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A Minicourse on Generalized Abelian Gauge Theory, Self-Dual Theories, and Differential Cohomology

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ABSTRACT: These are lecture notes for a minicourse given at the Simons Center for Geometry and Physics, Jan. 12-14, 2011. Version: Jan. 14, 2011

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1. Introduction and Summary

Introductory remarks

1. I have been given the task of explaining some of the physics background and physical intuition underlying physical applications of differential cohomology. Very little here will be new to physicists, especially at the beginning. On the other hand, we will assume some familiarity with differential cohomology.

2. This subject is frustrating because it is both trivial and difficult. It is "trivial" because ultimately we are just talking about free field theories. In some sense that is a bonus: Although I will not be mathematically rigorous, everything that follows is in principle susceptible of complete mathematical rigor. On the other hand, the subject is very difficult, at least for a physicist, because of the nontrivial role of topology.

3. Comment on the use of the word "theory." The word is used very imprecisely and sometimes hides some ignorance. Physicists often write an equation of motion, or an action, or a Hamiltonian and say that it defines a "theory." Sometimes the transition from one to the other picture can be nontrivial, although there are standard paradigms people keep in mind that makes them feel these transitions are always possible to carry out, at least in principle.

4. I have not given any references in these notes. The point of view I am expressing here is based on things I have learned from Dan Freed, Graeme Segal, and Edward Witten. It draws on various papers and projects with D. Belov, E. Diaconescu, J. Distler, D. Freed, R. Minasian, G. Segal, N. Seiberg, and E. Witten.

Here is an outline of topics one would want to cover:

1. Classical theory of a generalized Maxwell field.

2. Currents, Branes, and Dirac quantization

3. Quantum electric-magnetic duality: Partition function and Hilbert space

4. Coupling to electric and magnetic currents: Nature of the field and the anomalous electric current.

5. Anomalous theories in the presence of electric and magnetic current

6. Classical theory of self-dual fields

7. Quantum self-dual fields: Hilbert space

8. Splitting the partition function of the nonchiral field: Motivating the Chern-Simons theory

9. Chern-Simons theory: Edge state phenomenon

10. Quantization of CS theory associated to a differential cohomology theory on $X \times \mathbb{R}$: Identifying the quantum states of the CS theory with the conformal blocks of the self-dual field.

11. Quantization of Maxwell-Chern-Simons theory: Degenerate ground states ("Lowest Landau Levels") are identified with the conformal blocks of the self-dual field: This is an example of holography.

12. Action principle for the self-dual field: Period matrix in infinite dimensions is the action.

13. General formulation of a self-dual field from three pieces of data:

    - Pontryagin self-dual generalized cohomology theory
    - Isomorphism of electric and magnetic currents
    - Quadratic refinement of the bilinear form on electric and magnetic currents

14. Example: The RR field of type II string theory

15. Abelian gauge field of M-theory

16. Self-dual theories in six dimensions

2. Generalized Maxwell Theory and Electromagnetic Duality

2.1 Classical Generalized Maxwell Theory Without Sources

We will take a spacetime manifold M_n of dimension n. It may be compact or noncompact. It will be endowed with a smooth metric and we consider both Lorentz and Euclidean signature metrics. For simplicity we will assume M_n is oriented (we comment on the unoriented case below).

The simplest classical generalized Maxwell theory is a theory of a form field

$$F \in \Omega^{\ell}(M_n) \tag{2.1}$$

subject to the equations of motion:

$$\begin{aligned} dF &= 0 \\ d * F &= 0 \end{aligned} \tag{2.2}$$

Remarks:

1. Given a space-time splitting $M = S \times \mathbb{R}$ we can define electric and magnetic fields by decomposing
$$F = dtE + B \tag{2.3}$$
where $E \in \Omega^{\ell-1}(S)$ and $B \in \Omega^{\ell}(S)$ can depend on time. Substitution into (2.2) yields Maxwell-like equations. Maxwell's original case is $M_n = \mathbb{R}^{1,3}$ and $\ell = 2$.

2. All the cases $0 \leq \ell \leq 10$, $1 \leq n \leq 11$ are of physical interest in the context of M-theory, string theory, and supersymmetric field theories. Some cases are also of interest in condensed matter physics.

3. In classical physics we just want to know the solutions of the equations of motion. That is solved in principle for these linear PDE's. Solutions are harmonic forms. In particular, in Minkowski space we have wave solutions moving at the speed of light.

4. In classical physics we also want to define some physical observables. Of these the most important is the energy-momentum tensor: The energy-momentum tensor of a field configuration F is a symmetric quadratic form $T_F \in \mathrm{Sym}^2(T^* M_n)$:

$$T_F(v) = (\iota_v F, \iota_v F) - \frac{1}{2}(v, v)(F, F). \tag{2.4}$$

   For on-shell fields T_F is conserved. Here we use the norm $(\alpha, \beta) := \frac{\alpha \wedge *\beta}{\mathrm{vol}\,(g)}$. Note that $(\alpha, \beta) = (\beta, \alpha)$, always.

5. To a given solution we automatically have two deRham cohomology classes $[F] \in H^{\ell}_{dR}(M_n)$ and $[*F] \in H^{n-\ell}_{dR}(M_n)$. In the physics literature the word *flux* is used rather loosely. Sometimes it refers to the fieldstrength F and sometimes it refers to the periods of F, and hence it is sometimes used to refer to the cohomology classes defined by F. We will adopt the terminology that the *classical flux group* is the abelian group

$$\textit{Classical Flux Group} := H^{\ell}_{dR}(M_n) \oplus H^{n-\ell}_{dR}(M_n). \tag{2.5}$$

   The two summands are referred to as magnetic and electric fluxes, respectively.

2.1.1 Electric-Magnetic duality

Important symmetry of the situation: The classical theory of a generalized Maxwell field $F \in \Omega^{\ell}(M)$ is equivalent to the classical theory of a Maxwell field $\tilde{F} \in \Omega^{n-\ell}(M)$. At the level of the equations of motion this is obvious since we can set

$$\tilde{F} = *F \tag{2.6}$$

We must also check that this is a symmetry of the dynamics. Indeed under the mapping (2.6) we have

$$T_F = T_{\tilde{F}} \tag{2.7}$$

(To give a proof note that in Lorentzian signature $*^2 = -(-1)^{\ell(n-\ell)}$ and hence $(F, F) = -(\tilde{F}, \tilde{F})$. On the other hand, when we compute $(\iota_v \tilde{F}, \iota_v \tilde{F})$ it is clear from the identity expanding the product of epsilon tensors in a sum of terms involving $g_{\mu\nu}$ we clearly have

$$(\iota_v \tilde{F}, \iota_v \tilde{F}) = -(v, v)(F, F) + \mathit{const.}(\iota_v F, \iota_v F) \tag{2.8}$$

and therefore $T_{\tilde{F}}(v) = \mathit{const.}(\iota_v F, \iota_v F) - \frac{1}{2}(v, v)(F, F)$. On the other hand, the constant must be one since $T_{\tilde{F}}(v)$ is conserved for on-shell fields.)

2.1.2 Action Principle

To write an action principle, we must break some manifest symmetry, in particular electromagnetic duality. *IF* we solve $dF = 0$ then we put $F \in \Omega^{\ell}_{d}(M)$. When we do this $dF = 0$ is referred to as the *Bianchi identity* and $d * F = 0$ is regarded as an *equation of motion.*

We can derive this equation of motion as the Euler-Lagrange equation from the action principle:

$$S = \pi \int \lambda F * F \qquad\qquad F \in \Omega^\ell_d(M) \tag{2.9}$$

We will interpret the "coupling constant" λ physically below. The interpretation depends on ℓ and other choices. If λ is a constant then, until we discuss topology, its particular value is meaningless so long as it is nonzero: We can just reabsorb it into F.

The action is really a function of two fields: The field $F \in \Omega^\ell_d(M)$ and also the metric g on M. The Energy-momentum tensor comes from varying the metric (at fixed F):

$$\delta S = \int \mathrm{vol}\,(g) \frac{1}{2} \delta g^{\mu\nu} T_{\mu\nu} \tag{2.10}$$

2.1.3 Higher Rank Theories

A natural generalization is to take $F \in \Omega^\ell(M; V)$ where V a vector space. More generally V can be a $\mathbb{Z}$-graded vector space and we can take $F \in \Omega^*(M; V^*)$ of some fixed total degree. In order to formulate the energy momentum tensor we require a positive definite symmetric form on V.

We can define a richer set of actions in this case. If we solve the Bianchi identity so that $F \in \Omega^\ell_d(M; V)$ and choose a basis e_i for V so that $F = e_i F^i$ then we can write:

$$S = \pi \int \lambda_{ij} F^i * F^j + 2\pi \theta_{ij} F^i F^j \tag{2.11}$$

Remarks:

1. λ_{ij} symmetric and positive for unitarity.
2. If λ_{ij} is constant then it can be removed by a redefinition of F^i, before we take into account topology.
3. θ_{ij} is graded symmetric. If it is constant it has no effect on the equations of motion, but does affect the relation between canonical momentum and electric field in Hamiltonian mechanics.

One motivation for this generalization comes from Kaluza-Klein reduction: Once one has admitted that generalized Maxwell theories on general manifolds are interesting this generalization is inevitable:

Using the Kaluza-Klein idea a physicist would say:

The low energy effective field theory of a generalized Maxwell theory of an ℓ-form fieldstrength in $n + k$ dimensions on a manifold $M \times K$, with K compact and small is a generalized Maxwell theory where the fieldstrength is valued in

$$\oplus_{p+q=\ell} \Omega^p(M; \mathcal{H}^q(K)) \tag{2.12}$$

where $\mathcal{H}^q(K)$ is the real vector space of harmonic q-forms on K. The positive symmetric form is just the Hodge inner product.

2.1.4 Further Generalizations

1. Here, for simplicity, we have assumed M_n is oriented, but this is not necessary. If M_n is unoriented, or even unorientable then $*F$ can still be defined as a twisted differential form

$$*F \in \Omega^{n-\ell+\tau}(M_n) := \Omega^{n-\ell}(M_n; L) \tag{2.13}$$

where τ is the twisting of the orientation double cover, and L is the corresponding real line bundle.

2. Motivated by the importance of orbifolds and orientifolds in string theory the case where M is replaced by a groupoid X is also of physical interest. Again, for simplicity we will not discuss this here.

3. As a further generalization we could take $V \to M$ to be a bundle with flat connection. As an example V could be the real line bundle associated with the orientation double cover.

2.2 Electric currents

An electric current is a closed $n - \ell + 1$ form $J_e \in \Omega_d^{n-\ell+1}(M_n)$. With it we can modify the basic equations of motion to

$$\begin{aligned} dF &= 0 \\ d * F &= J_e \end{aligned} \tag{2.14}$$

Remarks:

1. That's what we see in nature, although there is of course backreaction. Here we are treating J_e as some externally prescribed source, and studying the reaction of the fields to those sources.

2. Let us stress that $F \in \Omega_d^{\ell}(M)$ is still a closed form, and to a solution of the equation of motion we can assign a definite magnetic flux.

3. On the other hand, we cannot assign a definite electric flux to a solution of the equations of motion. Instead, J_e is a closed differential form: $dJ_e = 0$ expresses charge conservation. Therefore J_e defines a cohomology class, but it is explicitly trivialized by $*F$, (so long as there exists a solution of the equations of motion). Nevertheless, J_e can in principle define a nontrivial relative cohomology class

$$q(J_e) := [J_e] \in H_{dR}^{n-\ell+1}(M_n, M_n^{-J_e}) \tag{2.15}$$

Here and in the following we will define

$$M_n^{-J_e} := M_n - \mathrm{Supp}(J_e) \tag{2.16}$$

We take (2.15) as a definition of the classical electric charge of the current J_e.

4. In physical situations we often have a spacetime splitting $M = S \times \mathbb{R}$ and the pullback of J_e to S at all times is compactly supported. In that case we can identify the group of electric charges as the compactly supported cohomology, or, more precisely:

$$\textit{Classical Electric Charge Group} := \mathcal{Q}_e := \ker \left[H^{n-\ell+1}_{dR,cpt}(S) \to H^{n-\ell+1}_{dR}(S) \right] \quad (2.17)$$

2.3 The relation of charge and flux groups

Now that we have introduced both charge and flux groups we should ask how they are related.

Return to $d * F = J_e$. Assume that J_e and $*F$ are defined on all of M. Then $dJ_e = 0$ so defines a class in $H^{n-\ell+1}_{dR}(M)$. But this class is clearly 0, if $*F$ is defined everywhere, because $*F$ explicitly trivializes it.

On the other hand, the pair $(J_e, *F)$ also defines a relative cocycle in $Z^{n-\ell+1}(M, M^{-J_e})$. Now that cocycle is trivialized by $(*F, 0) \in C^{n-\ell}(M, M^{-J_e})$.

Nevertheless $[(J_e, 0)] = -[(0, *F)]$ might define a nontrivial class. By our definition above $[(J_e, 0)]$ is the electric charge.

The electric charge is the kernel of the map ψ to $H^{n-\ell+1}_{dR}(M_n)$, so from the long exact sequence

$$\cdots \to H^{n-\ell}(M_n) \xrightarrow{\iota} H^{n-\ell}(M_n^{-J_e}) \xrightarrow{\delta} H^{n-\ell+1}(M_n, M_n^{-J_e}) \xrightarrow{\psi} H^{n-\ell+1}(M_n) \to \cdots \quad (2.18)$$

we express the group of electric charges in terms of the group of fluxes:

$$\mathcal{Q}_e \cong H^{n-\ell}_{dR}(M_n^{-J_e}) / \iota H^{n-\ell}_{dR}(M_n) \quad (2.19)$$

In physics parlance, we say that the "charge is measured by the flux at ∞." When $M = S \times \mathbb{R}$ we replace $M_n^{-J_e}$ by a "Gaussian sphere at infinity." Of course, if S is compact there is nowhere for the flux lines to end, and the total charge must vanish.

2.4 Electrically charged branes

Pseudo-definition of a p-brane: It is a physical extended object of p-spatial dimensions. The worldvolume is denoted $\mathcal{W}$ and is of $(p+1)$ spatial dimensions, since it is assumed that the object propagates forward in time (when M has a Lorentz signature metric). [1]

If the brane produces an electric current J_e whose support is (delta-function) localized on $\mathcal{W}$ we say it is electrically charged. The current J_e is proportional to the delta-function supported representative of the Poincaré dual of $\mathcal{W}$: [2]

$$J_e = q_e \delta(\mathcal{W} \hookrightarrow M_n) \quad (2.20)$$

[1] Physicists also consider "D-brane instantons" which are Wick rotations of branes to Euclidean space, and "S-branes" which live at a fixed time in Minkowski space, as well as "I-branes" which live at the intersections of brane worldvolumes (where there are typically new degrees of freedom).

[2] In order to avoid problems with singularities, we always imagine that our branes are thickened so that the current is smooth but supported in an arbitrarily small tubular neighborhood of $\mathcal{W}$.

The constant of proportionality q_e is one way to measure the electric charge of the brane.

The Poincaré dual $\delta(\mathcal{W}) \in \Omega^{n-(p+1)}(M_n)$ but $J_e \in \Omega^{n-\ell+1}(M_n)$ so we learn that for the generalized Maxwell field $F \in \Omega^{\ell}(M_n)$ the electrically charged branes are p-branes with

$$p = p_e := \ell - 2. \tag{2.21}$$

Remarks:

1. **The basic example**: We consider $M_n = \mathbb{R}^{1,n-1}$ with standard Minkowskian metric and orientation $dt \wedge d^{n-1}x$. The brane at a fixed time fills a p_e-dimensional hyperplane H_e in Euclidean space so $\mathcal{W} = \mathbb{R} \times H_e$. Let

$$D_e = n - (p_e + 1) = n - \ell + 1 \tag{2.22}$$

denote the number of transverse dimensions and let r be the Euclidean distance in the space $H^{\perp} \cong \mathbb{R}^{D_e}$ orthogonal to H. Then

$$F = \frac{q_e}{V_{D_e}} \frac{dt \wedge dr}{r^{D_e - 1}} \mathrm{vol}\,(H_e) \tag{2.23}$$

where V_D is the volume of the unit S^{D-1} sphere in Euclidean $\mathbb{R}^D$ (see Appendix A). Note there is an electric field but no magnetic field. [???? ♣ Do we need to orient H? ♣]

2. A basic physical example is the electron in Maxwell theory. The electron is structureless, so far as we know, except for its intrinsic spin and charge, but in some theories, like string theory, the brane carries its own geometric structures on its worldvolume $\mathcal{W}$.

3. Now, in the case of an electrically charged brane we can say there is a constant charge density $q_e \in H^0(\mathcal{W})$ on the brane and then we define the *charge of the brane* to be the class:

$$[J_e] = \iota_*(q_e) \in H^{n-\ell+1}_{dR}(M_n, M_n - \mathcal{W}) \tag{2.24}$$

is the pushforward to relative cohomology. Note that to apply the pushforward we need a Thom class on the normal bundle. Thus, the normal bundle must be oriented and hence $\mathcal{W}$ must be orientable.

4. As we discuss further in Section **** below, in type II string theory there are generalized Maxwell fields where the charge is quantized by K theory. In this case the analog of the constant charge density on the brane $\mathcal{W}$ is a K-theory class $Q_{\mathcal{W}}$ and the corresponding charge is a pushforward

$$[j] = \iota_*(Q_{\mathcal{W}}) \tag{2.25}$$

The requirement that the normal bundle be suitably oriented puts nontrivial restrictions on the brane configurations. The formula is further generalized to the differential theory to provide the actual RR current.

2.5 Magnetic current

Vacuum e-m duality suggests we should consider *magnetic sources*. By definition a magnetic current is a closed $\ell + 1$ form J_m. With it we can modify the equations of motion to

$$\begin{aligned} dF &= J_m \\ d * F &= 0 \end{aligned} \tag{2.26}$$

Remarks:

1. Of course, there is a completely parallel definition of magnetic charge and its relation to magnetic flux along the lines of (2.19) and we have:

$$\mathcal{Q}_m \cong H^{\ell}_{dR}(M_n^{-J_m})/\iota H^{\ell}_{dR}(M_n) \tag{2.27}$$

2. There is a parallel notion of magnetically charged brane. This will give a current which is proportional to a delta-function supported Poincaré dual for worldvolume $\mathcal{W}_m$ of a p-brane for

$$p = p_m := n - \ell - 2. \tag{2.28}$$

(Just change $\ell \to n - \ell$ in (2.21).) Again we can write the magnetic charge

$$[J_m] = \iota_*(q_m) \tag{2.29}$$

where $q_m \in H^0(\mathcal{W}_m)$ is the constant charge density along the brane worldvolume and ι_* is the pushforward to relative cohomology.

