Twisted Equivariant Matter

Gregory Moore, Rutgers University, SCGP, June 12 , 2013

References:

1. D. Freed and G. Moore, Twisted Equivariant Matter, **arXiv:1208.5055**

2. G. Moore, Quantum Symmetries and K-Theory,
Lecture Notes, Home page: Talk 44

Prologue & Apologia

A few years ago, Dan Freed and I were quite intrigued by the papers of Kitaev; Fu, Kane & Mele; Balents & J.E. Moore; Furusaki, Ludwig, Ryu & Schnyder; Roy; Stone, et. al. relating classification of topological phases of matter to ***K-theory***.

So, we spent some time struggling to understand what these papers had to say to us, and then we wrote our own version of this story. That paper is the subject of today's talk.

I'll be explaining things that you all probably know, but in a language which is perhaps somewhat unusual.

This is perhaps not a completely silly exercise because the new language suggests interesting (?) questions which might not otherwise have been asked.

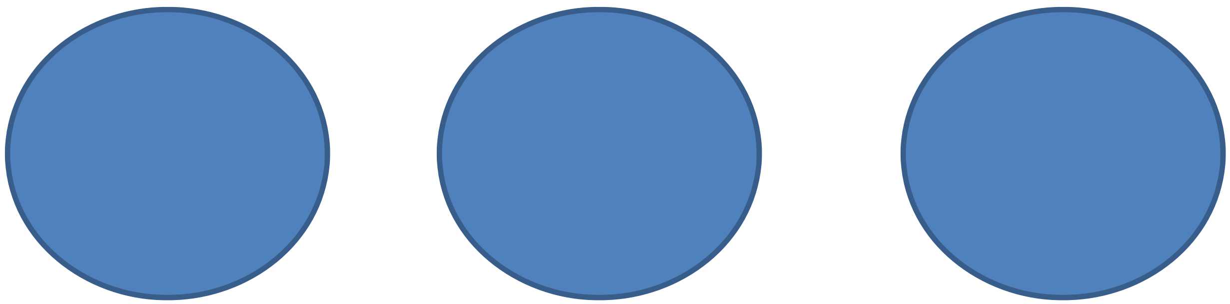
Main goal for today:

Give some inkling of how classification of topological phases can be made equivariant wrt physical symmetries, just using basic axioms of quantum mechanics.

and in particular for free fermions how one is led to twisted equivariant K-theory.

Physical Motivations -1

Topological phases = connected components of continuous families of gapped nonrelativistic QM systems



Restrict to physical systems with a symmetry group G and look at continuous families of systems with G-symmetry: We get a refined picture

Physical Motivations - 2

Suppose we have a (magnetic) crystallographic group:

$$1 \to \Pi \to G(C) \to P \to 1$$

In band structure theory one wants to say how the (magnetic) point group P acts on the Bloch wavefunctions. But this can involve tricky phases. If P(k) is a subgroup of P which fixes k how are the phase-choices related for different $P(k_1)$ and $P(k_2)$?

Textbooks deal with this in an ad hoc and unsatisfactory (to me) way.

The theory of twistings of K-theory provides a systematic approach to that problem.

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Symmetry in Quantum Mechanics: Wigner's Theorem

$\mathcal{H}$ Complex Hilbert space

$\mathbb{P}\mathcal{H}$ Pure states: Rank one projectors

$\mathrm{Aut}(\mathbb{P}\mathcal{H})$ Automorphisms preserving quantum overlaps $\mathrm{Tr}(P_1 P_2)$

Wigner's Theorem:

$$1 \to U(1) \to \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \to \mathrm{Aut}(\mathbb{P}\mathcal{H}) \to 1$$

$$\phi_{\mathcal{H}} \qquad \phi_{\mathbb{P}\mathcal{H}}$$

$$M_2 \cong \{\pm 1\}$$

Two components: unitary & antiunitary

Symmetry Group G of the Quantum State Space

$$
\begin{tikzcd}
1 \arrow[r] & U(1) \arrow[r] \arrow[d, equal] & G^{\tau} \arrow[r] \arrow[d, "\rho^{\tau}"'] \arrow[dd, dashed, bend left, "\phi_{G^{\tau}}"] & G \arrow[r] \arrow[d, "\rho"] \arrow[ddl, dashed, "\phi_{G}"'] & 1 \\
1 \arrow[r] & U(1) \arrow[r] & \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \arrow[r] \arrow[d, "\phi_{\mathcal{H}}"'] & \mathrm{Aut}(\mathbb{P}\mathcal{H}) \arrow[r] \arrow[dl, "\phi_{\mathbb{P}\mathcal{H}}"] & 1 \\
 & & M_2 \cong \{\pm 1\} & &
\end{tikzcd}
$$

Lighten Up!

We need to lighten the notation.

$$\tilde{g} \in G^{\tau} \quad \rightarrow \quad g \in G$$

denotes a lift of g (there are many: torsor for U(1))

$$\phi_{G^{\tau}}(\tilde{g}) = \phi_G(g) \in M_2 = \{\pm 1\}$$

So, just denote all four homomorphisms by ϕ; distinguish by context.

