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Lectures On The Physical Approach To Donaldson And Seiberg-Witten Invariants Of Four-Manifolds

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Abstract: These are lecture notes for lectures at the Simons Center for Geometry and Physics scheduled for March 22-24, 2017. A truncated version was delivered at the Pre-String-Math school in Hamburg, July 17-18, 2017. THESE NOTES ARE UNDER CONSTRUCTION. They are available at http://www.physics.rutgers.edu/∼gmoore/SCGP-FourManifoldsNotes-2017.pdf COMMENTS WELCOME. August 2, 2017

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1. Some Background History And The Plan Of The Lectures

Let us summarize some of the standard results on 4-manifolds. See the texbooks [8, 16, 17, 48] for details.

In these lectures X will always denote a compact, connected, orientable four-manifold without boundary.

1.1 Fundamental Group

First, if π is any finitely presented group then there is a compact 4-fold X with $\pi_1(X) \cong \pi$. [1] Since the word problem for groups is undecidable this means we cannot hope to classify all compact 4-manifolds. But we can still hope to understand simply connected 4-folds.

[1] To prove this theorem of Markov one takes a connected sum of $S^1 \times S^3$, one summand for each generator in the finite presentation of π. Choose a basepoint in the connected sum and a representative of each of the generators of the fundamental group. Now in the presentation of π each relation corresponds to a word in these generators, and hence corresponds to an embedded circle in the connected sum. Draw a tubular neighborhood around each of the embedded circles so that the neighborhood is $D \times S^1$. Displace these so that they do not intersect. Next do surgery to replace by neighborhoods of the form $\tilde{D} \times S^1$ where now the longitude becomes contractible in the disk $\tilde{D}$. Finally, using the Seifert-van Kampen theorem one can prove that the resulting manifold has $\pi_1(X) \cong \pi$. Four is the first dimension in which this happens: See Appendix D below.

1.2 Intersection Form

There is another interesting topological invariant, the intersection number. The intersection number of an oriented compact four-manifold gives an invariant:

$$H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \to \mathbb{Z} \tag{1.1}$$

It is an invariant of the homotopy type of X. Poincaré duality says that on free group $H^2(X;\mathbb{Z})/\mathrm{Tors}(H^2(X;\mathbb{Z}))$ of rank $b_2(X)$ it is a perfect pairing, and therefore corresponds to a symmetric integral unimodular bilinear form Q_X.

If α is a cohomology class Poincaré dual to $S(\alpha)$ then we the oriented intersection number can be written in several different ways:

$$S(\alpha)\cdot S(\beta) = \int_X \alpha\beta = \int_{S(\alpha)} \beta \tag{1.2}$$

The way the group $H^2(X;\mathbb{Z})/\mathrm{Tors}(H^2(X;\mathbb{Z}))$ will enter in our considerations is as a lattice in the vector space of DeRham cohomology classes

$$\bar{H}^2(X) \subset H^2_{DR}(X) \tag{1.3}$$

defined as the set of classes with integral periods. Of course this only makes sense when X has a differentiable structure. In that case

$$\bar{H}^2(X) \cong H^2(X;\mathbb{Z})/\mathrm{Tors}(H^2(X;\mathbb{Z})) \tag{1.4}$$

and

$$Q_X(\omega_1,\omega_2) := \int_X \omega_1 \wedge \omega_2 \tag{1.5}$$

The intersection form Q_X has signature $(+1^{b_2^+}, -1^{b_2^-})$. That is, if we consider it as a quadratic form on the real vector space $H^2(X;\mathbb{Z})\otimes\mathbb{R}$ then, after a suitable choice of basis it can be brought to this diagonal form. More invariantly, b_2^+ is the rank of the maximal sublattices in $H^2(X;\mathbb{Z})$ on which the restriction of Q_X is positive definite.

When X is oriented and has a Riemannian metric (or just a conformal structure) we can define a Hodge dual $* : \Omega^2(X) \to \Omega^2(X)$. It satisfies $*^2 = +1$ and so we can speak of <u>self-dual</u> forms:

$$*\omega = \omega \tag{1.6}$$

and <u>anti-self-dual</u> forms:

$$*\omega = -\omega \tag{1.7}$$

The Hodge theorem allows us to identify $H^2(X;\mathbb{R})$ with the space of harmonic two-forms $\mathcal{H}^2(X)$ and $*$ preserves this space. Then we can interpret b_2^+ as the dimension of the vector space of harmonic self-dual two-forms, and b_2^- as the dimension of the vector space of harmonic anti-self-dual two-forms.

Remark: Later on we will be working with various torsors of $\bar{H}^2(X)$ inside $H^2(X;\mathbb{R})$. For example, given an integral lift $\bar{w}_2(X)$ of $w_2(X)$ we will form

$$\Gamma_w := \frac{1}{2}\bar{w}_2(X) + \bar{H}^2(X) \subset H^2(X;\mathbb{R}) \tag{1.8}$$

Similar torsors associated to integral lifts of other mod-two classes will also play a role.

