A_{\mu \alpha}, \bar{\psi}^A_{\mu \dot{\alpha}}, \ldots \]

Topological twist: \[ \Rightarrow g_{\mu \nu}, \Psi_{\mu \nu}, \phi^\mu, \ldots \]

\[ Qg_{\mu \nu} = \Psi_{\mu \nu}, Q\Psi_{\mu \nu} = D_\mu \phi_\nu + D_\nu \phi_\mu, Q\phi^\mu = 0, \ldots \]

Superfields describe (Cartan model) for \( \text{diff}(X) \) — equivariant cohomology of \( \Omega^*(\text{Met}(X)) \)
Families Of Four-Manifolds – 2/5

\[ S = \{ Q, V \} + \text{const} \int tr F \wedge F \]

\[ T_{\mu\nu} = \{ Q, \Lambda_{\mu\nu} \} \quad D^\mu \Lambda_{\mu\nu} = \{ Q, Z_{\nu} \} \]

\[ Q \left( S + \int_X \text{vol}(g) \Psi^{\mu\nu} \Lambda_{\mu\nu} + \text{vol}(g) \phi^\mu Z_\mu \right) = 0 \]

(For a fixed volume form \( \text{vol}(g) \).)
Families Of Four-Manifolds – 3/5

\[ Z[g_{\mu\nu}, \Psi_{\mu\nu}, \phi^\mu] = \int d[A, \phi, \chi, \psi, \eta] \exp(S + \int_X \Psi^{\mu\nu} \Lambda_{\mu\nu} + \phi^\mu Z_\mu) \]

\[ Q \rightleftharpoons \text{closed } \text{diff}(X)\text{-equivariant} \]
\[ \text{differential form on } Met(X) \]

\[ \text{Diffeomorphism invariant} \]

\[ \text{Descends to cohomology class } \in H^*(\frac{Met(X)}{\text{Diff}(X)}) \]

\[ \text{Conjecture: These are the family Donaldson invariants} \]

\[ n \rightleftharpoons \text{parameter families of metrics have wall-crossing} \]
\[ \text{in the degree } n \text{ component for } b_2^+(X) \leq n + 1 \]
Suppose \( \lambda \in H^2(X; \mathbb{Z}) \) is ASD for a metric \( g^{(0)} \)

**Perturb:** \( g(t) = g^{(0)} + \sum_{\alpha=1}^{b} t^\alpha p_\alpha \)

\[
Z^{\text{sing}} \sim c \left( \frac{\lambda^2}{2} \right) \omega_{b-1} + d (\star)
\]

\( \omega_{b-1} \) angular form in \( t^\alpha \) around the point \( t=0 \).

For \( G = SU(2) \) \( c(n) \) are the coefficients of the same modular form that appears in the standard Donaldson WCF.
One can also couple $g_{\mu \nu}, \Psi_{\mu \nu}, \phi^\mu$ to the LEET around $\mathcal{U}_{\Lambda^2}$.

It is natural to expect that this will give the family SW invariants formulated by T.-J. Li & A.-K. Liu.

... and moreover that there is an analog of the Witten conjecture for the family Donaldson invariants.
Basic question:

There are lots of interesting superconformal theories. (Some of them don’t even have Lagrangian descriptions.)

Nevertheless, they can be topologically twisted and have Q-invariant operators.

Is this a source of new four-manifold invariants?
Superconformal Theories – 2/4

Important lesson from $SU(2)\ N_{fl} = 4$

$\tau(u; m_a)$ approaches a FINITE limit as $u \to \infty$

Completely changes the wall-crossing story.

$$\frac{d}{dg_{\mu\nu}} Z_u \sim \sum_{\ell,r} \int_{|u|=R} fr^{\ell} S^r \lim_{R\to\infty} du \ u^{\frac{\sigma+1+2\ell+r}{2}} \Theta_{\ell,r}(\tau_0)(1 + \text{Series} \ \frac{1}{u}, \frac{1}{\bar{u}})$$

$$\frac{1}{2} (\sigma + 1 + 2\ell + r) < -1$$

No wall-crossing at $b_2^+ = 1$

$$\frac{1}{2} (\sigma + 1 + 2\ell + r) \geq -1$$

Continuous metric dependence!
TFT fails utterly!!
Now consider $SU(2)$ $N_{f\ell} = 1$ at $m = m_*$.

$$\lim_{m \to m_*) Z_u \neq \int du \, d\bar{u} \lim_{m \to m_*) \text{Measure}(u, \bar{u}; m) := Z_u^*$$

$Z_u^*$ : continuous or no metric dependence from singularity!

Continuous metric dependence for $\sigma + r + 6 \ell < -7$

No metric dependence for $\sigma + r + 6 \ell \geq -7$

**Conjecture:** $Z_u^* + Z_{u_0} + \langle e^{p \, \phi_1 + \phi_2(S)} \rangle_{\text{AD3 Theory}}$

is topologically invariant except for wall-crossing at $u = \infty$ for $b_2^+(X) = 1$
The truth of this conjecture would suggest that the superconformal theories might provide new four-manifold invariants, at least in some range of $r$ & $\ell$.

The truth of this conjecture would then strongly motivate an investigation of the $u$-plane integral for general class S.

Much of the structure of $Z_u$ is known – follows pattern of higher rank.

Some important details remain to be understood more clearly.

Can, in principle, be derived from a 2d $(2,0)$ QFT derived from reduction of abelian 6d $(2,0)$ theory along a four-manifold.