ϕ-Twisted Extension

So define abstract notion of a ϕ-twisted extension of a $\mathbb{Z}_2$-graded group

$$\begin{tikzcd}
1 \arrow[r] & U(1) \arrow[r] & G^{\tau} \arrow[r] \arrow[dr, dashed, "\phi"'] & G \arrow[r] \arrow[d, dashed, "\phi"] & 1 \\
 & & & M_2 \cong \{\pm 1\} &
\end{tikzcd}$$

$$\lambda \tilde{g} = \tilde{g} \lambda^{\phi(\tilde{g})}$$

Wigner: Given a quantum symmetry group G of the states we get a ϕ-twisted extension of G with a ``ϕ-twisted representation''

$$\rho^{\tau}(\tilde{g}) = \begin{cases} \text{linear} & \phi(\tilde{g}) = +1 \\ \text{anti}-\text{linear} & \phi(\tilde{g}) = -1 \end{cases}$$

Symmetry of the Dynamics

If the physical system has a notion of time-orientation, then a physical symmetry group has a homomorphism

$$t : G \to M_2 \cong \{\pm 1\}$$

Time-translationally invariant systems have unitary evolution: U(s). Then G is a symmetry of the dynamics if:

$$\rho^\tau(\tilde{g}) U(s) \rho^\tau(\tilde{g})^{-1} = U(t(g)s) = e^{-it(g)sH/\hbar}$$

$$\rho^\tau(\tilde{g}) H = c(g) H \rho^\tau(\tilde{g})$$

$$c(g) := \phi(g) t(g)$$

Dynamics - Remarks

$$\boxed{c(g)\phi(g)t(g) = 1}$$

Which one is dependent on the other two depends on what problem you are solving.

If c(g) = -1 for any group element then Spec(H) is symmetric around zero

So, if H is bounded below and not above (as in typical relativistic QFT examples) then c=1 and ϕ=t (as is usually assumed).

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Gapped Systems

Definition: A QM system is gapped if 0 is in the resolvent set of H, i.e. 1/H exists and is a bounded operator.

Remark: In this case the Hilbert space is $\mathbb{Z}_2$-graded by sign(H):

$$\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$$

E < 0

E > 0

So, if G is a symmetry of a gapped quantum system we get a ϕ-twisted extension with:

$$\rho^\tau(\tilde{g}) = \begin{cases} \text{linear} & \phi(g) = +1 \\ \text{anti} - \text{linear} & \phi(g) = -1 \end{cases} \qquad \rho^\tau(\tilde{g}) = \begin{cases} \text{even} & c(g) = +1 \\ \text{odd} & c(g) = -1 \end{cases}$$

(ϕ, τ, c)-Twisted Representation of G

Again, this motivates an abstract definition:

Definition: A (ϕ, τ, c)-twisted Representation of G is:

1. A ϕ-twisted extension:

$$1 \to U(1) \to G^{\tau} \longrightarrow G \to 1$$

2. Together with a $\mathbb{Z}_2$-graded vector space V and a homomorphism $\rho^{\tau} : G^{\tau} \to \mathrm{End}(V)$

$$\rho^{\tau}(\tilde{g}) = \begin{cases} \text{linear} & \phi(g) = +1 \\ \text{anti} - \text{linear} & \phi(g) = -1 \end{cases} \qquad \rho^{\tau}(\tilde{g}) = \begin{cases} \text{even} & c(g) = +1 \\ \text{odd} & c(g) = -1 \end{cases}$$

Continuous Families of Quantum Systems with Symmetry

Define <u>isomorphic</u> quantum systems with symmetry type (G, ϕ, τ,c)

Define notion of a <u>continuous family</u> of gapped quantum systems with symmetry type (G, ϕ, τ, c)

$$(\mathcal{H}, H, G, \phi, \tau, c)_0 \sim (\mathcal{H}, H, G, \phi, \tau, c)_1$$

if there is a continuous family parametrized by [0,1] with end-systems isomorphic to systems 0 & 1.

$$\mathcal{TP}(G, \phi, \tau, c)$$

Set of homotopy classes of gapped systems with symmetry type (G, ϕ, τ, c)

Algebraic Structure -1

In general the only algebraic structure is given by combination of systems with the same symmetry type.

$$\mathcal{H}_{1+2} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

$$H_{1+2} = H_1 \otimes 1 + 1 \otimes H_2 \quad + \cdots$$

Not much explored....

.... might not be that interesting ...

Algebraic Structure: Free Fermions

$$\mathcal{H}^{\mathrm{Fock}} = \Lambda^* V \Longleftarrow \mathcal{H}^{\mathrm{DN}} \cong V \oplus \bar{V}$$

Monoid structure: $\mathcal{H}^{DN}_{1+2} = \mathcal{H}^{DN}_1 \oplus \mathcal{H}^{DN}_2$

Now define ``Free fermions with a symmetry''

``Group completion'' or ``quotient'' by a suitable notion of ``topologically trivial systems''

$\Longrightarrow$ Abelian group $\mathcal{RTP}(G, \phi, \tau, c)$

Under special assumptions about the symmetry type (G,ϕ,τ,c) $\mathfrak{RTP}$ can be identified with a twisted K-theory group.

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Equivariant K-theory of a Point

Now let G be compact Lie group. (It could be a finite group.)

$K_G(pt)$ is the representation ring of G. It can be defined in two ways:

Group completion of the monoid of finite-dimensional complex representations.