3. **The basic example**: We consider again $M_n = \mathbb{R}^{1,n-1}$ with standard Minkowskian metric and orientation $dt \wedge d^{n-1}x$. The brane at a fixed time fills a p_m-dimensional hyperplane H_m in Euclidean space so $\mathcal{W} = \mathbb{R} \times H_m$. Let

$$D_m = n - (p_m + 1) = \ell + 1 \tag{2.30}$$

Let $\omega_\perp$ denote the unit-normed volume form of the unit sphere in $H_m^\perp \cong \mathbb{R}^{D_m}$. Then

$$F = q_m \omega_\perp \tag{2.31}$$

Note that the electric field is zero and only the magnetic field is nonzero. Moreover,

4. As the basic example shows, on the complement of SuppJ_m we can define A with $F = dA$ locally. However there is no globally well-defined A and hence the field is topologically nontrivial.

3. Dirac Quantization

Everything we have done until now has been for the classical theory. When we introduce quantum mechanics we discover that charges and fluxes must be quantized. As we discuss further below, this will be interpreted as saying that the charge and flux groups should be lattices in the classical charge and flux groups.

3.1 Dirac quantization I: Quantization from the quantization of angular momentum

Dirac quantization comes when we include quantum mechanics.

In quantum mechanics angular momentum is a generator of $SU(2)$ and quantum states must be in representations of $SU(2)$ when that is a symmetry.

Therefore, let us consider the angular momentum of the electromagnetic field created by the presence of electric and magnetic branes.

This argument only works for $n \geq 4$.

3.1.1 Pair of dyons in $\mathbb{R}^{1,3}$ in Maxwell's theory

Consider pair of an electron and a monopole in $\mathbb{R}^3$. Draw field lines and see that there is nontrivial spin in the electromagnetic field from $\vec{E} \times \vec{B}$. Get Dirac quantization.

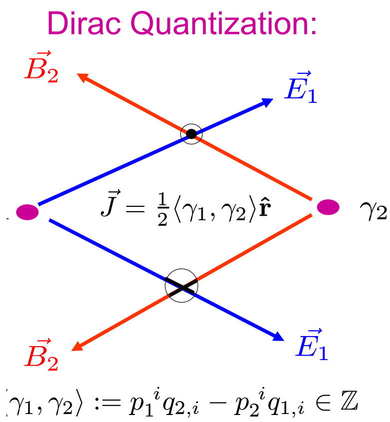

Figure 1: A pair of dyons in four-dimensions produces an electromagnetic field with spin around their axis.

Simple generalization: Two dyons of (magnetic, electric) charges (q_{m_i}, q_{e_i}) $i = 1, 2$ in $\mathbb{R}^{1,3}$. Computing the angular momentum of the electromagnetic field around the midpoint separating them

$$\vec{J}_{ij} = \int_{\mathbb{R}^3} d^3\vec{x} \left(x_i(\vec{E} \times \vec{B})_j - x_j(\vec{E} \times \vec{B})_i \right) \tag{3.1}$$

Find

$$\vec{J} = \frac{1}{c}(q_{m_1} q_{e_2} - q_{m_2} q_{e_1})\hat{r} \tag{3.2}$$

and therefore, by quantization of angular momentum

$$q_{m_1} q_{e_2} - q_{m_2} q_{e_1} = \hbar c \frac{s}{2} \tag{3.3}$$

where s is an integer.

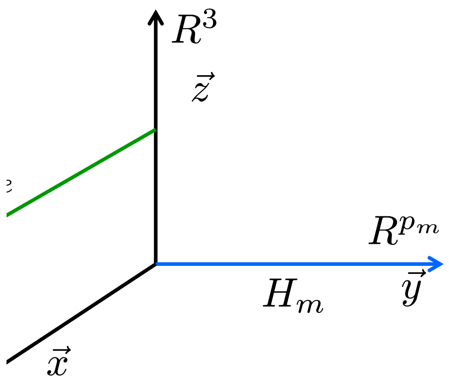

Figure 2: Two dual electric and magnetic branes at fixed time. An electric brane is located at $\vec{y} = 0$ and $\vec{z} = \vec{z}_0$, while a magnetic brane is located at $\vec{y} = 0$ and $\vec{x} = 0$. If $\vec{z}_0 = (0, 0, L) \in \mathbb{R}^3$ the field configuration produced by these branes carries angular momentum $q_e q_m$ for the generator of rotations in the 12 plane in $\vec{z}$-space.

3.1.2 Generalization

The above computation can be generalized. We take the two standard solutions mentioned above.

F_e is generated by an electric brane with $p_e = \ell - 2$ and worldvolume $\mathcal{W}_e = \mathbb{R} \times H_e$.

F_m is generated by a magnetic brane with $p_m = n - \ell - 2$ and worldvolume $\mathcal{W}_m = \mathbb{R} \times H_m$.

Note that the two branes at fixed time fill fill $p_e + p_m = (n-1) - 3$ spatial dimensions. Therefore, appear like two point particles in $\mathbb{R}^3$.

So, we divide space as $\mathbb{R}^3 \times \mathbb{R}^{p_e} \times \mathbb{R}^{p_m}$ with coordinates $(\vec{z}, \vec{x}, \vec{y})$ for the three factors.

We take H_m to be defined by the equations $\vec{x} = 0$ and $\vec{z} = 0$.

We take H_e to be defined by the equations $\vec{y} = 0$ and $\vec{z} = \vec{z}_0 = (0, 0, L)$.

Thus we have the figure FIGURE

We now compute the angular momentum in the total field

$$F = F_e + F_m = \frac{q_e}{V_{D_e}} \frac{dt d\rho_e}{\rho_e^{D_e - 1}} \mathrm{vol}\,(H_e) + q_m \omega_{H_m^\perp} \tag{3.4}$$

where $\rho_e^2 = \vec{y}^2 + (\vec{z} - \vec{z}_0)^2$

By symmetry, the only nonzero component of angular momentum is the generator of rotation in the 12 plane of the $\vec{z}$-space $\mathbb{R}^3$:

$$J_{12} = \int \mathrm{vol}\,(\mathbb{R}^{n-1})\,(z_1 T_{02} - z_2 T_{01}) \tag{3.5}$$

A fairly complicated computation (see Appendix B) shows, amazingly, that

$$J_{12} = q_e q_m, \tag{3.6}$$

and hence we recover once again Dirac quantization. ♣ FACTORS OF TWO ♣

3.1.3 Remarks

1. Case where $p_e = p_m$, $n = 2\ell$: Then we can have dyons carrying both electric and magnetic charge as in our EM example above. In general:

$$q_{m_1} q_{e_2} - (-1)^p q_{m_2} q_{e_1} = \hbar c \frac{s}{2} \tag{3.7}$$

Simple way to see that: Consider the case $\mathbb{R}^{1,3} \times T_1^p \times T_2^p$ where T^p is a p-dimensional torus. Then the first term comes from a magnetic brane wrapping T_1^p and an electric brane wrapping T_2^p while the second comes from the reversed order. But then there is an exchange of orientations when comparing the two situations $\mathrm{vol}\,(T_1^p)\mathrm{vol}\,(T_2^p) = (-1)^p \mathrm{vol}\,(T_2^p)\mathrm{vol}\,(T_1^p)$.

2. When there are several fields: Charges form a lattice with symmetric (antisymmetric) product according to $n = 0(4)$ or $n = 2(4)$.

3. ♣ BIG LEAP: was made here. We are assuming some knowledge of the quantization of the generalized Maxwell field theory. That there is a Hilbert space with some operators J_{ij} acting on it and the expression above in terms of the energy momentum tensor implements the rotational properties $[J_{ij}, \Phi] = \delta\Phi$.

3.2 Test branes and their actions: Emergence of Cheeger-Simons differential characters

Now consider a complementary picture: A test brane in external electromagnetic field

Consider the dynamics of a brane. It has an action principle $\int T \mathrm{vol}\,(h)$ given by the induced volume. T is the tension: energy per unit spatial volume.

Example: 0-branes have path integral measure

$$\exp[\frac{i}{\hbar}\int_{\mathcal{W}} T ds] \qquad ds = \sqrt{-(\frac{dx}{d\tau})^2} d\tau \tag{3.8}$$

Classically they move along geodesics. Now, if a 0-brane is electrically charged with charge q_e, it's motion is actually

$$\frac{d}{d\tau}\left(T\frac{dx_\mu}{ds}\right) = q_e F_{\mu\nu}\frac{dx^\nu}{d\tau}. \tag{3.9}$$

What term in the action reproduces this? We must modify

$$e^{\frac{i}{\hbar}\int_{\mathcal{W}} T ds} \chi(\mathcal{W}) \tag{3.10}$$

where $\chi(\mathcal{W})$ is a $U(1)$-valued function of the worldline $\mathcal{W}$.

1. Locality says it is in $Hom(Z_1(M), U(1))$
2. Equations of motion imply that if we vary $\mathcal{W} \to \mathcal{W}'$ so that $\mathcal{W}' - \mathcal{W} = \partial\mathcal{B}$ then

$$\chi(\mathcal{W}') = \chi(\mathcal{W}) e^{2\pi i \int_{\mathcal{B}} q_e F} \tag{3.11}$$

This is the original definition of a Cheeger-Simons character.

The argument generalizes to higher dimensions.

Thus, the coupling of the worldvolume of an electrically charged brane to the background gauge field defines a differential character, more precisely, $q_e F$ must be the fieldstrength of a differential character.

Remarks:

1. Physicists write
$$\chi(\mathcal{W}_p) = \exp 2\pi i \int_{\mathcal{W}_e} q_e A \tag{3.12}$$
so that the total action is
$$S = \int_{\mathcal{W}_e} T ds + 2\pi \int_{\mathcal{W}_e} q_e A \tag{3.13}$$

2. Of course, even with flat fields the extra phase $\chi(\mathcal{W}_e)$ has profound effects in quantum mechanics: The Aharonov-Bohm effect.

3.

3.3 Dirac Quantization II: Quantization from probe brane electric coupling

This is Dirac's original argument, in modern language.

We consider a magnetic brane with $J_m = q_m \delta(\mathcal{W}_m)$. This produces a definite external field F, as we have seen.

Next try to formulate the quantum mechanics of an electric brane with worldvolume $\mathcal{W}_e$ confined to move in $M_n - \mathrm{Supp} J_m$.

The problematic factor in the probe brane action
$$\exp 2\pi i \int_{\mathcal{W}_e} q_e A \tag{3.14}$$
is only well-defined in the presence of the magnetic brane if $q_e q_m \in \mathbb{Z}$. ♣ FIX FACTOR OF TWO HERE ♣

(Proof: Consider $\mathcal{W}_m$ to be located at $\vec{x} = 0$ and $\vec{z} = 0$ as in Figure ***. Consider a worldvolume of the form $\mathbb{R}^{p_e} \times \{z^a(\tau)\}$ where $z^a(\tau)$ is some trajectory γ in $\mathbb{R}^3$ not passing through $z = 0$ and $\mathbb{R}^{p_e}$ is the plane at $\vec{y} = 0$. We can fill in $\gamma = \partial D_\pm$ with two disks in the usual way. Then we claim that the discrepancy in the expnential is
$$\int_{\mathbb{R}^{p_e} \times S^2} q_e q_m \omega_{p_e+2} = q_e q_m \tag{3.15}$$
where ω_{p_e+2} is the unit volume form for the sphere in the transverse space $\mathbb{R}^{p_e} \times \mathbb{R}^3$. .... ♣ Some details to show here! ♣)

Remarks:

1. the normalization of charge used in the two arguments is slightly different, so we should not worry about the factor of two discrepancy.

2. In units where $e^2/\hbar c = 1/137$ is dimensionless and $g^2/(\hbar c)$ is dimensionless the correct normalization for the Dirac condition is

$$\frac{g^2}{\hbar c}\frac{e^2}{\hbar c} = \frac{s^2}{4} \qquad s \in \mathbb{Z} \tag{3.16}$$

and hence in nature the magnetic fine structure constant $\frac{g^2}{\hbar c} \sim \frac{137}{4}s^2$ would be rather large, and have large effects. Mysteriously, no magnetic monopoles have been observed in nature.

3.4 The quantized charge and flux group

What conclusions can we draw from the Dirac quantization condition?

In the above brane setup we have

$$\begin{aligned} \mathcal{Q}_e &= H^{n-\ell+1}(M_n, M_n^{-J_e}) \\ &\cong H^{n-\ell+1}(H_e^{\perp}, 0) \\ &\cong \mathbb{R} \end{aligned} \tag{3.17}$$

and similarly $\mathcal{Q}_m \cong \mathbb{R}$, so the classical charge group is $\mathbb{R} \oplus \mathbb{R}$.

Now, the set of *allowed charges* should be some subset of $\mathcal{Q}_e \oplus \mathcal{Q}_m \cong \mathbb{R} \oplus \mathbb{R}$ subject to two physical conditions:

- The set of allowed charges should be a subgroup. In particular, we have the "superposition principle": If q_e and q'_e are allowed charges then $aq_e + bq'_e$ are allowed charges for any $a, b \in \mathbb{Z}$. This follows from the linearity of the Maxwell equations.
- Dirac quantization: $q_e q_m \in \mathbb{Z}$ for any two allowed magnetic and electric charges.

We claim that Dirac quantization implies that the allowed values of electric and magnetic charges must form a lattice subgroup of the classical charge group $\mathbb{R} \oplus \mathbb{R}$. This is proved as follows:

For simplicity assume that both $\mathcal{Q}_e$ and $\mathcal{Q}_m$ are rank one, so the group of classical electric and magnetic charges is $\mathbb{R} \oplus \mathbb{R}$. Then we claim that: *Among the allowed positive charges there is a minimal one q_e^0 of which all other allowed electric charges are integral multiples.* To prove this consider two allowed charges q_e^0 and q_m^0 so that $q_e^0 q_m^0 = N_0$ for some nonzero integer N_0. Then, if q_e^1 is some other allowed charge it follows from Dirac quantization that $q_e^1/q_e^0 = s/t$ for some fraction s/t in lowest terms. Then by the superposition principle it follows that q_e^0/t is an allowed charge. But then by Dirac quantization N_0/t must be an integer. Thus, t is bounded and we can take q_e^0 so that $t = 1$, but then we have just learned that all other electric charges are integral multiples of q_e^0. Parallel arguments apply for the magnetic charges.

Similar arguments show that in general that the set of allowed charges should form a lattice subgroup of $\mathcal{Q}_e \oplus \mathcal{Q}_m$. What should this lattice subgroup be?

One natural guess is that it should be the image in relative deRham cohomology of

$$H^{\ell+1}(M, M^{-J_m}; \mathbb{Z}) \oplus H^{n-\ell+1}(M, M^{-J_e}; \mathbb{Z}) \tag{3.18}$$

meaning that, the group of allowed charges, as defined above, would be the image of this integral cohomology in the deRham group.

We will indeed investigate the quantum mechanics of the theory with charge quantization (6.1). However, we should not jump to conclusions that this is the only possibility. Any generalized cohomology theory can be mapped into deRham cohomology and will produce a lattice subgroup of deRham cohomology. It might be that the physics calls for other cohomology theories.

Example: The analog of the Cheeger-Simons coupling for a structureless brane but in type II string theory is the following: Suppose the RR field is topologically trivial so we may write the fieldstrength as

$$G = (d + H)C \in \Omega^{ev/od}(M_{10}) \tag{3.19}$$

then the worldvolume of a test brane will be $\mathcal{W}_{p+1}$ where p is ev/odd for IIA/IIB. Then the coupling is computed from perturbative string theory, or by anomaly computations to be

$$\exp 2\pi i \int_{\mathcal{W}_{p+1}} \left\{ e^{B} \mathrm{Tr} e^{F(\nabla)} \sqrt{\hat{A}} \right\} C \tag{3.20}$$

where

- $H = dB$ is trivialized on $\mathcal{W}_{p+1}$.
- ∇ is a connection on a vector bundle (the "Chan-Paton bundle") on $\mathcal{W}_{p+1}$.

Thus, the analog of $q_e \in H^0(\mathcal{W})$ becomes the expression in curly brackets. The point is that there is more structure on the branes than the structureless branes we considered above. The branes in string theory carry on them vector bundles with connections (more generally twisted vector bundles with twisted connections). Then, as we have said, the resulting RR charge in spacetime is

$$q_e(\check{j}_e) = \iota_*(q_{\mathcal{W}})) \tag{3.21}$$

where $q_{\mathcal{W}}$ is some (twisted) K-theory class of the bundle on the brane. Clearly, the image in deRham cohomology differs from the image of integral cohomology.

What does Dirac quantization say about the allowed fluxes? We would like to preserve the discussion of the relation between charges and fluxes involving the LES of relative cohomology. ("The charge is measured by the flux at infinity")

Therefore, a very natural proposal is that the quantum group of fluxes should be likewise a lattice subgroup of the classical group of fluxes $H^{\ell}_{dR}(M) \oplus H^{n-\ell}_{dR}(M)$, and moreover, whatever generalized cohomology group quantized the charges should also be used to quantize the fluxes.

For example, if we assume that the charge group is quantized by (6.1) then the quantum flux group would seem to be

$$H^{\ell}(M;\mathbb{Z}) \oplus H^{n-\ell+1}(M;\mathbb{Z}) \tag{3.22}$$

But this statement must be treated with care!

At first sight one might assume that (3.22) means that the DeRham cohomology class $[F]$ should be in the image of the inclusion

$$H^{\ell}(M;\mathbb{Z}) \to H^{\ell}(M;\mathbb{R}) \cong H^{\ell}_{dR}(M) \tag{3.23}$$

But this raises the immediate problem that the same cannot be simultaneously true for $[*F]$ in general, since the deRham cohomology class of $[*F]$ varies continuously as the metric varies.

We will resolve this puzzle, for generalized Maxwell theory, below in Section 6, where we will also discover some further subtleties in the statement (3.22).

4. The Generalized Maxwell Field as valued in Differential Cohomology

At this point we have argued that

- The electric coupling of (structureless) branes naturally leads to Cheeger-Simons differential characters.
- Charges and fluxes, as measured by differential form fieldstrengths, should live in some discrete lattice in deRham cohomology.