Typical element: $R_1 - R_2$, with R_1, R_2, finite-dimensional representations on complex vector spaces.

$\mathrm{Rep}_s(G)$: $\mathbb{Z}_2$-graded fin. dim. cplx reps (with <u>even</u> G-action);

$\mathrm{Triv}_s(G)$: Those with an odd automorphism P: $P\rho(g) = \rho(g) P$.

$$K_G(pt) = \mathrm{Rep}_s(G)/\mathrm{Triv}_s(G)$$

Twisting Equivariant K-theory of a point

There are very sophisticated viewpoints in terms of ``nontrivial bundles of spectra'' … but here twisting just amounts to changing some signs and phases in various defining equations.

We'll get a little more sophisticated later.

$$\rho(g_1)\rho(g_2) = \lambda(g_1, g_2)\rho(g_1 g_2)$$

$$\lambda(g_1, g_2) \in U(1)$$

Preserve associativity: $\Rightarrow$ λ Is a 2-cocycle

$$1 \to U(1) \to G^\tau \to G \to 1$$

An example of a ``twisting of the equivariant K-theory of a point'' is just an isomorphism class of a central extension of G

Now we can form a monoid of twisted representations (= projective representations of G = representations of G^{τ}) and group complete or divide by the monoid of trivial representations to get an abelian group:

$$K_G^{\tau}(pt) := K_{G^{\tau}}(pt)$$

Example

Consider a 2-dimensional Hilbert space $\mathscr{H} = \mathbb{C}^2$

$$1 \to U(1) \to U(2) \rtimes \mathbb{Z}_2 \to SO(3) \rtimes \mathbb{Z}_2 \to 1$$

$$1 \to U(1) \to U(2) \to SO(3) \to 1$$

$$K_{U(2)} \subset K_{SU(2)\times U(1)} \supset K_{SO(3)\times U(1)}$$

$$K_{SO(3)\times U(1)} \cong \mathbb{Z}[\delta^{\pm 1}, f^2 - 1]$$

$$K_{U(2)} \cong \mathbb{Z}[\delta^{\pm 1}, f]^+$$

Adding the other ingredients we saw from the general realization of symmetry in gapped quantum mechanics gives new twistings:

$$(\phi, c) : G \to M_{2,2} := \{\pm 1\} \times \{\pm 1\}$$

ϕ-twisted extension

$$1 \to U(1) \to G^{\tau} \to G \to 1$$

$$\lambda(g_1, g_2)\lambda(g_1 g_2, g_3) = \lambda(g_2, g_3)^{\phi(g_1)}\lambda(g_1, g_2 g_3)$$

$$\boxed{{}^{\phi}K_G^{\tau, c}(pt) := \mathrm{Rep}_s(G^{\tau}, \phi, c)/\mathrm{Triv}_s(G^{\tau}, \phi, c)}$$

``Trivial"

$$P\rho^{\tau}(g) = c(g)\rho^{\tau}(g)P$$

``Pairing of particles and holes"

Example

$$G = \langle \sigma \rangle \cong M_2$$

$\phi(\sigma) = +1$ $\quad 1 \to U(1) \to U(1) \times M_2 \to M_2 \to 1$

$\phi(\sigma) = -1$ $\quad 1 \to U(1) \to \mathrm{Pin}^+(2) \to M_2 \to 1$

$\phi(\sigma) = -1$ $\quad 1 \to U(1) \to \mathrm{Pin}^-(2) \to M_2 \to 1$

$K_{\mathbb{Z}_2}(pt) \cong \mathbb{Z} \oplus \mathbb{Z}\epsilon$ Sign rep.

${}^{\phi}K^{\tau+}_{\mathbb{Z}_2}(pt) \cong \mathbb{Z}[f]$ Real rep f = $\mathbb{R}^2$

${}^{\phi}K^{\tau-}_{\mathbb{Z}_2}(pt) \cong \mathbb{Z}[q]^-$ $\mathbb{H}$-rep. q=$\mathbb{H}$, odd powers

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Finite-Dimensional Systems

Continue to take G to be a compact group.

Restrict to finite-dimensional Hilbert space $\mathcal{H}$

$$\mathcal{TP}_{\rm finite}(G,\phi,\tau,c) = \mathrm{Rep}_s(G,\phi,\tau,c)$$

Proof: 1. Homotope H to $H^2 = 1$ & (twisted) reps of compact groups are discrete.

For $\mathcal{RTP}$: Use monoid structure provided by free fermions:

With a suitable notion of ``trivial system'' – perhaps justified by ``pairing of particles and holes'' we obtain:

$$\mathcal{RTP}_{\rm finite}(G,\phi,\tau,c) = {}^{\phi}K_G^{\tau,c}(pt)$$

The 10 CT-groups

$$M_{2,2} := \langle \bar{T}, \bar{C} \rangle \cong M_2 \times M_2$$

$$\phi(\bar{T}) = \phi(\bar{C}) = -1$$

5 subgroups:

$$\begin{tikzcd}
& \langle \bar{T}, \bar{C} \rangle & \\
\langle \bar{T} \rangle \arrow[ur] & \langle \bar{T}\bar{C} \rangle \arrow[u] & \langle \bar{C} \rangle \arrow[ul] \\
& 1 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

There are 10 possible ϕ-twisted extensions. They are determined by whether the lifts T,C satisfy:

$$T^2 = \pm 1 \qquad \text{and/or} \qquad C^2 = \pm 1$$

$$t(\bar{T}) = -1, t(\bar{C}) = +1 \qquad c(\bar{T}) = +1, c(\bar{C}) = -1$$

Theorem: The category of (A, ϕ, τ, c)-twisted representations, where A is a subgroup of $M_{2,2}$, is equivalent to the category of modules of real or complex Clifford algebras.

A	$\mathit{1}$	diag	$\{\pm1\}\times 1$	$\mathcal{C}$	$1\times\{\pm1\}$	$\mathcal{C}$	$\{\pm1\}\times 1$	$\mathcal{C}$	$1\times\{\pm1\}$	$\mathcal{C}$
T^2			$+1$	$+1$		-1	-1	-1		$+1$
C^2				-1	-1	-1		$+1$	$+1$	$+1$
R	$Cl_0^{\mathbb{C}}$	$Cl_{-1}^{\mathbb{C}}$	Cl_0	Cl_{-1}	Cl_{-2}	Cl_{-3}	Cl_{-4}	Cl_{-5}	Cl_{-6}	Cl_{-7}

Various versions of this statement have appeared in Kitaev; Ludwig et. al.; Fidkowski & Kitaev; Freed & Moore

It is also related to the Altland-Zirnbauer-Heinzner-Huckleberry solution of the free fermion Dyson problem. (See below.)

Relation to standard K-theories

There is in turn a relation between twistings of K-theories, central simple superalgebras, and simple degree shift of K-groups, so that in the very special case where

$$G^{\tau} = A^{\tau} \times G_0 \qquad G_0 = \ker(t, c)$$

A	1	diag	$\{\pm 1\} \times 1$	$\mathcal{C}$	$1 \times \{\pm 1\}$	$\mathcal{C}$	$\{\pm 1\} \times 1$	$\mathcal{C}$	$1 \times \{\pm 1\}$	$\mathcal{C}$
T^2			$+1$	$+1$		-1	-1	-1		$+1$
C^2				-1	-1	-1		$+1$	$+1$	$+1$
${}^{\phi}K_G^{t,c}$	$K_{G_0}^0$	$K_{G_0}^{-1}$	$KO_{G_0}^0$	$KO_{G_0}^{-1}$	$KO_{G_0}^{-2}$	$KO_{G_0}^{-3}$	$KO_{G_0}^{-4}$	$KO_{G_0}^{-5}$	$KO_{G_0}^{-6}$	$KO_{G_0}^{-7}$

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Digression: Dyson's 3-fold way

Dyson's Problem: Given a symmetry type (G,ϕ,τ) with c=1 and $\mathcal{H}$, what is the ensemble of commuting Hamiltonians?

Schur's lemma for irreducible ϕ-twisted representations: $Z(\mathcal{H}_\lambda)$ is a real associative division algebra.

Frobenius theorem: There are three real associative division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

$$\mathcal{H} \cong \oplus_\lambda N_\lambda \otimes \mathcal{H}_\lambda$$

$$Z(\mathcal{H}) \cong \oplus_\lambda \mathrm{End}(N_\lambda) \otimes D_\lambda$$

$$D_\lambda \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$$

Generalizes to 10-fold way

Given a symmetry type (G,ϕ,τ,c) and $\mathcal{H}$, what is the ensemble of <u>graded-commuting</u> Hamiltonians?

Schur's lemma for irreducible (ϕ,τ,c)-twisted representations: $Z_s(\mathcal{H}_\lambda)$ is a real associative super-division algebra.

Theorem (Wall, Deligne): There are ten real associative super-division algebras:

$$D_\lambda^s \in \{C\ell_{0,\pm 1,\pm 2,\pm 3}, \mathbb{C}\ell_{0,1}, \mathbb{H}\}$$

$$\mathcal{H} \cong \oplus_\lambda N_\lambda \otimes \mathcal{H}_\lambda$$

$$Z_s(\mathcal{H}) \cong \oplus_\lambda \mathrm{End}(N_\lambda) \otimes D_\lambda^s$$

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Noncompact groups?

When we generalize to noncompact groups and infinite-dimensional Hilbert spaces there are many possibilities.

One physically important case is where we consider a crystal

$$C \subset E := \mathbb{R}^d$$

A set of points C with atoms, spins or currents invariant under translation by a rank d lattice Π. Then C is invariant under the magnetic crystallographic group. So there is an extension:

$$1 \to \Pi \to G(C) \to P \to 1$$

$P \subset O(d) \times \mathbb{Z}_2$ is a finite group: ``Magnetic point group''

There are still too many possibilities for (ϕ,τ) to say something about $\mathcal{TP}\,(\phi,\tau)$ so we narrow it down using some more physics.

Bloch Theory -1

$$C \subset E := \mathbb{R}^d$$

Single electron approximation:

$$\mathcal{H} := L^2(E; W)$$

W is a finite dimensional vector space. e.g. for internal spin: $W \cong \mathbb{C}^2$

The Schrödinger operator H is invariant under G(C)

Bloch Theory -2

Now, because G(C) is a symmetry of the quantum system the Hilbert space $\mathcal{H}$ is a (ϕ,τ,c)-twisted representation of G(C) for some (ϕ,τ,c).