In view of these it is natural to try to define a quantum generalized Maxwell theory where we take the set of isomorphism classes of the generalized Maxwell field to be $\check{H}^{\ell}(M)$.

4.1 Translation to Physics Terminology

The differential cohomology group famously satisfies two compatible exact sequences:

$$0 \to \overbrace{H^{\ell-1}(M;\mathbb{R}/\mathbb{Z})}^{\text{flat}} \to \check{H}^{\ell}(M) \overset{\text{fieldstrength}}{\longrightarrow} \Omega^{\ell}_{\mathbb{Z}}(M) \to 0$$

$$0 \to \underbrace{\Omega^{\ell-1}(M)/\Omega^{\ell-1}_{\mathbb{Z}}(M)}_{\text{Topologically trivial}} \to \check{H}^{\ell}(M) \overset{\text{char.class}}{\longrightarrow} \underbrace{H^{\ell}(M;\mathbb{Z})}_{\text{Topological sector}} \to 0$$

The flat fields form a compact abelian group so we also have the useful exact sequence:

$$0 \to \overbrace{H^{\ell-1}(M;\mathbb{R})/H^{\ell-1}(M;\mathbb{Z})}^{\text{Wilson lines}} \to H^{\ell-1}(M;\mathbb{R}/\mathbb{Z}) \longrightarrow \underbrace{\mathrm{Tors}(H^{\ell+1}(M;\mathbb{Z}))}_{\text{Discrete Wilson lines}} \to 0$$

As indicated above, in physics we identify

1. $c \in H^\ell(M_n;\mathbb{Z})$: Dirac's quantization. We will refer to it as the *characteristic class* or *topological class.*

2. $F \in \Omega^\ell(M_n)$: Maxwell's fieldstrength Our normalization is that F will have *integral* periods, so that

$$c_{\mathbb{R}} = [F] \tag{4.1}$$

under the isomorphism $H^\ell(M;\mathbb{R}) \cong H^\ell_{dR}(M)$.

3. $H^\ell(M_n;\mathbb{R})/H^\ell(M_n;\mathbb{Z})$: Wilson lines.

4. Noncanonically $\check{H}^\ell(M)$ is a product of abelian groups of the form

$$T \times \Gamma \times V \tag{4.2}$$

where T is a connected torus, (the torus of Wilson lines), Γ is a discrete group, (the group of topological sectors $H^\ell(M;\mathbb{Z})$) and V is an infinite dimensional vector space. Physically, it corresponds to the "oscillator modes" of the field. It can be taken to be isomorphic to $\mathrm{Im} d^\dagger$.

Remarks

1. One thing which is not often appreciated by physicists that $\check{H}^\ell(M)$ is an abelian group. We will exploit this group structure in our discussion of the Hilbert space below in Section 6.

4.2 Examples

We now look at three simple examples. We will study more examples later.

4.2.1 $\ell = 1$: Periodic scalar

For $\ell = 1$, $\check{H}^1(M) = Map(M, U(1))$ is the space of identified with a periodic scalar fields on M: $\varphi : M \to U(1)$. $F = \frac{1}{2\pi i}\varphi^{-1}d\varphi$ is the fieldstrength. The integral periods are the winding numbers of φ around 1-cycles and are measured by the characteristic class $c \in H^1(M;\mathbb{Z})$. Flat field is the constant field φ, a constant phase. To be even more explicit, if we take $M = S^1$ then $\check{H}^1(S^1) = LU(1)$ is the famous loop group. Then

$$\varphi(\sigma) = \exp\left[2\pi i\phi_0 + 2\pi i w\sigma + \sum_{n\neq 0}\frac{\phi_n}{n}e^{2\pi i n\sigma}\right] \tag{4.3}$$

where $\sigma \sim \sigma + 1$ is a coordinate on the circle, illustrating very explicitly the meaning of the decomposition $T \times \Gamma \times V$ and identifying V with oscillator modes.

Now that fluxes are quantized the value of λ is meaningful. In this case we have a field mapping M to a circle of radius R and $\lambda = R^2$ is the radius of a circle.

4.2.2 $\ell = 2$: Quantum Maxwell

In this casee $\check{H}^2(M)$ is the set of isomorphism classes of principal $U(1)$ bundles over M with connection. c is the first chern class, $F = F(\nabla)$ is the curvature form of the connection ect.

In this case $\lambda = e^{-2}$ is the electromagnetic coupling.

4.2.3 $\ell = 3$: Gerbe connections

Appears in the WZW theory. The field theory of these is important in $n = 6, 10, 26$ because of the B field of string theory. Electric sources are strings.

See D.Freed Talk II for important refinement on superstring B-field.

4.3 The Groupoid of Fields

The differential cohomology summarizes the physical information in the generalized Maxwell field. However, in physics it is also important to have a notion of *locality*. For example, in the Segal axioms, if we glue together two cobordisms when we would also like to be able to glue together field configurations.

FIGURE OF JOINING COBORDISMS

In order to introduce locality we should consider the generalized Maxwell field to be an element of a groupoid – that is, we should consider the theory to be a *gauge theory*. In the groupoid the morphisms are gauge transformations and the set of isomorphism classes is $\check{H}^\ell(M)$.

It will be very important for us that all the objects have the automorphism group $H^{\ell-2}(M; U(1))$. Thus, we should actually regard $\check{H}^\ell(M)$ as a *stack*.

Example: As an example, one place where this is quite important is in the quantum definition of electric charge of a state. If spacetime is $M = \mathbb{R} \times S$ then there is a Hilbert space of quantum states $\Psi(\check{A})$. The automorphism group $H^{\ell-2}(S; \mathbb{R}/\mathbb{Z})$ acts on this Hilbert space and the characters define electric charge sections, since we interpret the automorphism group as the group of global gauge transformations:

$$\alpha \cdot \Psi(\check{A}) = e^{2\pi i \langle \alpha, Q \rangle} \Psi(\check{A}). \tag{4.4}$$

where $Q \in H^{n-\ell+1}(S; \mathbb{Z})$.

Examples of such groupoids include:

Example 1: The periodic scalar does not have automorphisms so we can take $\check{Z}^1(M) = Map(M, U(1))$. The only morphisms are the identity we we can take $\check{H}^1(M)$ to be a space, not a groupoid. ♣ CHECK! ♣

Example 2: One very useful model for $\check{Z}^2(M)$ is the groupoid of principal circle bundles with connection over M. The morphisms are isomorphisms of principal bundles (*not* necessarily preserving connection!).

Unlike nonabelian Yang-Mills theory, which is based on the groupoid of principal bundles with connection on M in the generalized abelian gauge theories there does not appear to be a distinguished geometric model for the groupoid of fields. Rather, at least for $\ell > 2$ there are several different equivalent groupoids no one of which seems particularly distinguished.

One example of a groupoid in the literature is the groupoid of cocycles $\check{Z}^\ell(M)$ constructed by Hopkins and Singer. We will use it below.

However, for physics, one would like to have a smaller groupoid which involves only continuous geometrical objects (as opposed to the integral and real cochains appearing in the Hopkins-Singer construction).

The way physicists (implicitly) think about these gauge theories is the following:

As a manifold, the differential cohomology group is a union of components, labeled by $a \in H^\ell(M;\mathbb{Z})$ each component being the quotient

$$\Omega^{\ell-1}(M)/\Omega_{\mathbb{Z}}^{\ell-1}(M) \tag{4.5}$$

In each component we choose a basepoint. E.g. for each $c \in H^\ell(M;\mathbb{Z})$ we choose a fieldstrength F_c and all other fieldstrengths in that component are

$$F = F_c + da \tag{4.6}$$

Thus we consider the "gauge field" in this component to be an ordinary form $a \in \Omega^{\ell-1}(M)$, globally defined, but subject to gauge transformations:

$$a \to a + \omega \qquad\qquad \omega \in \Omega_{\mathbb{Z}}^{\ell-1}(M) \tag{4.7}$$

When ω is exact then the physicists refer to these as *small gauge transforamtions* and when it is not exact they are called *large gauge transformations.*

Question for the Mathematicians: It would be nice if there were a groupoid $\check{Z}^\ell$ which had the properties:

1. The objects form a manifold with components $\check{Z}_c^\ell(M)$ labeled by $c \in H^\ell(M;\mathbb{Z})$.
2. Each component is the groupoid given by the action of a group on a manifold. Consideration of "open Wilson lines" suggests that this gauge group should be $\check{H}^{\ell-1}(M)$ so we would like the groupoid to be of the form $\check{Z}_c^\ell(M)//\check{H}^{\ell-1}(M)$, where the flat fields act trivially.

Notation: For the above reasons our notation for the differential cohomology class of a generalized Maxwell field will be $[\check{A}]$. It is meant to suggest that there is a "gauge potential" $\check{A}$ generalizing the usual one, which is an object in a groupoid - which we will call the groupoid of differential cocycles.

4.4 The Canonical Pairing

In the physics literature one often encounters terms in Lagrangians that look like

$$\int A_1 dA_2 \tag{4.8}$$

where A_i are differential form potentials for an abelian gauge theory. Of course, the most interesting situations occur when A_i are not globally well-defined. We comment on how this is (incorrectly) handled in the physics literature below. The correct procedure is the following.

1. The (rather subtle) graded ring structure

$$\check{H}^{\ell_1} \times \check{H}^{\ell_2} \to \check{H}^{\ell_1+\ell_2} \tag{4.9}$$

The fieldstrength and characteristic class multiply in the expected way:

$$F(\check{\chi}_1 \cdot \check{\chi}_2) = F(\check{\chi}_1) \wedge F(\check{\chi}_2) \qquad c(\check{\chi}_1 \cdot \check{\chi}_2) = c(\check{\chi}_1) \cup c(\check{\chi}_2) \tag{4.10}$$

The formula for the product of the holonomy is more subtle.

♣ GIVE SOME EXAMPLES. There is a nice description of the multiplication $\check{H}^1 \times \check{H}^2 \to \check{H}^2$ where we pull back a standard bundle with connection on $S^1 \times S^1$. It should have first Chern class = 1 and translation invariant fieldstrength and holonomy – ???? Are there similar nice models for low degree examples?? ♣

2. The theory of integration. In general if $\mathcal{X} \to \mathcal{P}$ is a family of compact oriented manifolds M of dimension n then we can define:

$$\int_{\mathcal{X}/\mathcal{P}}^{\check{H}} : \check{H}^s(\mathcal{X}) \to \check{H}^{s-n}(\mathcal{P}) \tag{4.11}$$

3. Now, using the fact that $\check{H}^1(pt) = \mathbb{R}/\mathbb{Z}$ we have the canonical pairing

$$\check{H}^{\ell} \times \check{H}^{n+1-\ell} \to \mathbb{R}/\mathbb{Z} \tag{4.12}$$

defined by

$$\langle [\check{A}_1], [\check{A}_2] \rangle := \int^{\check{H}} [\check{A}_1] \cdot [\check{A}_2] \tag{4.13}$$

4. An important special case of the pairing: If $[\check{A}_1]$ is topologically trivial then we may represent it by some $A_1 \in \Omega^{\ell_1 - 1}$ and then the pairing only depends on the fieldstrength of $[\check{A}_2]$: so pairing is

$$\langle [\check{A}_1], [\check{A}_2] \rangle = \int_M A_1 F_2 \mathrm{mod} \mathbb{Z} \tag{4.14}$$

so, in particular, if $[\check{A}_2]$ is also topologically trivial then

$$\langle [\check{A}_1], [\check{A}_2] \rangle = \int_M A_1 dA_2 \mathrm{mod} \mathbb{Z} \tag{4.15}$$

5. Another useful special case: $[\check{A}_1]$ is flat then it is represented by a class $\alpha_1 \in H^{\ell_1 - 1}(M, \mathbb{R}/\mathbb{Z})$ and then the pairing only depends on the characteristic class a_2 of $[\check{A}_2]$ and is given by

$$\langle [\check{A}_1], [\check{A}_2] \rangle = \int_M \alpha_1 a_2 \in \mathbb{R}/\mathbb{Z} \tag{4.16}$$

♣ INTRODUCE SEPARATE NOTATION FOR $U(1)$-VALUED PAIRING? ♣

6. Another useful special case arises when the differential characters $[\check{A}_1]$ and $[\check{A}_2]$ extend to a manifold $\mathcal{B}$ so that $\partial\mathcal{B} = M$. In this case we can apply a version of Stokes' theorem which says that

$$\int_M^{\check{H}} \chi = \int_{\mathcal{B}} F(\chi) \mathrm{mod}\mathbb{Z} \tag{4.17}$$

(and similarly in families).

7. Let us return to the way (4.8) is usually discussed in the physics literature. What is usually done is this: One assumes that $M = \partial\mathcal{B}$ is the boundary of some space and that it is possible to extend the fieldstrengths F_1 and F_2 to $\tilde{F}_1$ and $\tilde{F}_2$ on $\mathcal{B}$, and then we take $\int_{\mathcal{B}} \tilde{F}_1 \tilde{F}_2$. This is wrong for two reasons. First, the approach is limited by the existence of an appropriate cobordism, and in general such cobordisms do not exist. Moreover, and more seriously, even when such extensions exist in fact this definition is not acceptable since $\tilde{F}_i$ can be perturbed by forms with compact support on $\mathcal{B}$ away from the boundary. In general such perturbations will really change the value of the integral! What is true, and what follows from (4.10) and (4.17) is that if we extend the *differential characters* $[\check{A}_1]$ and $[\check{A}_2]$ from M to $\mathcal{B}$ when we can use the fieldstrengths $\tilde{F}_i$ of those extended characters to define the pairing to be $\int_{\mathcal{B}} \tilde{F}_1 \tilde{F}_2 \mathrm{mod}\mathbb{Z}$.

In the following sections we are going to show that the pairing on differential cohomology is very useful in writing actions, and we'll comment on some normalization issues.

Then we will show how the pairing is useful for formulating the quantum mechanical Hilbert space.

4.5 Generalized Maxwell-Chern-Simons Theory

The canonical pairing on differential cohomology classes allows us to introduce an important extension of generalized Maxwell theory in odd dimensions. Many applications involve higher rank theories so we consider this case. The action is

$$S = \pi \int \lambda_{ij} F^i * F^j + \pi \int^{\check{H}} k_{ij} [\check{A}^i] \cdot [\check{A}^j] \tag{4.18}$$

There are two separate cases

A. $n = 4s + 3$, $\ell = 2s + 2$:

With k_{ij} a symmetric matrix of integers.

Examples:

a.) $s = 0$: In AdS3 compactifications one typically gets such theories. Also relevant to QHE,

b.) $s = 1$: Holographic dual to $(2,0)$ theory.

Important point: Physics requires normalization of k_{ij} so that k_{ii} can be *odd*: Therefore we require extra structure such as spin structure ($s = 0$) or, for $s \geq 1$ an integral Wu structure according to Hopkins and Singer.

B. $n = 4s + 1$, $\ell = 2s + 1$:

Now k_{ij} is an antisymmetric matrix of integers.

Example: Type IIB String theory on $AdS5 \times M_5$.

4.6 $n = 11$, $\ell = 4$: Abelian gauge field of M-theory

Eleven-dimensional supergravity is very rigid: There is a unique supermultiplet and it contains an abelian 3-form gauge potential C with closed four-form fieldstrength $G \in \Omega^4_d(M)$. For topologically trivial C-fields, $C \in \Omega^3(M)$, the C-field enters the action through the terms: [3]

$$\exp\left[i\pi \int_M \lambda G * G + 2\pi i \int_M \left(\frac{1}{6} CGG - CI_8(g)\right)\right] \tag{4.19}$$

where $I_8(g)$ is an 8-form made from traces of the Ricci 2-form and representing

$$I_8 = \frac{4p_2 - p_1^2}{4 \cdot 48} \tag{4.20}$$

It is natural to try to represent the C field in terms of differential cohomology $\check{H}^4(M)$. However, the cubic term here would seem to be

$$\exp\left[2\pi i \frac{1}{6} \int^{\check{H}} [\check{C}] \cdot [\check{C}] \cdot [\check{C}]\right] \tag{4.21}$$

But this looks like nonsense since $\int^{\check{H}} [\check{C}] \cdot [\check{C}] \cdot [\check{C}]$ is only defined in $\mathbb{R}/\mathbb{Z}$. How can you divide by 6??

The remarkable story of how this apparent anomaly in 11-dimensional supergravity is resolved is explained in Section 11

5. Partition Functions

We now continue our investigation of the quantum theory, now interpreting the generalized Maxwell field as quantized by $\check{H}^\ell(M)$.

Strictly speaking, to define the "theory" we should consider the bordism category with background fields $g_{\mu\nu}, \check{j}_e, \check{j}_m$ coupling λ and really construct the anomalous field theory. (See Section **** for why it is an anomalous field theory and the meaning of some of these words.)

In this Section we settle for just understanding the partition function in the absence of currents.

5.1 The partition function

In the quantum theory, an important role is played by the *partition function.*

Let us assume M is compact and has Euclidean signature metric. For generalized Maxwell theory the partition function is then

$$Z(M_n; \lambda, g) = \int_{\check{H}^\ell(M_n)} \mu(\check{A}) e^{-S} \tag{5.1}$$

[3]The fieldstrength also couples to bilinears in the gravitinos. Thus it also enters in a Dirac operator coupled to a kind of superconnection which is linear in G.

We are working in Euclidean signature here. Moreover, $\mu(\check{A})$ is a naturally defined translation invariant measure on $\check{H}^\ell(M)$. It can be defined, formally, by observing that $T^*\check{H}^\ell(M_n) \cong \Omega_d^{n+1-\ell}(M_n)$, and the latter vector space inherits a natural Riemannian metric from the metric g on M_n. Formally, $\mu(\check{A})$ is the associated volume form.

The path integral (5.1) is a Gaussian path integral and is straightforward to evaluate:

$$Z_\ell(M_n;\lambda,g) = T_\ell \mathcal{N}_\ell \Theta_\ell \tag{5.2}$$

The theta function, the first of many we will encounter, is the sum over the classical solutions, subject to flux quantization, weighted by the classical action:

$$\Theta_\ell := \sum_{f\in\mathcal{H}^\ell_{\mathbb{Z}}(M_n)} e^{-\pi\int_{M_n}\lambda f * f} \tag{5.3}$$

The sum is over harmonic forms with integral periods.