$G(C)^{\tau}$ acts on Hilbert space $\mathcal{H}$

For simplicity assume Π acts without central extension (i.e. no magnetic field).

$$1 \to \Pi \to G(C)^{\tau} \to P^{\tau} \to 1$$

Bloch Theory -3

Now reinterpret $\mathcal{H}$ as the Hilbert space of sections of a twisted equivariant Hilbert bundle over the Brillouin torus.

$$\psi(\bar{k}, x+R) = e^{ik\cdot R}\psi(\bar{k}, x)$$

$$R \in \Pi \qquad\qquad \bar{k} \in T^*$$

$$T^* := E^*/\Pi^*$$

Brillouin torus; Irreps of Π

Bloch Theory - 4

$$T := E/\Pi$$ Dual torus

Poincare line bundle: $\mathcal{L} \to T^* \times T$

Sections of $\mathcal{L}$ are equivalent to Π-equivariant functions:

$$T^* \times E \longrightarrow \mathbb{C}$$

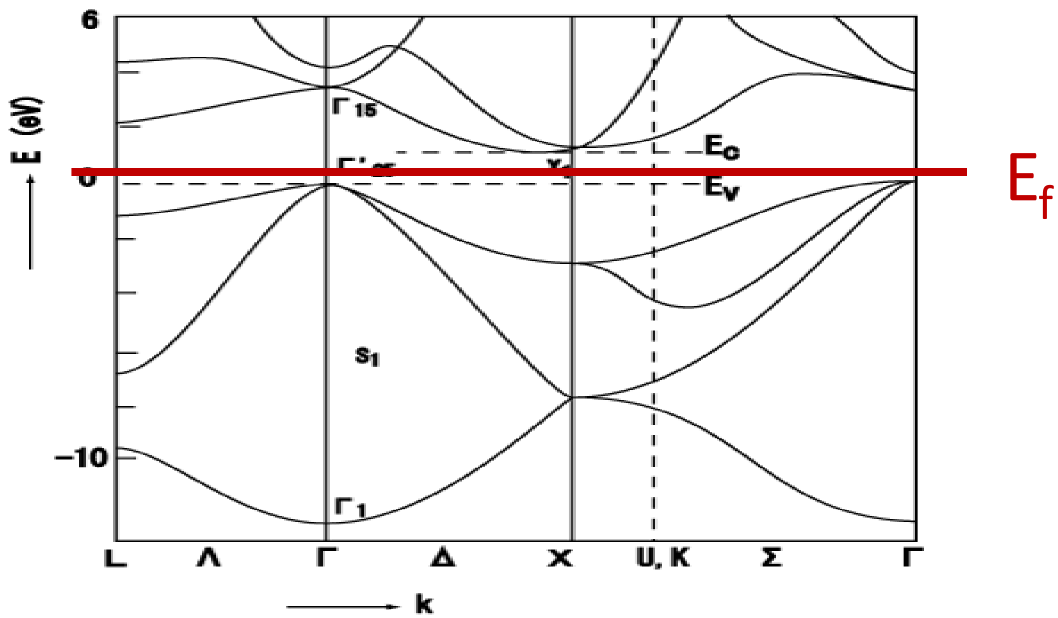
$$\psi(\bar{k}, x + R) = e^{ik\cdot R}\psi(\bar{k}, x)$$

$$\bar{k} \in T^* \qquad \mathcal{E}_{\bar{k}} := L^2(T; \mathcal{L}_{\bar{k}})$$

$$\mathcal{H} := L^2(E; W) \cong L^2(T^*; \mathcal{E} \otimes W)$$

Insulators

The Hamiltonian H defines a continuous family of self-adjoint operators on $\mathcal{E}$. This gives the usual band structure:



In an insulator there wil be a gap, hence an energy E_f so that we have a direct sum of Hilbert bundles:

$$\mathcal{E} \cong \mathcal{E}^- \oplus \mathcal{E}^+$$

$$E < E_f \qquad\qquad E > E_f$$

Equivariant Insulators: Two Cases

$\mathcal{E}^-$ and $\mathcal{E}^+$ are finite and infinite-dimensional Hilbert bundles over T*, respectively.

For some purposes it is useful to focus on a finite number of bands above the Fermi level and make $\mathcal{E}^+$ a finite-dimensional bundle.

Thus, there are two cases: $\mathcal{E}^+$ has finite or infinite dimension.

Through the Fourier transform to Bloch waves this translates into $\mathcal{E}^-$ and $\mathcal{E}^+$ being ``twisted equivariant bundles over the groupoid T*//P " .

We next spend the next 10 slides explaining this terminology.