The prefactor $\mathcal{N}_\ell(g)$ is the Gaussian integral on the quadratic fluctuations around the classical solutions. It is slightly elaborate to evaluate because there are gauge-invariances for gauge invariances ("ghosts for ghosts"). The result is similar to analytic torsion, but does have nontrivial smooth variation with the metric:

$$\log \mathcal{N}_\ell = \frac{1}{2}\sum_{s=0}^{\ell-1}(-1)^{\ell-s}\log\left(\frac{L_s}{V_s^2}\right) \tag{5.4}$$

$$L_s = \det(d^\dagger d|_{\Omega^s\cap \mathrm{Im}\ d^\dagger}) \tag{5.5}$$

$$V_s = \mathrm{vol}\,(\mathcal{H}^s/\mathcal{H}^s_{\mathbb{Z}}) \tag{5.6}$$

Finally $T_\ell := |TorsH^{\ell+1}(M_n;\mathbb{Z})|$ is the order of the torsion group. This is an overall factor since shifting the field by a topologically nontrivial flat field does not change the action.

5.2 Quantum Electric-Magnetic duality in vacuum

Back to electromagnetic symmetry of the vacuum equations: Justify this quantum mechanically.

Now, with quantized fluxes, the coupling constant λ in the actions above become meaningful. We will argue that in the quantum theory electromagnetic duality between a generalized Maxwell theory of an ℓ-form and an $n-\ell$-form requires

$$\lambda\tilde{\lambda} = \hbar^2 \tag{5.7}$$

In terms of the partition function in the absence of currents this means that:

$$\frac{Z_\ell(M;\lambda,g)}{\mathcal{V}_\ell} = \frac{Z_{n-\ell}(M;\lambda^{-1},g)}{\mathcal{V}_{n-\ell}} \tag{5.8}$$

where we will find it useful to introduce the following notation: We denote the group of flat fields by

$$\mathcal{T}^\ell(M) := H^{\ell-1}(M;\mathbb{R}/\mathbb{Z}) \tag{5.9}$$

or just $\mathcal{T}^\ell$ for short. Using the metric $g_{\mu\nu}$ this compact abelian group inherits a measure from the identification of the connected component with $\mathcal{H}^{\ell-1}(M)/\mathcal{H}^{\ell-1}_{\mathbb{Z}}(M)$. Denote the volume by $\mathcal{V}_\ell$.

One can give a direct proof of (5.8) along the following lines.

1. One relates $\mathcal{N}_\ell$ to $\mathcal{N}_{n-\ell}$ using standard Hodge theory.
2. Then the relation between the theta functions can be separately verified using the Poisson summation formula.

   ♣ Carry it out. There are some extra prefactors of λ which should be cleaned up. ♣

However, there is a formal argument, much loved by the physicists because it gives further insight, and in principle can be used to establish the full duality of theories - that is, the duality for all amplitudes, not just the partition function.

We can given the standard argument (upgraded to include differential cohomology) [4] as follows:

We begin by considering the path integral

$$\mathcal{Z} = \frac{1}{\mathrm{vol}\,\check{H}^\ell}\int_{\check{H}^\ell}\mu(\check{A})\int_{\check{H}^{n-\ell}}\mu(\check{A}_D)\int_{\Omega^\ell}\mu(G)e^{-\pi\int\lambda\mathcal{F}*\mathcal{F}+2\pi i\int GF(\check{A}_D)} \tag{5.10}$$

where $\mathcal{F} := F(\check{A}) - G$. There is a "gauge invariance" where we shift $\check{A}$ by an arbitrary element of $\check{H}^\ell$, and then shift G by the fieldstrength of that element, so we have divided by the volume of this "gauge group."

Now, if we first integrate over G, we are doing a Gaussian integral on a vector space. The saddle point value is $F(\check{A}) - G = i\lambda^{-1} * F(\check{A}_D)$ and after doing the Gaussian integral we find

$$\begin{aligned}\mathcal{Z} &= \frac{1}{\mathrm{vol}\,\check{H}^\ell}\int_{\check{H}^\ell}\mu(\check{A})\int_{\check{H}^{n-\ell}}\mu(\check{A}_D)e^{-\pi\int\lambda^{-1}F(\check{A}_D)*F(\check{A}_D)+2\pi i\int F(\check{A})F(\check{A}_D)} \\ &= Z_{n-\ell}(M;\lambda^{-1},g)\end{aligned} \tag{5.11}$$

On the other hand, if we first integrate over $\check{A}_D$ then we get a "periodic delta function" of G which constrains G to have support on $\Omega^\ell_{\mathbb{Z}}(M)$:

$$\int_{\check{H}^{n-\ell}}\mu(\check{A}_D)e^{2\pi i\int GF(\check{A}_D)} = \mathrm{vol}\,(H^{n-\ell-1}(M,U(1)))\int_{\Omega^\ell_{\mathbb{Z}}}\mu(G_0)\delta(G-G_0) \tag{5.12}$$

Next, we do the G integral to replace G by G_0 in $\mathcal{F}$. However, now we can use the translation invariance of the measure $\mu(\check{A})$ to shift away any $G_0 \in \Omega^\ell_{\mathbb{Z}}(M)$. Therefore, after shifting away G_0 we are left with

$$\mathrm{vol}\,(\Omega^\ell_{\mathbb{Z}}) = \frac{\mathrm{vol}\,(\check{H}^\ell)}{\mathrm{vol}\,(H^{\ell-1}(M,U(1)))} \tag{5.13}$$

[4]The standard argument is in Rocek-Verlinde; Rocek and who? and Witten "Abelian S-duality"

and therefore doing the path integral $\mathcal{Z}$ in this order leaves us with:

$$\mathcal{Z} = \frac{\mathcal{V}_{n-\ell}}{\mathcal{V}_\ell} \int_{\check{H}^\ell} \mu(\check{A}) e^{-\pi\lambda \int F(\check{A}) * F(\check{A})} \tag{5.14}$$

Therefore we conclude the result (5.8).

Remarks:

Generalized theories: $Sp(2n, R)$ or $O(p, q; R)$ symmetry.

5.3 Examples

5.3.1 $\ell = 1$: Periodic scalar

$\lambda = R^2$ is the radius of a circle. In $n = 2$ the dual theory is again a periodic scalar. This is T-duality. Generalize to higher rank.

5.3.2 $\ell = 2$: Quantum Maxwell

$\lambda = e^{-2}$ is the electromagnetic coupling. In $n = 4$ the dual theory is again $\ell = 2$. This is S-duality. ♣ IMPORTANT TO GENERALIZE TO HIGHER RANK. ♣

♣ Need to clarify:

1. Powers of λ in the Gaussian integral over G. This accounts for the Im τ^χ dependence and $\tau^{\chi+\sigma}\bar{\tau}^{\chi-\sigma}$ found in Witten. Note it is there in the functional integrals from the contributions of the volumes on Harmonic tori.

2. Include currents and boundaries: Show that this basic argument can really be used to give an isomorphism of complete theories.

♣

6. Hilbert Space: EM Duality and Pontryagin-Poincare Duality

Let us see how the Hilbert space formulation respects EM duality.

We take spacetime to be $M = \mathbb{R} \times S$, where S is space, which we take to be compact and oriented.

Let us recall that the charge group was taken to be the real image in DeRham cohomology of

$$H^{\ell+1}(M, M^{-J_m}; \mathbb{Z}) \oplus H^{n-\ell+1}(M, M^{-J_e}; \mathbb{Z}) \tag{6.1}$$

Classically, in the absence of charges, to a solution of the equations of motion we could associated an element of the classical flux group

$$H^\ell_{dR}(M_n) \oplus H^{n-\ell}_{dR}(M_n) \tag{6.2}$$

and, from the LES one might rush to the conclusion that the quantum flux group – in the case where the generalized Maxwell field has isomorphism class in $\check{H}^\ell(M_n)$ – should be the real image in deRham cohomology of

$$H^\ell(M_n; \mathbb{Z}) \oplus H^{n-\ell}(M_n; \mathbb{Z}) \tag{6.3}$$

However, this statement is problematic because if $[F]$ has quantized periods then $[*F]$ will not have quantized periods for generic metrics, because if the metric varies then $[*F]$ will not remain constant, in general.

We will show how, in the careful formulation of the Hilbert space, the give a definition of quantum flux sectors which avoids this problem.

So now we proceed with the quantization.

From the action principle we derive the relation between the classical field and the canonical momentum:

$$\Pi = 2\pi\lambda(*F)|_S \tag{6.4}$$

the phase space is

$$T^*\check{H}^\ell(S) = \check{H}^\ell(S) \times \Omega_d^{n-\ell}(S) \tag{6.5}$$

and standard quantization should give $L^2(\check{H}^\ell(S))$ with respect to some measure on $\check{H}^\ell(S)$. [5]

The Heisenberg relations are as follows: Π and F become operator-valued $(n-\ell)$ and ℓ-forms, respectively and we have the relations

$$[\int_S \omega_1 F, \int_S \omega_2 \Pi] = i\hbar \int_S \omega_1 d\omega_2 \tag{6.6}$$

for differential forms ω_1, ω_2 of the appropriate degrees.

While straightforward, this raises two issues:

- Of course, we have broken manifest EM duality. Quantum EM duality suggests that we should have an isomorphic description in terms of $L^2(\check{H}^{n-\ell}(S))$. How does this work?
- There are no general rules in quantum physics for quantizing disconnected phase spaces.

We can solve both problems by exploiting the fact that $\check{H}^\ell(S)$ is an abelian group.

Recall that for an abelian group G with a translationally invariant measure, $L^2(G)$ can be viewed as the unique irrep of the Heisenberg group $Heis(G \times \widehat{G})$, where $\widehat{G}$ is the Pontryagin dual group. G acts by translation and $\widehat{G}$ acts by multiplication:

$$(T_{g_0}\psi)(g) := \psi(g + g_0).$$

$$(M_\chi\psi)(g) := \chi(g)\psi(g)$$

[5] We are going to be cavalier about issues of functional analysis here, believing that in this Gaussian theory such points can be dealt with completely rigorously. Roughly speaking our wavefunctionals should have Gaussian falloff for large fieldstrengths: $\psi(\check{A}) \sim \exp[-\int_S \kappa F * F]$ where κ is some positive constant. We choose a basic such falloff for the groundstate and construct the whole Hilbert space by action of the operators $F(\check{A})$ and Π on that state. The story should be similar to G. Segal's discussion of the Hilbert space of a massive scalar field. An important related issue we are not discussing here is the issue of polarization. We are studying representations of the Heisenberg algebra with energy bounded from below.

then

$$T_{g_0} M_\chi = \chi(g_0) M_\chi T_{g_0}.$$

from which we obtain the cocycle.

We can apply this in our case because of the beautiful fact that if S is compact and oriented then we have Poincaré-Pontryagin duality: The canonical pairing

$$\check{H}^\ell(S) \times \check{H}^{n-\ell}(S) \to \mathbb{R}/\mathbb{Z} \tag{6.7}$$

is in fact a perfect pairing. Therefore $\check{H}^{n-\ell}(S)$ is the Pontryagin dual group and we can apply the above construction.

Next, the Stone-von Neumann theorem guarantees that we have a unique irrep of this group where the central $U(1)$ acts by scalars. Therefore

The Hilbert space $\mathcal{H}(S)$ of the theory on a spatial slice S is, up to isomorphism, the unique SvN irrep of

$$\mathrm{Heis}(\check{H}^\ell(S) \times \check{H}^{n-\ell}(S))$$

with the cocycle given by canonical pairing. This is a manifestly EM dual formulation of Hilbert space!

Open Problem: What happens when S is noncompact? In this case Poincare-Pontryagin duality would be

$$\check{H}^\ell_{cpt}(S) \times \check{H}^{n-\ell}(S) \to \mathbb{R}/\mathbb{Z} \tag{6.8}$$

but this breaks $\ell \to n - \ell$ symmetry. Is there a version of "L^2 differential cohomology" with a pairing

$$\check{H}^\ell_{L^2}(S) \times \check{H}^{n-\ell}_{L^2}(S) \to \mathbb{R}/\mathbb{Z} \tag{6.9}$$

which is a perfect pairing.

Moreover, in applications to flux quantization we have finite *energy density* rather than finite energy. Nevertheless, electromagnetic duality should hold.

6.1 Quantum definition of flux and flux sectors

How shall we define electric flux quantum mechanically. Classically it is $[*F]$, quantum mechanically the canonical momentum is

$$\Pi = 2\pi\lambda(*F)|_S \tag{6.10}$$

Now, $\hbar^{-1}\Pi$ is the Hermitian generator of translations on $\check{H}^\ell$ and hence

Definition: A state ψ of definite electric field $[\check{\mathcal{E}}] \in \check{H}^{n-\ell}(S)$ is a translation eigenstate on $\check{H}^\ell(S)$, i.e.

$$\forall \check{\phi} \in \check{H}^\ell(S) \qquad \psi(\check{A} + \check{\phi}) = \exp\left(2\pi i \int_S^{\check{H}} \check{\mathcal{E}} * \check{\phi}\right)\psi(\check{A})$$

Now, states of definite electric field are not normalizable, and what we generally care about is only states of definite electric flux.

Observe that $\check{\mathcal{E}}_1$ and $\check{\mathcal{E}}_2$ are continuously connected (i.e. have the same characteristic class) if and only if

$$\int_S^{\check{H}} \check{\phi}_f * \check{\mathcal{E}}_1 = \int_S^{\check{H}} \check{\phi}_f * \check{\mathcal{E}}_2 \qquad\qquad \forall \check{\phi}_f \in H^{\ell-1}(S, \mathbb{R}/\mathbb{Z})$$

Definition: A state of definite electric flux is an eigenstate under translation by flat characters $H^{\ell-1}(S, \mathbb{R}/\mathbb{Z}) \subset \check{H}^\ell(S)$, i.e.

$$\forall \phi_f \in H^{\ell-1}(S, \mathbb{R}/\mathbb{Z}) \qquad\qquad \psi(\check{A} + \check{\phi}_f) = \exp\left(2\pi i \int e\phi_f\right)\psi(\check{A})$$

for some $e \in H^{n-\ell}(S, \mathbb{Z})$.

We have used a specific duality frame to motivate our definition, but now we can make it EM duality invariant:

The cocycle vanishes on the compact subgroup of flat fields $H^{\ell-1}(S; \mathbb{R}/\mathbb{Z})$ so we can lift it to an isomorphic subgroup in $\mathrm{Heis}(\check{H}^\ell(S) \times \check{H}^{n-\ell}(S))$. Then we define the quantum electric fluxes to be the characters of this action, so we have a grading:

$$\mathcal{H} = \oplus_{e \in H^{n-\ell}(S;\mathbb{Z})} \mathcal{H}_e \tag{6.11}$$

Similarly, we can lift the group of dual flat fields $H^{n-\ell-1}(S; \mathbb{R}/\mathbb{Z})$ to $\mathrm{Heis}(\check{H}^\ell(S) \times \check{H}^{n-\ell}(S))$ and define the quantum magnetic fluxes to be the characters of $H^\ell(S; \mathbb{Z})$:

$$\mathcal{H} = \oplus_{m \in H^\ell(S;\mathbb{Z})} \mathcal{H}_m \tag{6.12}$$

If we realize the Hilbert space as $L^2(\check{H}^\ell(S))$ then there are clearly sectors labeled by the components of $\check{H}^\ell(S)$. If $m \in H^\ell(S; \mathbb{Z})$ labels a component then a quantum state with support on this component is acted on by the multiplication operator corresponding to $\eta_m \in H^{n-\ell-1}(S; \mathbb{Z})$ by the phase $e^{2\pi i \int \eta m}$ and hence is in a state of definite magnetic flux, according to our definition.

Now, however we can observe an interesting point: The lifts of $H^{\ell-1}(S; \mathbb{R}/\mathbb{Z})$ and $H^{n-\ell-1}(S; \mathbb{R}/\mathbb{Z})$ to $\mathrm{Heis}(\check{H}^\ell(S) \times \check{H}^{n-\ell}(S))$ do *not* commute. Rather if

$\mathcal{U}_E(\eta_e)$ Translation operator by $\eta_e \in H^{\ell-1}(S; \mathbb{R}/\mathbb{Z})$

$\mathcal{U}_M(\eta_m)$ Translation operator by $\eta_m \in H^{n-\ell-1}(S; \mathbb{R}/\mathbb{Z})$

then, by our explicit cocycle we see that the group commutator is

$$[\mathcal{U}_E(\eta_e), \mathcal{U}_M(\eta_m)] = \exp\left(2\pi i \int_S \eta_e \beta \eta_m\right) \tag{6.13}$$

is the torsion pairing.

We can conclude:

The Hilbert space can be simultaneously graded by the fluxes modulo torsion

$$\mathcal{H} = \oplus_{\bar{e},\bar{m}} \mathcal{H}_{\bar{e},\bar{m}} \tag{6.14}$$

but it cannot in general be simultaneously graded by electric and magnetic fluxes.

Example 1 If we take $\ell = 1$ and $S = S^1$ so that we have the periodic scalar field on the circle then (6.14) is the usual grading by "winding" (the magnetic flux) and "momentum" (the electric flux).

Example 2 If we consider Maxwell theory, $\ell = 2$ on a Lens space $L_k = S^3/\mathbb{Z}_k$.

Flux groups $H^2(L_k;\mathbb{Z}) = \mathbb{Z}_k$ are all torsion.

Flat fields $H^1(L_k;\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_k$

The Heisenberg group extension, restricted to the flat fields is the standard Heisenberg group

$$0 \to \mathbb{Z}_k \to \mathrm{Heis}(\mathbb{Z}_k \times \mathbb{Z}_k) \to \mathbb{Z}_k \times \mathbb{Z}_k \to 0$$

♣ WHAT HAPPENS IF YOU ADD A θ-ANGLE? ♣

6.2 Experimental realization?

Since the above effect exists in ordinary Maxwell theory one might ask if it can be experimentally tested.

One immediately runs into a serious difficulty: It is not possible to have an embedded three-manifold in $\mathbb{R}^3$ with torsion in its cohomology. One proof can be found in the appendix of [6] uses the key property that $S^3 - Y$ would be another embedded 3-manifold with $X \cap Y = \partial X = \partial Y$ and $X \cup Y = S^3$. Then one combines the relative cohomology sequences for $Y, \partial Y$, together with the coefficient sequence, and excision to show that the image of $H^1(Y, \partial Y; \mathbb{R}/\mathbb{Z}) \to H^2(Y, \mathbb{Z})$ is zero.

Now in [7] it is argued that a certain combination of Josephson junctions allows one to "resolve" lines of double points of immersed three-manifolds, at least in so far as the Maxwell gauge field is concerned. Thus, using superconductors and Josephson junctions one can impose boundary conditions on the electromagnetic field which simulate Maxwell theory on a three-manifold with torsion. The groundstates of the Maxwell field should be degenerate and form a representation of the Heisenberg group. It is possible this could be useful for quantum computation.