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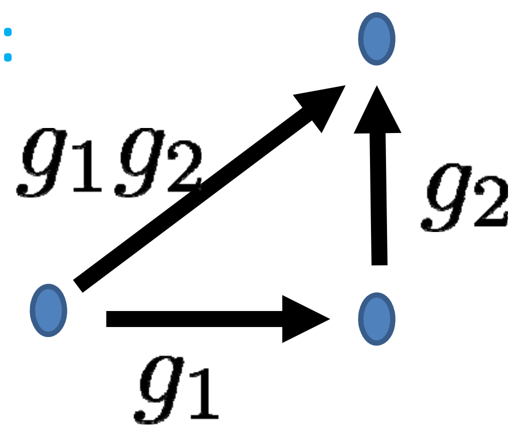
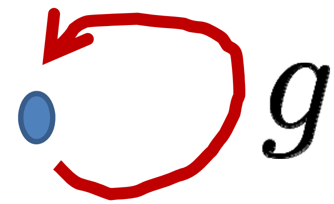
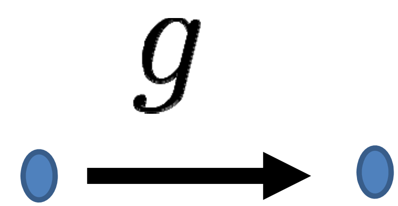
Groups As Categories

Now we want to give a geometrical interpretation to (ϕ,τ,c)-twisted representations of G.

Suppose G is a group, and think of it as acting on a point:







(Two points are identified)

Therefore, we have a category:

One object: 

Morphisms are group elements

Axioms of a category are equivalent to existence of identity and associativity.

(This is a special category because all morphisms are invertible.)

Central Extensions as Line Bundles

Now, for every group element g we give a complex line L_g, together with a product law:

$$\lambda_{g_2,g_1} : L_{g_2} \otimes L_{g_1} \to L_{g_2 g_1}$$

$$\lambda_{g_2,g_1} : \ell_{g_2} \otimes \ell_{g_1} \to \lambda(g_2, g_1)\ell_{g_2 g_1}$$

Require associativity:

$$L_{g_3} \otimes L_{g_2} \otimes L_{g_1} \to L_{g_3 g_2 g_1}$$

This defines a line bundle over the space of morphisms with a product law. Then G^{τ} is the associated principal U(1) bundle over G.

ϕ-Twisted Extensions

For a complex vector space V define notation:

$$^{\phi}V := \begin{cases} V & \phi = +1 \\ \overline{V} & \phi = -1 \end{cases}$$

Now each arrow, g, has ϕ(g) = ± 1 attached and we modify the product law to

$$\lambda_{g_2,g_1} : {}^{\phi(g_1)}L_{g_2} \otimes L_{g_1} \to L_{g_2 g_1}$$

Require associativity:

$${}^{\phi(g_2 g_1)}L_{g_3} \otimes {}^{\phi(g_1)}L_{g_2} \otimes L_{g_1} \to L_{g_3 g_2 g_1}$$

(ϕ,τ,c)-twisted representations, again

First we use the homomorphism $c: G \to M_2$ to give a $\mathbb{Z}_2$-grading to the lines L_g.

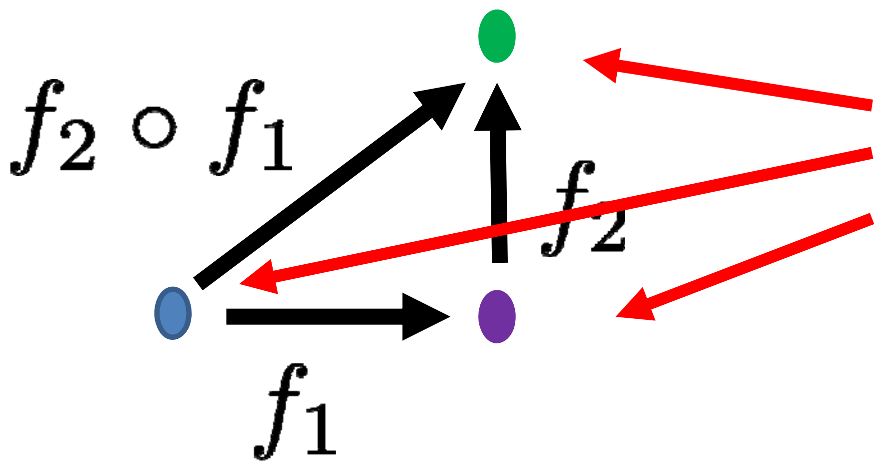
A (ϕ,τ,c)-twisted representation is a $\mathbb{Z}_2$-graded vector space V ("vector bundle over a point") together with a $\mathbb{C}$-linear and even isomorphism:

$$\rho_g : {}^{\phi(g)}\left(L_g \otimes V\right) \to V$$

$$\rho^\tau(\tilde{g}) = \begin{cases} \text{linear} & \phi(g) = +1 \\ \text{anti} - \text{linear} & \phi(g) = -1 \end{cases} \qquad \rho^\tau(\tilde{g}) = \begin{cases} \text{even} & c(g) = +1 \\ \text{odd} & c(g) = -1 \end{cases}$$

Groupoids

Definition: A groupoid $\mathcal{G}$ is a category all of whose morphisms are invertible



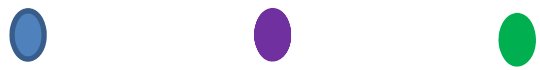
Points = objects are now NOT identified.

Example: Group G acts on a topological space X.