7. Quantum theory in the presence of electric and magnetic currents

7.1 Coupling to electric current

Let us now return to the theory in the presence of purely electric current.

We can still write an action principle. $dF = 0$ so $F = dA$ locally.

Action for F in presence of external electric current:

$$\exp[i\pi \int \lambda F * F + 2\pi i \int J_e A] \tag{7.1}$$

This immediately raises some problems:

1. A is not unique. If we transform A by a small gauge transformation $A \to A + d\epsilon$ then for M closed we can integrate by parts and use $dJ_e = 0$ so all is well.

[6] Kitaev, Moore, Walker

[7] Kitaev, Moore, Walker

2. However, if we shift $A \to A + \omega$ where ω is closed with (nonzero) integral periods this will not work. We conclude that J_e should itself have integral periods.

3. Finally, A might not be globally well-defined.

We have seen that we should view A rather as a locally defined potential for a differential cohomology class $[\check{A}] \in \check{H}^{\ell}(M_n)$. Our three problems are neatly solved if we view J_e as the *fieldstrength* of a differential cohomology class [8]

$$[\check{j}_e] \in \check{H}^{n-\ell+1}(M_n) \tag{7.2}$$

♣ ?? Should it be in "compactly supported" differential cohomology? That is where the topological class was to be located. ?? ♣

So now we can invoke differential cohomology to give us a well-defined electric coupling:

$$\exp[2\pi i \langle [\check{j}_e], [\check{A}] \rangle] = \exp\left(2\pi i \int^{\check{H}} [\check{j}_e] \cdot [\check{A}]\right) \tag{7.3}$$

which reduces to the usual expression when $[\check{A}]$ is topologically trivial by (4.14).

Remarks:

1. We have two complementary views on the electric coupling. In the case when the current is sourced by an electrically charged brane $\mathcal{W}_e$ of (2.20) the electric couplings are related by:
$$\exp\left(2\pi i \int_M J_e A\right) = \exp 2\pi i \int_{\mathcal{W}_e} q_e A \tag{7.4}$$

2. As an example showing that this is a nontrivial extension of the usual coupling consider $\varphi \in \check{H}^1(M)$. We take $\dim M = 2$ so that the electric current defines a class $[\check{j}_e] \in \check{H}^2(M)$, and can be considered to be the isomorphism class of a line bundle with connection ∇, where J_e is the curvature $F(\nabla)$ of the connection ∇. For topologically trivial field configurations $\varphi = e^{2\pi i \phi}$ with a well-defined logarithm $\phi : M \to \mathbb{R}$ we can write the electric coupling as
$$\exp[2\pi i \int_M \phi F(\nabla)] \tag{7.5}$$
and we therefore recognize a form of "background charge" familiar in conformal field theory. Note that integration over the translation by a flat field $\phi_0 \in \mathbb{R}/\mathbb{Z}$ shows that the path integral is zero unless $\int_M F(\nabla) = 0$. This is the quantum mechanical implementation of the classical statement that J_e must be trivialized by the on-shell fields.

[8] At this point we are begin inconsistent with our notational convention. J_e is the fieldstrength, not the "potential" for the electric current. Since $\check{j}_e$ is external and we will never need to know its "potential" this will be ok.

3. Continuing with our example, formulating the electric coupling in terms of differential cohomology has consequences not visible in the usual naive formulation of the coupling. For example, if the electric current is such that ∇ is a flat connection, then the pairing only depends on the characteristic class of φ which gives $a \in H^1(M;\mathbb{Z})$ measuring the winding number. Choosing a basis of cycles on M the electric coupling then becomes $\prod_i h_i^{w_i}$ where h_i are the holonomies of ∇ around the basis cycles and w_i are the winding numbers.

4. If we look for the stationary points of (7.1) (where we vary F within $\check{H}^\ell$ so $\delta F = d\delta a$) then the stationary equations are

$$d * F = (-1)^{n(\ell-1)}\lambda^{-1} J_e \tag{7.6}$$

so we have redefined the normalization of electric current to be more in line with topology.

7.1.1 Partition function in the presence of electric current

In the path integral

$$Z_\ell(M;\lambda, g, \check{j}_e) = \int_{\check{H}^\ell} \mu(\check{A}) e^{-\pi\int \lambda F*F + 2\pi i \int^{\check{H}} \check{A}\cdot\check{j}_e} \tag{7.7}$$

we can shift $\check{A}$ by a flat field: The kinetic term does not change, but the second term changes under $\check{A} \to \check{A} + \check{\phi}_f$ by $\exp(2\pi i \int \phi_f c(\check{j}_e))$. Therefore, the integral over the compact abelian group $H^{\ell-1}(M;\mathbb{R}/\mathbb{Z})$ kills the partition function unless $c(\check{j}_e) = 0$. This is the quantum mechanical version of the statement that there is no solution of the equation of motion unless the electric current is trivializable.

The partition function in the presence of current is simply

$$Z_\ell(M_n, \lambda, g, \check{j}_e) = e^{-(-1)^n \pi\lambda^{-1}\int J_e * (dd^\dagger)^{-1} J_e} Z_\ell(M_n, \lambda, g) \tag{7.8}$$

♣ This is quite a surprise. I was sure it would involve a nontrivial modification of the theta function ♣

The proof of this goes as follows:

1. Action: $\exp[-\pi\int \lambda F * F + 2\pi i (-1)^{\ell(n-\ell+1)} \int^{\check{H}} \check{j}_e \cdot \check{A}]$

2. Since $\check{j}_e$ is topologically trivial, $J_e = d\kappa_e$, where $\kappa_e \in \Omega^{n-\ell}/\Omega^{n-\ell}_{\mathbb{Z}}$ so the action becomes:

$$-\pi \int \lambda F * F + 2\pi i (-1)^{\ell(n-\ell+1)} \int \kappa_e F \tag{7.9}$$

3. Local gauge potential in flux sector $c \in H^\ell(M_n;\mathbb{Z})$: $F = f_c + da$.

4. Variation wrt a: $\delta S = \int -2\pi\lambda(d\delta a) * (f_c + da) + 2\pi i (-1)^{\ell(n-\ell+1)} \int \kappa_e d(\delta a)$

5. Simplify sign: $\delta S = \int -2\pi\lambda(d\delta a) * (f_c + da) + 2\pi i (-1)^\ell \int d(\delta a)\kappa_e$

6. Rewrite it as

$$\int d\left(-2\pi\lambda\delta a*(f_c+da)+2\pi i(-1)^{\ell}\delta a\kappa_e\right)$$

$$-\int\left(-2\pi\lambda(-1)^{\ell-1}\delta ad*(f_c+da)-2\pi i\delta aJ_e\right)$$

7. So stationary equation is:

$$d*(f_c+da)=i\lambda^{-1}(-1)^{\ell}J_e \tag{7.10}$$

8. Now the next step is slightly tricky, and it is easy to get the wrong sign in the saddle point action if it is not handled correctly. Together with $d(f_c+da)=0$ we see that we should take $f_c\in\mathcal{H}^{\ell}_{\mathbb{Z}}(M_n)$ to be harmonic with integer periods. Then, since $J_e\in \mathrm{Im}d$, $*J_e\in\mathrm{Im}d^{\dagger}$ and we can solve for the saddle-point value of $a\in\mathrm{Im}d^{\dagger}$ as:

$$a_{sp}=i\lambda^{-1}(-1)^{\ell}(*d*d)^{-1}*J_e=i\lambda^{-1}(-1)^{\ell(n-\ell)+1}(d^{\dagger}d)^{-1}*J_e \tag{7.11}$$

9. Now, if we plug back into the action to find the stationary value we find it is a sum of three terms:

$$\int\left(-\pi\lambda f_c*f_c+2\pi i(-1)^{\ell(n-\ell+1)}\kappa_e f_c\right) \tag{7.12}$$

$$\int(-2\pi\lambda da_{sp}*f_c) \tag{7.13}$$

$$\int\left(-\pi\lambda da_{sp}*da_{sp}+2\pi i(-1)^{\ell(n-\ell+1)}\kappa_e da_{sp}\right) \tag{7.14}$$

Now, the cross term (7.13) vanishes after integration by parts, since f_c is harmonic. The last term evaluates to

$$-\pi(-1)^n\int\lambda^{-1}J_e(d^{\dagger}d)^{-1}*J_e=-\pi(-1)^n\lambda^{-1}J_e*(dd^{\dagger})^{-1}J_e \tag{7.15}$$

In the first term, if we write $\kappa_e=d^{\dagger}\zeta_e+\kappa_e^h$ where κ_e^h is harmonic then we get

$$\int\left(-\pi\lambda f_c*f_c+2\pi i(-1)^{\ell(n-\ell+1)}\kappa_e^h f_c\right) \tag{7.16}$$

Now, if we insist on $\check{j}_e$ being in differential cohomology then the harmonic piece of the trivialization should have integer periods. In that case we can drop this second term.

10. If, however, we allow κ_e^h to be harmonic, but not with integer periods, then we get the expected Theta function with an insertion.

7.2 Nature of field in the presence of magnetic current

Now let us consider again adding sources. We have argued that in quantum mechanics the generalized Maxwell field should be viewed as defining an element of differential cohomology.

What happens to this picture if we add a magnetic current? Then

$$dF = J_m \tag{7.17}$$

If J_m is nonzero we clearly can no longer regard F as a the fieldstrength of a differential cohomology class.

On the other hand, $dJ_m = 0$ and the (relative, or compactly supported) cohomology class $[J_m] \in H^{\ell+1}(M, M - \mathrm{Supp}J_m; \mathbb{Z})$ is quantized, so we can certainly view J_m as the fieldstrength of some differential cohomology class $[\check{j}_m]$ of degree $\ell + 1$. When we discussed the electric coupling in Section 7.1 above we also saw that it was useful to view the electric current as describing a class $[\check{j}_e] \in \check{H}^{n-\ell+1}(M)$.

If we view J_m as the fieldstrength of a differential class then one natural interpretation of (7.17) is that there is a theory of differential *cocycles* and differential *cochains* and that the magnetic current is "really" a differential *cocycle*, denoted $\check{j}_m$ which is being trivialized by a differential *cochain* $\check{A}$.

One way to make this precise is to use the Hopkins-Singer model of differential cohomology using the homotopy fiber product of chain complexes. Then

$$\check{j}_m = (c_m, h_m, J_m) \in \check{C}^{\ell+1}(M) := C^{\ell+1}(M, \mathbb{Z}) \times C^{\ell}(M, \mathbb{R}) \times \Omega^{\ell+1}(M) \tag{7.18}$$

and the field $\check{A}$ is now

$$\check{A} = (c_f, h_f, F) \in \check{C}^{\ell}(M) = C^{\ell}(M, \mathbb{Z}) \times C^{\ell-1}(M, \mathbb{R}) \times \Omega^{\ell}(M) \tag{7.19}$$

and

$$\delta\check{A} = \check{j}_m \tag{7.20}$$

entails three equations, one of which is (7.17). [9]

If we regard the cocycle $\check{j}_m \in \ker\delta := \check{Z}^{\ell+1}$ then the difference of two solutions to (7.20) is an arbitrary differential cocycle $\check{Z}^{\ell}(M)$, so the groupoid of fields is a torsor for differential cocycles of degree ℓ. The isomorphism classes of this groupoid form a torsor for $\check{H}^{\ell}(M)$.

Remarks:

1. We will be studying the theory as a "function" of the external currents so the picture we should have is that we have a family of fieldspaces fibered over $\check{Z}^{\ell+1}$ where the fiber is a torsor for $\check{Z}^{\ell}(M)$. We will denote it by $\check{Z}^{\ell}_{\check{j}_m}(M)$ and the set of isomorphism classes by $\check{H}^{\ell}_{\check{j}_m}(M)$. **** FIGURE ****

[9] ♣ What is the physical interpreation of the other two equations?

7.2.1 The example of $\ell = 1$

The presence of magnetic current changes the geometric interpretation of the generalized Maxwell field. A nice example of this is given by the case of $\ell = 1$ along the following lines. [10] Recall the description in Section **??**. Now, there is a very nice geometrical groupoid representing $\check{Z}^2(M)$, namely, the groupoid of principal $U(1)$ bundles with connection on M. With this interpretation, a magnetic current $\check{j}_m$ is a a principal $U(1)$ bundle $P_m \to M$ with connection ∇ and fieldstrength $J_m \in \Omega^2_{\mathbb{Z}}(M)$. In this case (7.20) means that the fieldstrength (of $\check{j}_m$) is globally trivialized by the fieldstrength of the Maxwell field: $J_m = dF$. Therefore $F \in \Omega^1(M)$ should be viewed as a globally well-defined one-form which is the connection form of P_m relative to a particular section $s : M \to P_m$:

$$F = \frac{1}{2\pi i} s^{-1} \nabla s \tag{7.21}$$

Both F and s are globally well-defined and in particular s is a global trivialization $s : M \to P_m$. Thus, in this example introducing magnetic current has turned our $U(1)$ valued function φ into a trivializing section of a principal $U(1)$ bundle.

7.2.2 Action and partition function

Suppose now there is background magnetic current j_m but no electric current.

At the level of fieldstrengths we can trivialize the magnetic current with some $\kappa_m \in \Omega^\ell(M)$ and two trivializations differ by $\Omega^\ell_{\mathbb{Z}}$. Then the fieldstrengths can be written as

$$F = \kappa_m + \Delta F \tag{7.22}$$

where $\Delta F \in \Omega^\ell_{\mathbb{Z}}(M)$.

The space of fields is disconnected and the set of components is a torsor for $H^\ell(M;\mathbb{Z})$. We can use the metric to choose a distinguished trivialization $\bar{\kappa}_m \in \mathrm{Im} d^\dagger$. However, it will be interesting to consider $\kappa_m = \bar{\kappa}_m + \kappa^h_m$, where κ^h_m is harmonic. Then we can label components by $c \in H^\ell(M;\mathbb{Z})$ and write

$$F = \bar{\kappa}_m + f_c + da \tag{7.23}$$

Then the action is

$$\pi \int \lambda(\bar{\kappa}_m + f_c + da) * (\bar{\kappa}_m + f_c + da) \tag{7.24}$$

The evaluation of the partition function applies as before and we get

$$\begin{aligned} Z_\ell(M_n, \lambda, g, \check{j}_m) &= e^{-\pi \int \lambda \bar{\kappa}_m * \bar{\kappa}_m} Z_\ell(M_n, \lambda, g) \\ &= e^{-\pi \int \lambda J_m * (dd^\dagger)^{-1} J_m} Z_\ell(M_n, \lambda, g) \end{aligned} \tag{7.25}$$

where, in evaluating the classical action in sector c we have used the property that $\bar{\kappa}_m \in \mathrm{Im} d^\dagger$ so that $\int \bar{\kappa}_m * f_c = 0$.

Note that

[10] D. Freed, K-Theory in Quantum Field Theory

1. The prefactor in the action is just

$$e^{-\pi \int \lambda J_m * (dd^\dagger)^{-1} J_m}$$

   in accord with electro-magnetic duality. ♣ UP TO A SIGN $(-1)^n$!!! ♣

2. We noted that we could have allowed a harmonic piece κ_m^h. Then we would have gotten a shifted theta function:

$$\sum_{f \in \mathcal{H}^\ell_{\mathbb{Z}}} e^{-\pi \int \lambda (f_c + \kappa_m^h) * (f_c + \kappa_m^h)} \tag{7.26}$$

   and this is the Poisson summation dual of the theta function with insertion κ_e^h. If the periods of κ_m^h are integral, we can shift it away, in accord with electromagnetic duality with (7.8).

♣ Need to discuss how to differentiate along space of magnetic currents.
♣

7.3 Simultaneous electric and magnetic current

We have now argued that the electric and magnetic currents should be viewed as differential cocycles

$$\begin{aligned} \check{j}_e &\in \check{Z}^{n-\ell+1}(\mathcal{X}) \\ \check{j}_m &\in \check{Z}^{\ell+1}(\mathcal{X}) \end{aligned} \tag{7.27}$$

The classical equations are

$$\begin{aligned} dF &= J_m \\ d * F &= J_e \end{aligned} \tag{7.28}$$

and the solutions are a torsor for harmonic forms.

Now we want to consider the quantum theory.

We want to study the partition functions and Hilbert spaces (and more generally, all correlation functions) as "functions" of the couplings, metric, and now, also the external currents $\check{j}_e, \check{j}_m$. Together these parameters form a space (a groupoid) which we will denote $\mathcal{P}$.

We then form a fiber space

$$\mathcal{X} \to \mathcal{P} \tag{7.29}$$

whose fibers are spacetimes M equipped with coupling, metric, and external currents.

We have discussed how the fields of our theory are fibered over $\mathcal{P}$. Now, how should we interpret the electric coupling (7.3) in the presence of magnetic current?

In the Hopkins-Singer paper a theory of integration of differential chains is developed. The natural generalization of Stokes theorem holds. It is natural then, to interpret the pairing (7.3) on the level of cochains.

In the absence of magnetic current $\check{A} \in \check{Z}^\ell(M)$ and $\check{j}_e \in \check{Z}^{n-\ell+1}(M)$ so integration over the fibers of the family gives

$$\int_{\mathcal{X}/\mathcal{P}}^{\check{H}} \check{j}_e \cdot \check{A} \in \check{Z}^1(\mathcal{P}) \tag{7.30}$$

But

$$\check{Z}^1(\mathcal{P}) \cong Map(\mathcal{P}, \mathbb{R}/\mathbb{Z}) \tag{7.31}$$

so this makes sense.

Now, in the presence of magnetic current $\check{A}$ is no longer a cocycle but just a cochain, but in the Hopkins-Singer picture we can still make sense of

$$\int_{\mathcal{X}/\mathcal{P}}^{\check{H}} \check{j}_e \cdot \check{A} \in \check{C}^1(\mathcal{P}) \tag{7.32}$$

Moreover, as we have said, Stokes theorem holds and we can write

$$\delta \int_{\mathcal{X}/\mathcal{P}}^{\check{H}} \check{A} \cdot \check{j}_e = \int_{\mathcal{X}/\mathcal{P}}^{\check{H}} \check{j}_m \cdot \check{j}_e \in \check{Z}^2(\mathcal{P}) \tag{7.33}$$

Now, the groupoid $\check{Z}^2(\mathcal{P})$ is equivalent to the groupoid of line bundles with connection on $\mathcal{P}$. If we take this viewpoint we can then return to the example of Section 7.2.1 and interpret $\exp[2\pi i \int \check{A}\check{j}_e]$ as a nonflat trivialization of the line bundle with connection given (up to a sign) by $\int_{\mathcal{X}/\mathcal{P}}^{\check{H}} \check{j}_m \cdot \check{j}_e$.