Objects = X, Morphisms = X x G. Groupoid denoted $\mathcal{G}$=X//G

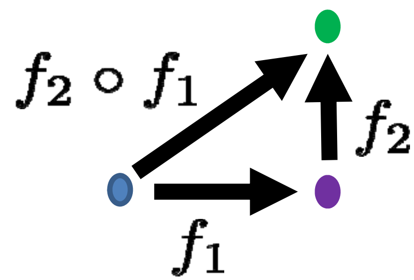
$$x \xrightarrow{g} g \cdot x$$

Composable Morphisms

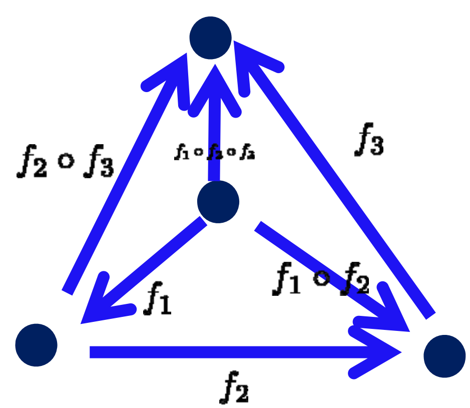
$\mathcal{G}_0$ 

$\mathcal{G}_1$ 

$\mathcal{G}_2$ $\{(f_1,f_2): \mathrm{end}(f_1)=\mathrm{beg}(f_2)\}$



$\mathcal{G}_3$ $\{(f_1,f_2,f_3): \mathrm{end}(f_i)=\mathrm{beg}(f_{i+1})\}$



etc.

Twisting K-theory of a groupoid

Homomorphism of a groupoid $\mathcal{G} \to M_2$: $\mu : \mathcal{G}_1 \to M_2$

$$\mu(f_2 \circ f_1) = \mu(f_2)\mu(f_1) \quad (f_1, f_2) \in \mathcal{G}_2$$

Definition: Let $\mathcal{G}$ be a groupoid with homomorphisms ϕ,c: $\mathcal{G} \to M_2$. A (ϕ,c)-twisting of the K-theory of $\mathcal{G}$ is:

a.) A collection of $\mathbb{Z}_2$-graded complex lines L_f, $\forall$ f$\in \mathcal{G}_1$, $\mathbb{Z}_2$-graded by c(f).

b.) A collection of $\mathbb{C}$-linear, even, isomorphisms (data on $\mathcal{G}_2$):

$$\lambda_{f_2,f_1} : {}^{\phi(f_1)}L_{f_2} \otimes L_{f_1} \to L_{f_2 \circ f_1}$$

c.) Satisfying the associativity (cocycle) condition (on $\mathcal{G}_3$)

Remarks

We define a twisting of K-theory of $\mathcal{G}$ before defining the K-theory!

We will think of twistings of T*// P as defining a symmetry class of the band structure problem.

These twistings have a nice generalization to a class of geometrical twistings given by bundles of central simple superalgebras and invertible bimodules.

Isomorphism classes of such twistings, for $\mathcal{G}$ = X//G are given, as a set, by

$$H^0_G(X;\mathbb{Z}_2) \times H^1_G(X;\mathbb{Z}_2) \times H^3_G(X;\mathbb{Z}_\phi)$$

(There are yet more general twistings,....)

Definition of a (ϕ, τ, c)-twisted bundle

a.) A complex $\mathbb{Z}_2$-graded bundle over $\mathcal{G}_0$.

b.) A collection of $\mathbb{C}$-linear, even isomorphisms over $\mathcal{G}_1$.

$$x_1 \xrightarrow{f} x_2$$

$$\rho_{x,f} : {}^{\phi(f)}\left(L_f \otimes V_{x_1}\right) \to V_{x_2}$$

c.) Compatibility (gluing) condition on $\mathcal{G}_2$.

Definition of twisted K-theory on a groupoid

Isomorphism classes of ν-twisted bundles form a monoid $\mathrm{Vect}^{\nu}(\mathcal{G})$ under $\oplus$.

$\mathrm{Triv}^{\nu}(\mathcal{G})$ is the submonoid of bundles with an odd automorphism P: V $\rightarrow$ V

$$K^{\nu}(\mathcal{G}) := \mathrm{Vect}^{\nu}(\mathcal{G})/\mathrm{Triv}^{\nu}(\mathcal{G})$$

$$K^{\nu}(X//G) := {}^{\phi}K_{G}^{\tau,c}(X)$$

is a generalization of equivariant KR-theory with twistings and groupoids.

For X=pt recover previous description.

1. Introduction
2. Review of symmetry in quantum mechanics
3. Gapped systems and (reduced) topological phases
4. Equivariant twisted K-theory of a point
5. Finite-dimensional systems and the 10 CT-groups
6. Digression: Dyson's 3-fold way and Altland-Zirnbauer
7. Bloch Theory
8. Equivariant twisted K-theory of a groupoid.
9. Back to Bloch

Back to Bloch

$$\mathcal{H} \cong L^2(T^*; (\mathcal{E}^+ \oplus \mathcal{E}^-) \otimes W)$$

The magnetic point group P^τ acts on $\mathcal{H}$ to define a twisted equivariant bundle over T* with a ***canonical twisting*** ν_{can}

	$\mathcal{E}^+$ FINITE	$\mathcal{E}^+$ INFINITE
$\mathcal{TP}$	$\mathrm{sVect}^{\nu_{can}}(T^*//P)$	projects onto: $\mathrm{sVect}^{\nu_{can}}(T^*//P)$
$\mathcal{RTP}$	$K^{\nu_{can}}(T^*//P)$	$K^{\nu_{can}}(T^*//P)$

Relation to more standard K-groups

Take $\mathcal{E}^+$ to be infinite-dimensional, and ***assume***:

$$(G(C)^\tau, \phi, c) \cong (A^\tau, \phi, c) \times G_0$$

A	1	$\langle \bar{T} \rangle$	$\langle \bar{T} \rangle$
T^2		$+1$	-1
$K^{\nu_{can}}(T^*//P)$	$K^{\nu_0}_{P_0}(T^*)$	$KR^{\nu_0}_{P_0}(T^*)$	$KR^{\nu_0-4}_{P_0}(T^*)$

Cases studied in the literature

$$G(C) = \langle \bar{P} \rangle \times \Pi \qquad K^0_{\mathbb{Z}_2}(T^*)$$
$$\phi(\bar{P}) = t(\bar{P}) = +1$$

Turner, et. al.

$$G(C) = \langle \bar{T} \rangle \times \Pi \qquad KR^{-4}(T^*)$$
$$\phi(\bar{T}) = t(\bar{T}) = -1$$

Kane, et. al.

$$G(C) = \langle \bar{T}, \bar{P} \rangle \times \Pi \qquad KO^{-4}_{\mathbb{Z}_2}(T^*)$$

Remark: Kane-Mele and Chern-Simons invariants descend to $\mathscr{RTP}$ and are equal. KO invariant refines Kane-Mele invt.

Example: Diamond Structure

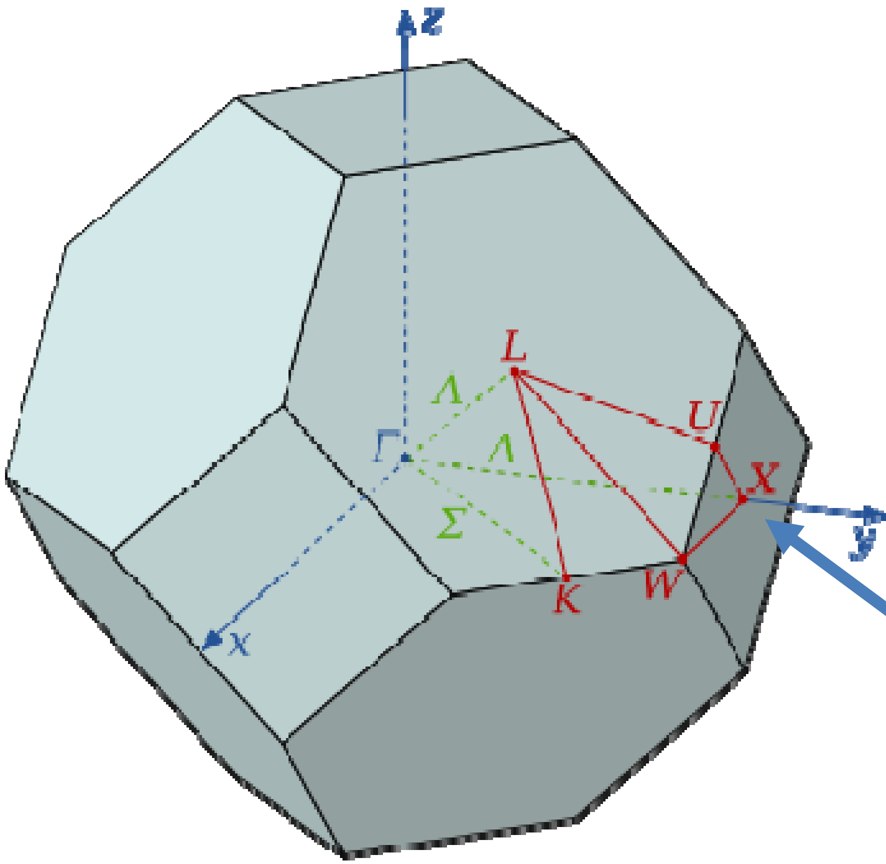
$P = \langle I \rangle \times \mathrm{Cubic} \qquad K^{\nu_{can}}(T^*//P) = ?$

Localization:

$$S^{-1}K^{\nu_{can}}(T^*//P) \cong S^{-1}K_P^{\iota^*(\nu_{can})}((T^*)^I)$$

$(T^*)^I$ 8 Fixed Points under k → - k

$\Gamma \quad \cup$ Orbit of 4 L-points $\cup$ Orbit of 3 X-points



$$\widetilde{\mathbb{Z}_2 \times D_4}$$

$$\Big\downarrow$$

$$P(X) \cong \mathbb{Z}_2 \times D_4$$

A K-theory invariant which is an element of $\mathrm{Rep}(\widetilde{\mathbb{Z}_2 \times D_4}) \otimes \mathbb{Z}[\frac{1}{2}]$

Things To Do

Compute the canonical twisting and the equivariant K-groups for more elaborate (nonsymmorphic) magnetic crystallographic groups.

Relate K-theory invariants to edge-phenomena and entanglement spectra.

Are there materials which realize twistings other than the canonical twisting? They would have to be exotic.