Now, if we want to do the path integral we integrate over the torsor $\check{H}^\ell_{\check{j}_m}(M)$ with the action

$$Z_\ell(M; \lambda, g, \check{j}_e, \check{j}_m) = \int_{\check{H}^\ell_{\check{j}_m}(M)} \mu(\check{A}) e^{-\pi \int \lambda F * F + 2\pi i \int^{\check{H}} \check{j}_e \cdot \check{A}} \tag{7.34}$$

When integrating along a fiber $\check{Z}_{\check{j}_m}$ the electric coupling can be trivialized (there is no curvature along the fiber) and hence we obtain a partition function which is also a section of the line bundle $\int_{\mathcal{X}/\mathcal{P}}^{\check{H}} \check{j}_m \cdot \check{j}_e$.

♣ This is a tricky point. Discuss it more carefully. ♣

Thus, we conclude that in the presence of electric and magnetic current the partition function is a section of a line bundle over the space of electric and magnetic currents and not an ordinary function.

Example: One helpful example is the case $\ell = 1$ discussed in Section 7.2.1 above. In this case the magnetic current is a principal $U(1)$ bundle $P \to M$ with connection ∇ while the electric current is a cocycle $\check{j}_e \in \check{Z}^n(M)$. We can take the example where the fieldstrength J_e is a delta-function supported Poincaré dual to a set of points $\wp_a$:

$$J_e = \sum_a q_a \delta(\wp_a) \qquad\qquad q_a \in \mathbb{Z} \tag{7.35}$$

(with $\sum q_a = 0$ if $M = S \times \mathbb{R}$ with S compact). Then the electric coupling to a field $s : M \to P$ is

$$\otimes_a (s(\wp_a))^{q_a} \tag{7.36}$$

which is manifestly a section of a line bundle.

7.3.1 Explanation for physicists:

As we have explained the components of fieldspace are a torsor for $H^\ell(M;\mathbb{Z})$. In each component we choose a trivialization F_0 of J_m and then we can write fields with respect to this basepoint as

$$F = F_0 + da \tag{7.37}$$

How should we write the electric coupling? One possibility is to write the action

$$\pi \int \lambda(F_0 + da) * (F_0 + da) + 2\pi \int J_e a \tag{7.38}$$

But there are problems with this choice:

1. clearly a depends on the choice of F_0. Different choices differ by a phase: If $F_0 \to F_0 + d(\psi)$ then the electric coupling changes by an overall phase $e^{-2\pi i \int J_e \psi}$.

2. We might try to make better sense by choosing a trivialization G_0 of J_e, $dG_0 = J_e$. Then after integrating by parts and adding a constant $\int F_0 G_0$ (which depends on trivialization) we get: [11]

$$\pi \int \lambda(F_0 + da) * (F_0 + da) + 2\pi \int G_0 F \tag{7.39}$$

But this depends on the trivialization G_0, and moreover F has hidden J_m dependence because $dF = J_m$.

What we can take away from these expressions is that:

1. We need to make a choice about what we mean by "the same field" when we change J_m. That is, we need to choose a connection on the space of field configurations which is fibered over the space of magnetic currents.

2. With this choice now consider a small square of currents. So we choose $\tilde{J}_m$ on $\Delta^1 \times M$ interpolating between J_m^0 and J_m^1 and similarly for $\tilde{J}_e$. Then the *change* in the electric coupling should be given by

$$\chi_{J_m^1, J_e(t)} = \chi_{J_m^0, J_e(t)} \exp[2\pi i \int J_e \tilde{F}] \tag{7.40}$$

where $\tilde{F}$ is a choice of trivialization $d\tilde{F} = \tilde{J}_m$. Similarly,

$$\chi_{J_m(t), J_e^1} = \chi_{J_m(t), J_e^0} \exp[2\pi i \int \tilde{J}_e \tilde{F}(t)] \tag{7.41}$$

Therefore, there is holonomy around the square

$$(\int_{\Delta \times M} \tilde{J}_e \tilde{F}^1 + \int J_e^0 \tilde{F}) - (\int J_e^1 \tilde{F} + \int \tilde{J}_e F^0) = \int_{\square \times M} \tilde{J}_e \wedge \tilde{J}_m \tag{7.42}$$

Example: For $\ell = 1, n = 2$ this is just the $U(1)$ WZW model coupled to external gauge fields of both chiralities. To see this identify $F_0 \to A$ take $J_e = dB$, where A, B are one-forms and write $a \to \phi$. Then, in light-cone variables the action (7.39) is just

$$\int_M 2\pi\lambda \partial_+\phi \partial_-\phi + 2\pi(\lambda A_- - B_-)\partial_+\phi + 2\pi(\lambda A_+ + B_+)\partial_-\phi + 2\pi\lambda A_+ A_- \tag{7.43}$$

Examples: Write some explicit partition functions as functions of the currents.

[11] sign of cocycle term can change

7.3.2 Explicit partition function

$$Z_\ell(M;\lambda, g, \check{j}_e, \check{j}_m) = \int_{\check{H}^\ell_{\check{j}_m}(M)} \mu(\check{A}) e^{-\pi\int \lambda F * F + 2\pi i \int^{\check{H}} \check{j}_e\cdot \check{A}} \tag{7.44}$$

Again, we can consider the subintegration over the shifts by the flat fields $\check{\phi}_f$. This projects the partition function to have support on $\check{j}_e$ with $c(\check{j}_e) = 0$, and hence the coupling can be written more explicitly ???? ♣ Presumably it is only the characteristic class in the vertical direction which is trivial ♣

7.4 Hilbert spaces in the presence of electric and magnetic current

A similar story should hold for the Hilbert space. We now take a family $\mathcal{S} \to \mathcal{P}$ whose fibers are spatial slices S equipped with electric and magnetic currents.

Then the presence of these currents allows us to construct

$$\int_{\mathcal{S}/\mathcal{P}} \check{j}_e \check{j}_m \in \check{H}^3(\mathcal{P}) \tag{7.45}$$

This should be interpreted as a gerbe with connection over $\mathcal{P}$.

The physical interpretation would be that one can only construct a projective Hilbert space in these anomalous theories and we therefore have a family of projective Hilbert spaces. These determine a gerbe class which should be (7.45).

Open Problem: Construct these projective Hilbert spaces and the corresponding gerbe with connection. Demonstrate the electro-magnetic duality. An important point will be the use of the polarization. We want positive energy representations. The Hamltonian is $H = \int_S (\lambda^{-1}\Pi * \Pi + \lambda F * F)$. But here $F = F_0 + F_q$ where F_0 is a classical solution in the presence of currents, F_q is the quantum operators and there is a similar shift of Π. The classical energy might need to be regularized, and the polarization will change as we change the background currents. Thus the Hilbert spaces will be related to each other by Bogoliubov transformations.

7.5 Generalization of the Segal Axioms: Anomalous Field Theories

In this section we will be careful about the use of the word "theory." We will formulate the generalized Maxwell theory in the presence of electric and magnetic currents in the Atiyah-Segal framework of axioms for a field theory.

Let $\mathcal{F}^{(n)}$ be an n-dimensional field theory in the sense of Segal. So it is a functor from a geometric category of n-dimensional spacetimes (morphisms) equipped with some geometric structures to a linear category. In particular $\mathcal{F}^{(n)}$ on closed n-manifolds is a partition function, on closed $(n-1)$-manfiolds is a linear space of quantum states, and we imagine that it has been extended to be defined on lower dimensional manifolds.

In our case the geometric category is going to be manifolds endowed with electric and magnetic currents. Now we claim that the resulting field theory is not a field theory in the usual sense, but rather an *anomalous field theory.* To define this notion we first define two other notions from the literature on topological field theory:

1. An n-dimensional field theory $\tilde{\mathcal{F}}^{(n)}$ valued in an (ordinary) $(n+1)$-dimensional field theory $\mathcal{F}^{(n+1)}$ would be one where

$$\tilde{\mathcal{F}}^{(n)}(X) \in \mathcal{F}^{(n+1)}(X) \tag{7.46}$$

for all X. In particular, the partition function of an n-dimensional field theory is a vector in the Hilbert space of an $(n+1)$-dimensional field theory. So there is a vector space of partition functions.

2. An *invertible field theory* is ....

3. Finally we can define an *anomalous n-dimensional field theory* is an n-dimensional field theory valued in a invertible $(n+1)$-dimensional

In the presence of simultaneous electric and magnetic current we have an anomalous theory:

In concrete terms -

1. The partition function is valued in the Hilbert space of a topological theory one dimension higher.

2. The Hilbert space is replaced by a family of projective Hilbert spaces with connection.

8. Self-Duality and Chern-Simons Theory

Main goals:

1. Explain baby example of "holography"
2. Explain crucial role of quadratic refinement.

8.1 Classical self-dual fields

$n = 4s+2$, $\ell = 2s+1$, Lorentzian signature $*^2 = 1$:

We replace the defining equations of generalized Maxwell theory by the pair of equations: [12]

$$\begin{aligned} dF &= 0 \\ F - *\epsilon F &= 0 \end{aligned} \tag{8.1}$$

where $\epsilon = 1$ for the SD and $\epsilon = -1$ for the ASD cases. Note that the equation of motion and the Bianchi identity are now the same.

Energy-momentum tensor: $T(F^+)$.

Waves: $n = 2, \ell = 1$: Left or right moving. But not both.

$n = 6, \ell = 3$: On $\mathbb{R}^{1,1} \times K$, left-movers times SD Harmonic 2-forms. rightmovers times ASD Harmonic 2-forms.

$n = 6, \ell = 3$: On $\mathbb{R}^{1,3} \times C$, right-polarized times SD Harmonic 1-forms. leftpolarized times ASD Harmonic 1-forms.

[12]♣ ?? Change notation to $\mathcal{F}$?

8.1.1 Higher Rank Theories

In the higher rank case we take $F \in \Omega^\ell(M; V)$. In order to write the self-duality equations we require an extra structure:

1. $n = 0\text{mod}4$: $I^2 = -1$, complex structure on V allows us to write the self-duality equation: $F = \pm(* \otimes I)F$. (e.g. Seiberg-Witten theory)

2. $n = 2\text{mod}4$: $I^2 = +1$, an involution, or equivalently, a projection operator on V allows us to write: $F = \pm(* \otimes I)F$. (e.g. Narain theory)

For a physical theory one needs to write an energy-momentum tensor. Recall that previously this required us to endow V with a positive symmetric form. Now, we demand that this is compatible with the extra structure of I:

1. For $n = 0\text{mod}4$ we require compatibility between the positive quadratic form $g(v, w)$ and the complex structure: $g(Iv, Iw) = g(v, w)$. This allows us to define an symplectic form $\omega(v, w) := g(v, Iw)$. The space of coupling constants is then the Seigel upper half-plane: $Sp(V)/U(V)$.

2. For $n = 2\text{mod}4$ we require an orthogonal structure compatible with the involution so that $g(v) = \langle v, Iv \rangle$ is a positive definite metric. The space of coupling constants is then the Grassmannian: $O(p, q))/O(p) \times O(q)$.

After making choices can write actions for $n = 0\text{mod}4$ and also for $n = 2\text{mod}4$ if $p = q$.

8.2 Challenges for Quantum Theory

Let us return to the case of a single self-dual $\ell = 2s+1$ form in $4s+2$ dimensions. We want now to discuss the quantum theory. There are three problems which immediately arise:

1. The Obvious action is zero $\int F * F = \int FF = 0$. Nevertheless, there is an action, but it involves further choices, as described below.

2. $[F] = [*F]$: How can flux be quantized !?

3. NB. Now $J_e = J_m$ (more generally, there is an isomorphism between differential cohomology groups where electric and magnetic currents are defined.) Therefore we have simultaneous presence of electric and magnetic current and the theory will be anomalous.

8.3 Three (related) strategies to solve the problem

8.3.1 Splitting the nonchiral theory

Physically we expect the theory of a nonchiral $\ell = 2s + 1$ form in $4s + 2$ dimensions to "factorize" as a self-dual and anti-self-dual theory.

To make this intuition plain consider for example the theory on $M = \mathbb{R} \times S$. If we work with topologically trivial fields $F = da$ then a satisfies the wave equation:

$$d_M^\dagger d_M a = 0 \tag{8.2}$$

but in this dimension and degree the waveoperator $d_M^\dagger d_M$ splits:

$$\partial_t^2 + d^\dagger d = (\partial_t - *d)(\partial_t + *d) \tag{8.3}$$

where d and Hodge $*$ refer to S. In particular, on a $2s$ form $*d : \Omega^{2s}(S) \to \Omega^{2s}(S)$ so it makes sense to define chiral and antichiral waves:

$$\begin{aligned}(\partial_t - *d)a_L &= 0 \\ (\partial_t + *d)a_R &= 0\end{aligned} \tag{8.4}$$

The general solution will be a sum of chiral and anti-chiral solutions

Closely related to this, in first quantization the (reductive part of the) little group is $SO(n-2) = SO(4s)$. The usual representation Λ^{2s} now splits into irreducibles:

$$\Lambda^{2s} \cong \Lambda_+^{2s} \oplus \Lambda_-^{2s} \tag{8.5}$$

because $*^2 = +1$ in Euclidean signature on $\mathbb{R}^{4s}$.

Also, if we decompose a nonchiral fieldstrength $F = F^+ + F^-$ into its chiral and anti-chiral parts:

$$F^\pm = \frac{1}{2}(F + \pm * F) \tag{8.6}$$

then (somewhat nontrivially)

$$T[F] = T[F^+] + T[F^-], \tag{8.7}$$

so, at least naively, we expect the dynamics of the modes to decouple.

For all these reasons it becomes interesting to try to "split" the Hilbert space of the non-self-dual field and also the partition function of the non-self-dual field coupled to external electric and magnetic currents. We will discuss that further in Sections *** below.

8.3.2 Holographically Dual Pure Chern-Simons Theory

As we will show, when we try to split the nonchiral partition function in the presence of external currents we are led to the conclusion that the partition function should be a section of a line bundle on $\check{H}^{2s+s}(M)$.

Moreover, that line bundle is precisely the line bundle defined by a corresponding Chern-Simons action on a manifold Y with boundary $\partial Y = M$.

2. Landau levels of massive Maxwell-Chern-Simons theory in $4s+3$ dimensions. (Example of holography.)

3. Long-distance/topological limit: Holographically dual Pure Chern-Simons Theory in $4s+3$ dimensions.

8.4 The Hilbert Space of the Self-Dual Field

Natural way to split in half: we should aim to define some kind of Heisenberg extension of $\check{H}^{\ell}(S)$.

But how can we do this? It is not so obvious what to take as the cocycle since it is not naturally a product of a group and its Pontryagin dual.

Our approach will be based on the following

Theorem: A central extension of an abelian group A is uniquely determined up to isomorphism by a skew and alternating bihomomorphism

$$s : A \times A \to U(1)$$

This is the commutator function.

For example for our previous nonself-dual field $A = \check{H}^{\ell}(X) \times \check{H}^{n-\ell}(X)$ we have the explicit cocycle $\exp[2\pi i \langle \check{A}, \check{A}_D \rangle]$ and hence the commutator function:

$$s(([\check{A}_1], [\check{A}_1^D]), ([\check{A}_2], [\check{A}_2^D])) = \exp\left[2\pi i \big(\langle [\check{A}_2], [\check{A}_1^D] \rangle - \langle [\check{A}_1], [\check{A}_2^D] \rangle\big)\right] \tag{8.8}$$

This of course suggests that to define a Heisenberg group extension of a single copy $\check{H}^{\ell}(S)$ we should take

$$s(\check{A}_1, \check{A}_2) = \exp[2\pi i \langle \check{A}_1, \check{A}_2 \rangle] \tag{8.9}$$

But:

1. Not alternating.

2. $\langle \check{A}, \check{A} \rangle = \frac{1}{2} \int_S a \nu_{2s}$ where ν_j is the Wu class.

Theorem A': $\mathbb{Z}_2$-graded Heisenberg group is determined by s skew and bimultiplicative

Theorem B' unique $\mathbb{Z}_2$-graded SvN representation. $\mathbb{Z}_2$ grading is given by Wu class.

Example 1: $Heis(\check{H}^1(S^1))$: Bosonization. $\nu_0 = 1$.

Example 2: Consider the chiral 2-form $\check{A}$ in $\dim M = 6$. Suppose $M = T^2 \times M_4$. Consider a vertex operator

$$V(\Sigma) = e^{2\pi i \int_{\Sigma} \check{A}}$$

for $\Sigma \in Z_2(M_4)$. This will be fermionic for $\Sigma \cdot \Sigma$ odd and bosonic for $\Sigma \cdot \Sigma$ even. So, for example, if $M_4 = dP_n$ then in the limit $\mathrm{vol}\,(dP_n) \ll \mathrm{vol}\,(T^2)$ the theory will reduce to a theory of free fermions.

Open Problems:

1. Only constructed up to isomorphism! Need an explicit cocycle to construct the Heisenberg group. Want to do this naturally. Also want to do it in the presence of external current.

2. Elucidate the sense in which the $\mathbb{Z}_2$-grading means we have "fermionic" states in the higher dimensional examples. Is there a sense in which the standard spin-statistics intuition is not correct?

8.4.1 Relation to splitting of non-self-dual field

Intuitively we expect from the above remarks that the Hilbert space of the nonself-dual field should "factorize" between Hilbert spaces of self-dual and anti-self-dual fields. At the level of oscillator representations in topologically trivial situations this is fairly straightforward.

EXPLAIN.

Now we explain how to achieve this splitting in general, from a more invariant point of view.

First, we explain a general construction we can make with self-dual abelian groups:

Suppose A is an abelian Lie group with an invariant measure and moreover it is self-dual $A \cong \widehat{A}$ by a nondegenerate pairing $\langle\cdot,\cdot\rangle : A \to \mathbb{R}/\mathbb{Z}$. Suppose finally that $\tilde{A}$ is a Heisenberg extension of A with *commutator function* (not cocycle!) $s(a,b) = \exp[2\pi i\langle a,b\rangle]$. In this situation $L^2(A)$ is, of course a representation of $\mathrm{Heis}(A \times \widehat{A})$, but is also a representation of "half" of that Heisenberg group in the following sense: For $a \in A$ we can define the operators:

$$\begin{aligned}(T_a\psi)(b) &= \psi(b+a)\\ (M_a\psi)(b) &= e^{2\pi i\langle a,b\rangle}\psi(b)\end{aligned} \tag{8.10}$$

The T_a define a representation of A (not $\tilde{A}$) on $L^2(A)$, as do the M_a. Of course, we have

$$T_a M_b = e^{2\pi i\langle b,a\rangle} M_b T_a \tag{8.11}$$

Now, for any pair of integers k,ℓ we can define the operators:

$$\rho_{k,\ell}(a) := T_{ka}M_{ka} \tag{8.12}$$

These define a representation of a *degenerate* extension $\tilde{A}_{k,\ell}$ of A on $L^2(A)$ with commutator function

$$\begin{aligned}\rho_{k,\ell}(a)\rho_{k,\ell}(b) &= T_{ka}M_{ka}T_{\ell a}M_{\ell b}\\ &= e^{2\pi ik\ell\langle b,a\rangle}T_{ka}T_{\ell b}M_{ka}M_{\ell b}\\ &= e^{2\pi ik\ell\langle b,a\rangle}T_{\ell b}T_{ka}M_{\ell b}M_{ka}\\ &= e^{2\pi i(2k\ell)\langle b,a\rangle}T_{\ell b}M_{\ell b}T_{ka}M_{ka}\\ &= e^{2\pi i(2k\ell)\langle b,a\rangle}\rho_{k,\ell}(b)\rho_{k,\ell}(a)\end{aligned} \tag{8.13}$$

Note particularly that we cannot get the basic commutator function $\langle a,b\rangle$ defining the nondegenerate Heisenberg extension $\widetilde{A}$. We cannot take k or ℓ to be half-integral if there are nontrivial elements of order 2 in A. Instead we obtain a representation of an extension $\tilde{A}_{k,\ell}$ with commutator $2k\ell\langle a,b\rangle$. Thus the center of $\tilde{A}_{k,\ell}$ is isomorphic to $U(1) \times Z_{2k\ell}$ where $Z_N \subset A$ is the subgroup of elements of order N. The irreducible representations where $U(1)$ acts canonically are therefore labeled by the characters $\widehat{Z_{2k\ell}}$.

Now, a computation similar to (8.13) shows that $\rho_{k,\ell}$ and $\rho_{k,-\ell}$ define sets of operators which are in each others *commutant*. Thus we have a finite index subgroup $\tilde{A}_{k,\ell} \times \tilde{A}_{k,-\ell} \subset \mathrm{Heis}(A \times \widehat{A})$, and we can decompose the Hilbert space in terms of irreps of this subgroup:

$$\oplus_{\alpha,\beta}N_{\alpha,\beta}\overline{\mathcal{H}_\alpha} \otimes \mathcal{H}_\beta \tag{8.14}$$

where $\overline{\mathcal{H}_\alpha}$ is a representation of $\tilde{A}_{k,-\ell}$ and $\mathcal{H}_\beta$ is a representation of $\tilde{A}_{k,\ell}$ and α, β run over the characters $\widehat{Z_{2k\ell}}$. $N_{\alpha,\beta}$ are integer degeneracies (intertwiners) determined by ????

Now, we apply all this to $A = \check{H}^\ell(S)$. As we have discussed, $L^2(A)$ is the Hilbert space of the *non-chiral* field. The operators $\rho_{k,\ell}(\check{A})$ are generalizations of vertex operators of 1+1-dimensional conformal field theory. The new element is that when $\lambda = p/q$ then the "vertex operators" can be factorized into "chiral vertex operators" which define analogs of the extended chiral algebras of 2d rational conformal field theory. Roughly speaking, $\rho_{q,p}(\check{A})$ only change the dependence of the wavefunction on the modest satisfying $(\partial_t + *d)\delta\check{A} = 0$ while $\rho_{q,-p}$ only depends on the quantum modes satisfying $(\partial_t - *d)\delta\check{A} = 0$. Then (8.14) becomes a decomposition of the Hilbert space into representation spaces of the two "chiral and antichiral algebras."

♣ EXPLAIN THIS BETTER. IT WAS ALSO NOT DONE SO WELL IN FMS ♣

8.5 Splitting the partition function of a nonchiral theory

Let us return to the action in the presence of electric and magnetic current (7.39), but now rename things calling $F_0 \to \mathcal{A}$ and $a \to C$.

$$S = \pi \int_M \lambda(\mathcal{A} + dC) * (\mathcal{A} + dC) + 2\pi \int J_e C \tag{8.15}$$

Here $\mathcal{A} \in \Omega^{2s+1}(M)$ and $C \in \Omega^{2s}(M)$.

For a self-dual theory we should identify the magnetic and electric currents. The magnetic current has entered because $\mathcal{A} + dC$ is a trivialization of J_m. Now we identify electric and magnetic current, or more precisely take $J_e = \epsilon \frac{k}{2} d\mathcal{A}$. Then, for a special coupling:

$$\lambda = \frac{k}{2} \tag{8.16}$$

only the self-dual or anti-self-dual part of φ couples to, or "sees" the external current A.

In Lorentzian signature this clear since we can rewrite (8.15) as

$$S = \pi k \int_M \left(\frac{1}{2} dC * dC + \mathcal{A}(*dC + \epsilon dC) + \frac{1}{2} \mathcal{A} * \mathcal{A} \right) + \pi k \epsilon \int_{\partial M} \mathcal{A} C \tag{8.17}$$

If we denote our fieldstrength $R = dC$ then only the ϵ self-dual part $*R + \epsilon R$ couples to the external field $\mathcal{A}$. Thus, the $\mathcal{A}$-dependence "probes" only the dynamics of the self-dual field.

Remarks:

1. In Euclidean signature we have instead....
2. Note that the action is not "gauge invariant" under $\mathcal{A} \to \mathcal{A} + d\epsilon$, $C \to C - \epsilon$, but transforms like

$$e^{-i\pi k \epsilon \int_M J_e \epsilon} \tag{8.18}$$

But this is precisely the transformation of a Chern-Sions action on Y where $\partial Y = M$:

$$e^{\pi i k \int \mathcal{A} d\mathcal{A}} \to e^{\pi i k \int \mathcal{A} d\mathcal{A}} e^{-i\pi k \epsilon \int_M J_e \epsilon} \tag{8.19}$$

This suggests that the partition function should be valued in the line bundle defined by the Chern-Simons action on Y.

3. Indeed, because of self-duality there is simultaneous electric and magnetic current. Therefore, because of what we explained in Sections **** the partition "function" $Z(M; k, g, j)$ should be a section of a line bundle. This is the same line bundle as that defined by the Chern-Simons form for a gauge theory one degree higher on a bounding manifold with $\partial Y = M$.

8.5.1 Explicit sum over fluxes: Deriving the conformal blocks

Sum over fluxes of nonchiral theory is a sum of perfect squares (of theta functions) This can be turned into perfect square by including a quadratic refinement. Example of chiral scalar in 1+1.

This motivates the importance of the quadratic refinement!

8.6 The edge state phenomenon: Quantization on $Disk \times \mathbb{R}$

$Y = D \times \mathbb{R}$. D is the two-dimensional disk. (More generally D is $4s + 2$-disk.)

Consider quantization of $N \int AdA$ on the disk. Gauge transformations $A \to A + \omega$, $\omega \in \Omega^1_{\mathbb{Z}}(Y)$.

Need boundary conditions:

$$\delta \int_M AdA = \int 2\delta AF + \int_{\partial M} \delta A \wedge A. \tag{8.20}$$

With some choices of boundary condition and gauge group the gauge modes ω in bulk become *chiral* propagating physical modes on the boundary. If, e.g. we impose $A_\partial = *A_\partial$ then the boundary modes are self-dual.

Recall computation of Elitzur et. al. Integrate over A_0, impose Gauss law via flatness. Substitute $A = U^{-1}dU$. Evaluate the action and find the *chiral* WZW action on the boundary.

Alternative view: Quantization of flat $U(1)$ gauge fields on the disk with suitable gauge group gives Hilbert space $\mathcal{H}(D)$ given by basic representation of $\widehat{LU(1)}_{2N}$. Important that gauge group is $Map(D, U(1))$ with boundary values fixed to 1. If we have unrestricted maps $D \to U(1)$ then we kill the singletons.

To get level 1: Spin CS.

N.B. $q_\alpha(A) = \frac{1}{2}\int AdA$ is quadratic refinement (depending on spin structure).

One natural way to determine Hamiltonian for the edge modes: Take Maxwell-Chern-Simons on space with metric having a pole at the boundary: Singletons of AdS/CFT. (Return to this later.)

♣ Explain Witten's revised viewpoint on the edge modes and what consequences it has. ♣

8.7 Quantization of Pure Chern-Simons Theory

1. Chern-Simons field on Y_{4s+3} is $\check{A} \in \check{Z}^{2s+2}(Y)$.

2. Action is $\exp[i\pi k \int_{Y_{4s+3}} \check{A}\cdot\check{A}]$. When k is odd we need extra strucutre, such as integral Wu structure.

3. Consider the case where Y_{4s+3} has a boundary, viewed as a collar neighborhood $\mathbb{R}\times X_{4s+2}$. We study the Hilbert space $\mathcal{H}(X_{4s+2})$ and in particular we are interested in the wavefunctions Ψ of the Chern-Simons theory as functions of the boundary values $\check{A}_\partial \in \check{Z}^{2s+s}(X_{4s+2})$.

4. Now, when Y has a boundary the Chern-Simons action $\exp[i\pi k \int_{Y_{4s+3}} \check{A}\cdot\check{A}]$ is not a complex number but rather an element of a line bundle with connection. The line bundle is $\mathcal{L}_{CS} \to \check{Z}^{2s+2}(X_{4s+2})$ and the action

$$\exp[i\pi k \int_{Y_{4s+3}} \check{A}\cdot\check{A}] \in \mathcal{L}_{CS}|_{\check{A}_\partial} \tag{8.21}$$

5. Therefore, if we consider the formal path integral

$$\Psi(\check{A}_\partial) = \int_{\check{H}^{2s+2}(Y):\check{A}|_X=\check{A}_\partial} \mu(\check{A}) e^{i\pi k \int^{\check{H}} \check{A}\cdot\check{A}} \tag{8.22}$$

we are adding points in the same line, and the resulting path integral $\Psi(\check{A}_\partial)$ is valued in $\mathcal{L}_{CS}|_{\check{A}_\partial}$.

6. Although we are formally dealing with a topological field theory, if we choose a quantization scheme that depends on the metric then our Hilbert space of states will in general depend on the metric and define a projectively flat bundle over the space of metrics.

8.7.1 Background charge and tadpole constraint

Automorphisms are Poincare dual to center of the quadratic form.

Gauss law implies wavefunction is concentrated on components of $\check{H}^{2\ell+2}(X)$ such that

$$k(a-\mu)=0 \tag{8.23}$$

In the case of differential cohomology μ is 2-torsion. Typically there is just one component.

8.7.2 Explicit wavefunctions: Theta functions

Over $Met(X)\times \check{Z}^{2\ell+2}(X)$ we have a Chern-Simons line bundle with connection $(\mathcal{L}_{CS}, \nabla_S)$.

This is the prequantum line bundle.

Tangent space to $\check{Z}$ is isomorphic to $\Omega^{2\ell+1}(X)$ and the curvature is $-2\pi i\omega$, where ω is the canonical symplectic form.

For simplicity, assume that we work in the component with $c(\check{A})=0$. That is, assume k is even or $\mu=0$.

Now $*$ induces a complex structure on $\Omega^{2\ell+1}(X)$ making it a Kahler manifold.

Holomorphy + Gauss law gives defining equation of a theta function times a section of a determinant line bundle determining the metric dependence.

Important: Explain how to write the theta functions when μ is nonzero.

8.7.3 The holographic identity

We claim that the SD theory in $4s+2$ dimensions of level k defines a quantum field theory valued in the Chern-Simons theory of level k in $4s+3$ dimensions. This is an expression of the "holographic relation" between these to.

Let us work in the simplified situation To be precise, the holographic dictionary is that

Holographic dictionary: $\partial Y = X$, $\check{A}_\partial$ is the current $\check{j}$ of the self-dual theory. Denoting the gauge field of the self-dual theory by $\check{C}_{sd}$ our basic claim is that we can identify partition functions on X with wavefunctions of the Chern-Simons theory through

$$\Psi(\check{A}_\partial; g) = \left\langle e^{2\pi i \int^{\check{H}}_{X_{4s+2}} \check{A}_\partial \check{C} + \int g^{\mu\nu} T_{\mu\nu}} \right\rangle_{\mathrm{self-dual\ theory}} \tag{8.24}$$

Remarks:

1. As a function of g there should be a projectively flat connection and the above wavefunctions should be parallel sections. The energy momentum tensor defines the connection.

2. If we insert topology or operators in the "bulk" of Y then we change the resulting state on the boundary X. There should be a one-one correspondence between the operators which change the state of the CS theory and conjugate operators in the self-dual theory. This is the basic statement of holography.

8.8 Approach via Maxwell-Chern-Simons theory in $4s+3$ dimensions

The way this comes up in AdS compactifications is that the GAGT is a Maxwell-Chern-Simons theory.

Examples:

a.) AdS_3 compactifications: Gukov, Martinec, Moore, Strominger.

b.) AdS_5 compactifications: Witten, AdS/CFT and TFT ; Belov-Moore, AdS-Singletons

c.) AdS_7 compactifications: M-theory 3-form reproducing self-dual form in 6d.

This gives a somewhat more concrete approach.

Once again, considerations analogous to those in Section 8.7 show that the wavefunctions $\Psi(\check{A}_\partial, g)$ are sections of $(\mathcal{L}_{CS}, \nabla_{CS})$.

We again have a Gauss law, and let us suppose that it constrains the support of the wavefunction to the topologically trivial sector $c(\check{A}_\partial) = 0$.

We may then use the metric to factorize the configuration space

$$\left(\mathcal{H}^{2s+1}(X)/\mathcal{H}^{2s+1}_{\mathbb{Z}}(X)\right) \times \mathrm{Im} d^\dagger \tag{8.25}$$

The Hilbert space also factorizes. There is a unique groundstate as a function on $\mathrm{Im}d^\dagger$ (the "oscillator modes") while there is a line bundle with connection and curvature $k\omega$ on the torus of harmonic forms.

The latter is a "Landau-level problem" and there is famously a degenerate space of groundstates, which can be written explicitly in terms of theta functions.

Simple calculusof $A = \partial_z - \bar{z}, A^\dagger = \partial_{\bar{z}} + z$ etc.

At long distances, the physics of this Maxwell-Chern-Simons theory is governed by that of the topological Chern-Simons theory. Explain that large τ evolution of $d\tau^2 + e^{2\tau} d\vec{x}^2$ (AdS metric) projects onto the groundstates.

The advantage of this approach is that the Hamiltonian of the "singleton modes" is determined from that of the MCS theory. Also the metric dependence and derivation of the theta functions is clearer.

8.9 Self-Dual partition function for $k = 1$

Z section of line bundle over universal intermediate jacobian with complex structure from $*$ over space of metrics on X. (naively conformal structures, but there is a conformal anomaly).

Holomorphic quantization gives usual theta functions on intermediate jacobian. So $Z = \mathcal{N}\Theta$. $\mathcal{N}$ has "metric dependence"

Quillen norm squared of $\mathcal{N}$ is "Cheeger's half-torsion."

♣ Explain how gravitational anomalies are captured by Hopkins-Singer ♣

$$\frac{1}{8}\int (\check{\lambda}\check{\lambda} - \check{L})$$

8.10 An action principle for the SD field

From the theta function we learn that the *on-shell* action is the period matrix in a particular duality frame. Different maximal Lagrangian splittings give equivalent theta functions, but different period matrices. Thus, we expect the action to depend on a choice such as a Lagrangian subspace.

This is the paradigm for the action of the SD field: An infinite-dimensional analog of the period matrix.

8.10.1 Action in the topologically trivial case

Let us consider the simplest case where there is no nontrivial topology. For example we could consider the theory on $\mathbb{R}^{1,4s+1}$. The space $V := \Omega^\ell(M_n)$ with $\ell = 2s + 1$ has a symplectic structure

$$\omega(\phi_1, \phi_2) := \int_M \phi_1 \wedge \phi_2 \tag{8.26}$$

1. First, as in the ordinary nonself-dual theory, to formulate an action we restrict attention to fields $R \in V_{cl} := \Omega^\ell_d(M)$. We will vary within this space. Note that it is a Lagrangian subspace of V. We call the fieldstrength R so that it will not be confused with the classical self-dual field we see in the semiclassical physics of this theory. We will do the path integral over closed fields R modulo gauge transformations by

2. Now, we *choose* another Lagrangian subspace $V_m \subset V$, assumed to be maximal Lagrangian and transversal to V_{cl} (i.e. $V_{cl} \cap V_m = \{0\}$) and moreover

$$V = V_m \oplus *V_m \tag{8.27}$$

is a decomposition into maximal Lagrangian subspaces. It is important to demand that V_m and $*V_m$ are transverse. (This condition can be slightly relaxed.)

3. Now, given (8.27) there is a unique decomposition of any $R \in V_{cl}$ as

$$R = R_m + R_e \tag{8.28}$$

with $R_m \in V_m$ and $R_e \in V_e := *V_m$.

4. The Lorentzian signature action for the ϵ-self-dual field is then

$$S = \pi \int (R_e * R_e + \epsilon R_e R_m) \tag{8.29}$$

There are two nice features of this action:

5. First, the action is stationary iff the ϵ-self-dual field $\mathcal{F} := R_e - \epsilon * R_e$ is closed:

$$d\mathcal{F} = 0 \tag{8.30}$$

Thus, the set of stationary points of the action is the set of solutions of the self-dual equations of motion for $\mathcal{F}$.

6. The second nice feature is that if we consider the action as a functional of both the metric and the field R then, varying the metric holding R fixed the action varies into

$$\delta S = \frac{\pi}{2} \int \mathrm{vol}\,(g) \delta g^{\mu\nu} T(\mathcal{F})_{\mu\nu} \tag{8.31}$$

where $T(\mathcal{F})_{\mu\nu}$ is the standard energy-momentum tensor for the ϵ-self-dual field $\mathcal{F} := R_e + \epsilon * R_e$.

7. The proof that the variation of the action gives (8.30) goes as follows:

$$\begin{aligned}
\delta S &= \pi \int 2\delta R_e * R_e + \epsilon \delta R_e R_m + \epsilon R_e \delta R_m \\
&= \pi \int 2\delta R_e * R_e + 2\epsilon \delta R_e R_m \\
&= 2\pi \int \delta R (*R_e + \epsilon R_m) \\
&= 2\pi \int d(\delta c)(*R_e + \epsilon R_m)
\end{aligned} \tag{8.32}$$

Where in the second line we used the fact that both R and δR are in V_{cl}, which is Lagrangian. In the third line we notice that $*R_e + \epsilon R_m \in V_m$, and hence we can replace δR_e by δR. In the fourth line we use the fact that variations of R in V_{cl} are exact. Now integration by parts gives $d(*R_e + \epsilon R_m) = 0$. Finally, $dR = dR_e + dR_m = 0$, so $d(*R_e + \epsilon R_m) = 0$ is equivalent to $d(*R_e - \epsilon R_e) = 0$ which is equivalent to $d(R_e - \epsilon * R_e) = 0$.

Remarks:

1. A similar discussion holds in Euclidean signature. In this case, it is automatically tree that $*V_m$ is transverse to V_m.

2. It is not difficult to take into account the possibility that spacetime has nontrivial cohomology.

3. The action is an infinite-dimensional version of the period matrix, and relies on a very elementary general construction given three pairwise transverse Lagrangian subspaces. ♣ EXPLAIN THIS MORE. Belov-Moore, 0605038, Remark 6.1. Try to make it comprehensible... ♣

4. The viewpoint presented here neatly resolves the difficulty mentioned in item 2 in Section 8.2. The periods of R can be quantized, while those of the closed self-dual fieldstrength $\mathcal{F}$ will be discrete, but will vary continuously with metric.

5. A common choice of V_m is given by choosing some timelike direction ξ and letting V_m be the forms anniliated by $\iota(\xi)$. It is commonly said that there is no Lorentz invariant action for the self-dual field, but this is not really true, as the following example shows. Let us consider the self-dual scalar on $\mathbb{R}^{1,1}$ so $*dx^{\pm} = \pm dx^{\pm}$. We can choose V_m to be of the form

$$V_m = \{R = f(x)dx^+ + (G \cdot f)(x)dx^-\} \tag{8.33}$$

where $f(x)$ is any suitably normalizable function and $(G \cdot f)(x) = \int G(x,y)f(y)d^2y$. Then V_m will be Lagrangian if $G(x,y) = G(y,x)$ is symmetric, and it will be Lorentz invariant if $G(\phi_\lambda(x), y) = \lambda^{-2}G(x, \phi_\lambda^{-1}(y))$, where $\phi_\lambda(x^+, x^-) = (\lambda x^+, \lambda^{-1}x^-)$. An example of such a kernel function would be $G(x,y) = (x^+ - y^+)^{-2}$. This leads to the action for a self-dual scalar field

$$S = \pi \int \partial_+\phi\partial_-\phi d^2x - \pi \int d^2x d^2y \partial_-\phi(x)G^{-1}(x,y)\partial_-\phi(y) \tag{8.34}$$

Note that it is nonlocal.

Open Problem: It is possible that with a suitable definition of "local" there is no local and Lorentz invariant Lagrangian for the self-dual field. We believe that the above actions are the most general possible actions for the self-dual field, so the problem is reduced to showing that there is no "local" and Lorentz invariant choice for V_m.

8.11 Quantum "spin" Chern-Simons theory in 3mod4 dimensions

Give analog of Belov-Moore for all dimensions:

k_{ij} determines an integral lattice K.

If K is even there is a well-defined TFT. If not we must specify some additional topological data: Integral Wu structure.

The quantum theory should only depend on the data of a finite group $D(K) = K^*/K$ with $\mathbb{Q}/\mathbb{Z}$-valued bilinear form and a quadratic refinement of that form.

Nice observation of Kapustin-Saulina; Freed, Hopkins, Lurie, Teleman; Banks-Seiberg: When $D(K)$ splits as $A \oplus \widehat{A}$ with A an Lagrangian subgroup the theory is equivalent to a gauge theory with finite gauge group A: Prove this for all dimensions, including the spin theories.

9. A General Theory of Self-Dual Fields

9.1 General self-dual fields

Proposal for data of general self-dual field based on Pontryagin self-dual generalized cohomology theory

9.2 Hopkins-Singer Quadratic Functor

Families $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$: Quadratic refinement for the self-dual cohomology theory for $\mathcal{Z}$ determines the rest: CS action on $\mathcal{Y}$ and line bundle for partition function for $\mathcal{X}$.

9.3 Formulation

Combine Dan's old orientifolds draft with my orientifold texnotes section on this.

10. RR fields and differential K-theory

10.1 The perturbative RR fields

10.2 The arguments for K-theoretic quantization

10.3 RR current in differential K-theory

Answer clearly the quantization of RR fluxes and charges in a way useful to people like Tomasiello. Give quantization of fluxes as $[G] = (\mathrm{ch}(x + \frac{1}{2}\theta)\sqrt{\hat{A}})_{\mathbb{R}}$ and explain about θ. Then explain about fluxes in the presence of charges. We have given the general principles above, but spell it out here.

10.4 Self-Duality

quadratic refinement for type II. $\int^{KO} \bar{\check{j}}\check{j}$.

10.5 B-fields and twisted K-theory

10.6 Orbifolds and Orientifolds: Equivariant K-theory and KR theory

Now $\check{j}$ is in twisted differential KR. It is nontrivial to write the quadratic refinement and compute its center.

10.7 Tachyons and a physical interpretation of the model of differential K-theory in terms of bundles with connection.

11. The abelian gauge field of M-theory

11.1 How the Chern-Simons Term Makes Sense

Let us return to the puzzle we mentioned in Section 4.6: For topologically trivial C-fields the "Chern-Simons term" in the 11-dimensional supergravity action is

$$\Phi(C) = \exp\left(2\pi i \int_{M_{11}} \frac{1}{6} CGG - CI_8\right) \tag{11.1}$$

It is not obvious how to extend this to a well-defined function for topologically nontrivial C-fields.

Since there is only one gauge fieldstrength $G \in \Omega^4(M_{11})$ in M-theory we expect that the C-field should be quantized by $\check{H}^4(M)$ and that $\Phi(C)$ should extend to a *cubic* refinement of the trilinear form

$$\check{H}^4 \times \check{H}^4 \times \check{H}^4 \to \mathbb{R}/\mathbb{Z} \tag{11.2}$$

The definition of this cubic refinement is a remarkable story, due mostly to Witten, with some finishing touches provided in [13]

1. We use the strange fact that $K(\mathbb{Z}, 4)$ is an approximation to BE_8 up to the 15-skeleton. Thus, an integral class $a \in H^4(M; \mathbb{Z})$ for M of dimension 11 (or 12) is uniquely associated with the isomorphism class of an E_8 bundle on M.

2. Using this observation one can make a groupoid of differential cocycles $\check{Z}^4(M)$ in terms of triples $\check{C} = (P, A, c)$ where P is an E_8 bundle on M, A is a connection on P, and $c \in \Omega^3(M)$ is a globally well-defined 3-forms. Among other things, there is a morphism $(P, A, c) \to (P, A', c')$ if

$$c' - c = CS(A, A') \tag{11.3}$$

   so $\mathrm{tr}\, F^2 + dc$ is gauge invariant.

3. Subtlety: Cancelation of anomalies on M2 branes means that actually $\check{C}$ should be a cochain trivializing a canonical cocycle $\check{W}_5$ representing $w_4 \in H^4(M; \mathbb{R}/\mathbb{Z})$ (viewed as a flat differential class of degree three). In the HS model this canonical cocycle is $\check{W}_5 = (0, \frac{1}{2}\lambda(g), 0)$, where $\lambda(g) = \frac{1}{16\pi^2}\mathrm{tr}\, R^2(g)$ so we require $\delta\check{C} = \check{W}_5$. Thus, there is a background magnetic current $\check{W}_5$ and $\check{C}$ if a field which trivializes that background magnetic current. The groupoid of C-fields is a torsor for the groupoid of cocycles $\check{Z}^4$.

4. One result of this subtlety - the only one we really need - is that we should define the fieldstrength to be

$$G = \mathrm{tr}\, F^2 - \frac{1}{2}\mathrm{tr}\, R^2 + dc \tag{11.4}$$

[13] Diaconescu, Freed, Moore; Freed and Moore

and in particular $[G] = a_R - \bar{p}_1/4$ is the magnetic Dirac quantization law. Note that w_4 has an integral lift λ so we can speak unambiguously of the topological class $c \in H^4(M_{11};\mathbb{Z})$ of the C-field.

5. Now we claim that the cubic Chern-Simons term in topologically nontrivial fields is

$$\Phi(\check{C}) = \exp[2\pi i(\frac{\xi(D_A)}{2} + \frac{\xi(D_{RS})}{4}) + 2\pi i I_{local}] \tag{11.5}$$

where D is the Dirac operator,

$$\xi(D) = \frac{1}{2}(\eta(D) + h(D)) \tag{11.6}$$

and

$$I_{\rm local} = \int_{M_{11}} \left(\frac{1}{2}cG^2 - \frac{1}{2}cdcG + \frac{1}{6}c(dc)^2 - cI_8(g)\right)$$

is an integral over globally well-defined forms.

6. Why should (11.5) have anything to do with the original expression $\Phi(C)$ of 11-dimensional supergravity? To see this we consider its variation, and that is most easily done by considering the APS index theorem, which will related the variation of the eta invariants to the variation of an index density in 12 dimensions.

7. The relevant index density is

$$\left[\frac{1}{2}i(D_A) + \frac{1}{4}i(D_{RS})\right]^{(12)}$$

Now a remarkable computation shows that

$$\left[\frac{1}{2}i(D_A) + \frac{1}{4}i(D_{RS})\right]^{(12)} = \frac{1}{6}G_Z^3 - G_Z I_8(g_Z) - d(i_{\rm local})$$

where $\partial Z = M$.

8. The extra division by 2 of $\xi(D_{RS})$ means that actually, over the space of metrics modulo diffeomorphisms Φ is a section of a line bundle with a connection with order two holonomy. However, in supergravity there are also fermions (the "gravitino") and

$$\text{Pfaff}(D_{RS})\Phi(\check{C}) \tag{11.7}$$

turns out to be a section of a geometrically trivializable with a *canonical* section.

11.1.1 The self-induced electric current and electric charge

Because of the trilinear term the theory of the C-field, even when freezing the 11-dimensional metric $g_{\mu\nu}$, is not a free field theory but has equation of motion

$$d * G = \frac{1}{2}G^2 - I_8 \tag{11.8}$$

However, if we scale up the metric to go to long distances: $g_{\mu\nu} \to t^2 g_{\mu\nu}$ then the equation of motion becomes

$$d * G = t^{-3}(\frac{1}{2}G^2 - I_8) \tag{11.9}$$

so the theory becomes free at long distances. So, we will proceed with our investigation of charges and fluxes since these should be "measurable" at long distances.

In particular, because of the equation (11.8), the C-field self-induces electric current $\check{j}_e$. We expect that to be an element of $\check{H}^8(M_{11})$. It is easy to construct

$$2\check{j}_e = \check{C} \cdot \check{C} - 2\check{I}_8 \tag{11.10}$$

but the division by two is tricky. DFM constructed the charge $q(\check{j}_e)$. It only depends on the topological class c of $\check{j}_e$ so we denote it as $q(c)$. Then $q(c)$ is a distinguished integral lift of

$$\frac{1}{2}c(c - \lambda) + 30\hat{A}_8$$

that exists when M_{11} has a spin structure. In particular, the Gauss law says that if $\partial M_{11} = N_{10}$ is closed then $q(c) = 0$.

11.2 Extension to manifolds with boundary

An important extension in physics is to the case where $M_{11} = \mathbb{R} \times S$ but the spatial slice S is not closed and has a boundary at *finite* geodesic distance. In this case $\Phi(\check{C})$ is a section of a nontrivial line bundle over the boundary data.

Following an important construction by Horava and Witten, Freed and Moore showed that if each boundary component of S is provided with an E_8 bundle with connection together with suitable fermions then the product of fermion determinants times $\Phi(\check{C})$ has a canonical trivialization.

11.3 Compactification on a circle bundle: Recover the AHSS differential $Sq^3 + H$

One of the most important "dualities" in string/M-theory is in fact the duality between M-theory and Type II string theory.

This duality states that M-theory on a circle bundle $S^1 \to M_{11} \to M_{10}$ should be "equivalent" to type II string theory on M_{10}.

Even at this vague level, that raises a significant puzzle: The GAGT in M-theory is that of a single gauge field quantized by $\check{H}^4(M_{11})$. However, the GAGT's of IIA string theory consist of

1. A gauge field quantized by $\check{H}^3(M_{10})$ (or, more properly by $\check{R}^{-1}(M_{10})$.
2. A RR gauge field quantized by differential $\check{K}^\tau(M_{10})$ where τ is a twisting (which includes the degree).

One piece of the resolution of this puzzle is the following.

Suppose $\pi : M_{10} \times S^1 \to M_{10}$ and suppose the class $c(\check{C}) = \pi^*(\bar{c}) + \bar{h} \cup \Theta$. Then for two choices $\bar{c}_1, \bar{c}_2$

$$\pi_*(q_e(c_1) - q_e(c_2)) = (Sq^3 + \bar{h})(\bar{c}_1 - \bar{c}_2) \tag{11.11}$$

$$\pi_*(2q_e(c)) = \bar{h}(2\bar{c} - \lambda) \tag{11.12}$$

The significance of this is that $Sq^3 + \bar{h}$ is the first nontrivial differential in the AHSS sequence relating K-theory to ordinary cohomology. Thus, from the vanishing of electric charge in M-theory we derive the statement that the integral quantization of the four-form flux in type II theory should have a twisted K-theory lift. By self-duality the four-form is related to the 8-form.

Strangely the 2-form flux comes from the KK reduction of the metric on the circle bundle, and self-duality relates it to the 8-form flux.

The 0-form flux (and its 10-form dual) are even more mysterious, and do not appear to have an interpretation in M-theory.

11.4 Quantization of electric flux

Page charge story.

12. Superconformal theories in six dimensions

12.1 The six-dimensional tensormultiplet

12.2 What we know about the nonabelian $(2,0)$ theories

12.3 Seiberg-Witten theory

Compactify partially twisted theory on $R^4 \times C$.

12.4 Higher rank tensormultiplet theories

12.5 Approach from M-theory Holographic dual on $AdS_7 \times M_4$

Get self-dual $(2,0)$ theory.

12.6 Holographically dual 7-dimensional theories

A. Volume forms on $\mathbb{R}^D$ with Euclidean metric

$\mathbb{R}^D$ with Euclidean metric with standard orientation $d^Dx = dx^1 \cdots dx^D$.

Radial coordinates:

$$d^Dx = r^{D-1} dr \Omega_{D-1} \tag{A.1}$$

$$\begin{aligned} \Omega_{D-1} &= \frac{1}{r^D} \iota(x^i \frac{\partial}{\partial x^i}) d^Dx \\ &= \sum_{i=1}^{D} (-1)^{i-1} \frac{x^i dx^1 \wedge \cdots \widehat{dx^i} \cdots \wedge dx^D}{r^D} \end{aligned} \tag{A.2}$$

Unit volume form on unit sphere S^{D-1}:

$$\omega_{D-1} := \frac{1}{V_D} \Omega_{D-1} \tag{A.3}$$

where

$$V_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} \tag{A.4}$$

Hodge $*$:

$$*dr = r^{D-1}\Omega_{D-1} \tag{A.5}$$

Useful alternative version ($D \neq 2$):

$$*d\left(\frac{1}{r^{D-2}}\right) = -(D-2)\Omega_{D-1} \tag{A.6}$$

Now obviously

$$d\omega_{D-1} = \eta(\vec{0}) = \delta^D(\vec{0})d^D x \tag{A.7}$$

so

$$d\left(\frac{1}{V_D} * \frac{dr}{r^{D-1}}\right) = \eta(\vec{0}) = \delta^D(\vec{0})d^D x \tag{A.8}$$

B. Computation of the spin of the field produced by a dual pair of electric and magnetic branes

We write

$$F_e = \frac{q_e}{2V_{D_e}} \frac{dt \wedge d\rho_e^2}{\rho_e^{D_e}} dx^{1\ldots p_e} \tag{B.1}$$

$$F_m = \frac{q_m}{V_{D_m}} \frac{1}{\rho_m^{D_m}} \iota\left(z_a \frac{\partial}{\partial z_a} + x_i \frac{\partial}{\partial x_i}\right) dz^{123} dx^{1\ldots p_e} \tag{B.2}$$

Now

$$\begin{aligned} T_{0a} &= \left(\iota(\frac{\partial}{\partial t})F, \iota(\frac{\partial}{\partial z_a})F\right) \\ &= \left(\iota(\frac{\partial}{\partial t})F_e, \iota(\frac{\partial}{\partial z_a})F_m\right) \\ &= \pm\frac{1}{2^?} \frac{q_e q_m}{V_{D_e} V_{D_m}} \frac{1}{\rho_e^{D_e} \rho_m^{D_m}} \epsilon_{abc} z^b z_0^c \end{aligned} \tag{B.3}$$

♣ NEED TO CHECK SIGN AND FACTOR OF 2 ♣

Therefore

$$\begin{aligned} J_{12} &= \int d^3\vec{z} d^{p_e}\vec{x} d^{p_m}\vec{y}\, (z_1 T_{02} - z_2 T_{01}) \\ &= \pm\frac{1}{2^?} \frac{q_e q_m}{V_{D_e} V_{D_m}} L \int d^3\vec{z} d^{p_e}\vec{x} d^{p_m}\vec{y} \frac{(z_1^2 + z_2^2)}{\rho_e^{D_e} \rho_m^{D_m}} \\ &= q_e q_m \end{aligned} \tag{B.4}$$

♣ Need to check the integrals actually work! ♣

C. Some notational choices

1. $\mathcal{H}^q(M)$: the real vector space of harmonic q-forms on M

2. M a manifold, usually of dimension n.

3. $\Omega^\ell(M)$ the topological vector space of all smooth differnetial forms of degree ℓ on M.

4. $\Omega^\ell_d(M)$ the topological vector space of all closed differnetial forms of degree ℓ on M.

5. $\Omega^\ell_{\mathbb{Z}}(M)$ the topological vector space of all differnetial forms of degree ℓ on M with integral periods. Note $\Omega^\ell_{\mathbb{Z}}(M) \subset \Omega^\ell_d(M)$.