1.3 Whitehead Theorem

In 1949 J.H.C. Whitehead introduced the notion of CW decomposition of manifolds to classify homotopy type. In [35] Milnor observed that an interesting consequence is that two simply connected oriented four-manifolds X_1, X_2 are homotopy equivalent iff $Q_{X_1} \cong Q_{X_2}$. To prove this one notes that the cell-decomposition of a simply connected four-fold is

$$\bigvee_i S_i^2 \vee D^4 \tag{1.9}$$

and is determined by the homotopy class of a map $f : S^3 \to \bigvee_i S_i^2$, which can be related to the intersection matrix. [2]

1.4 Serre's Theorem

Thus we come to the classification of integral unimodular forms. Serre gave a nice classification in the *indefinite* case.

	Indefinite	Definite
even	$mE_8 \oplus nH$,$m \in \mathbb{Z}$, $n > 0$	$1, 2, 24, > 10^7, \ldots$
odd	$m\langle +1 \rangle \oplus n\langle -1 \rangle$	too many

The even definite forms only exist in dimension 0 modulo 8. We have listed the number of inequivalent ones for the first few cases. The unique lattice in dimension 8 is the E_8 root lattice.

H denotes the even integral form on $\mathbb{Z} \oplus \mathbb{Z}$ given by

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.10}$$

1.5 Freedman's Theorem: Homeomorphism Type

Michael Freedman achieved a stunning breakthrough in a 1982 paper:

For all unimodular integral forms Q there is a simply connected compact orientable topological manifold X with $Q \cong Q_X$. Moreover,

1. If Q is even then there is a unique such X up to homeomorphism.

2. If Q is odd then there are exactly two homeomorphism types and at most one of them can be smooth.

As an example of how breathtaking this is note that for $Q = 0$ this proves the (four-dimensional, topological) Poincaré conjecture. For $Q = 1$ we have $X = \mathbb{C}P^2$ but there must be another manifold, "fake $\mathbb{C}P^2$" which is homeomorphic to $\mathbb{C}P^2$ but does not admit a smooth structure!

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[2] In more detail, the homotopy class of such a map can be characterized as follows. For each S_i^2 we choose a point p_i that is not the basepoint of the wedge product. We can choose p_i to be a regular value of f so that the preimage under f is a knot in S^3. There is then a linking matrix for these knots (with diagonal matrices being the self-linking number). This linking matrix is closely related to the intersection matrix on $H^2(X, \mathbb{Z})$. I thank J. Morgan and G. Segal for explanations of this fact.

1.6 Donaldson's Theorems: Diffeomorphism type

Almost simultaneously with Freedman's work (the papers were published in 1983) Simon Donaldson announced some equally striking theorems.

First, if X admits a smooth structure and Q_X is definite, then it must be diagonal: $m\langle 1\rangle$ with nonzero $m \in \mathbb{Z}$.

Remarkable corollaries include the fact that the manifold corresponding to $2E_8$ does not admit a smooth structure. (All previous known tests - notably Rokhlin's theorem - admitted the possibility that it might.) Similarly, if Q is odd and definite and nonstandard (i.e. not a diagonal matrix of all $+1$ or all -1) then neither of the two homeomorphism types admits a smooth structure.

Second, Donaldson introduced his famous polynomial invariants. These are a sequence of polynomial function on $H_0(X) \oplus H_2(X)$ which are invariants of the *smooth structure* of X. An example of striking statement that follows from his invariants is that one cannot write an algebraic surface as a nontrivial connected sum $X_1 \# X_2$ with $b_2^+(X_i) > 0$. In particular, it cannot be written as a connected sum of two algebraic surfaces. [3]

Donaldson's construction used nonabelian gauge theory for rank one gauge groups $G = SU(2)$ and $G = SO(3)$. He defined the polynomials using the intersection theory on the moduli space of anti-self-dual connections on principal G bundles over X. Now, the equation

$$F^+ = F + *F = 0 \tag{1.11}$$

makes use of a Riemannian metric. But the dependence on the metric drops out *except* for manifolds with $b_2^+ = 1$. In this case the polynomials are piecewise constant in the space of metrics but jump across walls. A wall-crossing formula for how they jump across walls is completely understood. We will come back to these important facts.

Donaldson's invariants were used to prove some striking facts about the smooth structures of 4-manifolds. The world of topological four-manifolds can be quite wild. There are, for example, continuously infinitely many different differentiable structures on $\mathbb{R}^4$.

After all this progress – Freedman and Donaldson both received the 1986 Fields medal – it was natural to wonder if physics was playing an important role. After all, Donaldson was using nonabelian gauge theory and instantons. Nonabelian gauge theory and instantons play a major role in modern particle theory.

This is where Witten enters. In 1988 he gave a quantum field theoretic description of Donaldson polynomials [53]. We will describe it in detail (in part following a particularly beautiful approach to Witten's paper introduced by M. Atiyah and L. Jeffrey [3]). In a word: The partition function Z_{DW} on a four-fold X with certain observables added to the action is a generating function for all the Donaldson polynomials on X. Technically Witten's QFT is known as the "N=2 supersymmetric extension of G Yang-Mills theory" where G is simple and rank one. ♣Comment on motivation from M. Atiyah [2] ? ♣

Witten's interpretation was beautiful - it was the genesis of the concept of topological twisting and more broadly of topological field theory - but it was not clear what could be

[3] Recall that $b_2^+(X)$ is the rank of the maximal positive definite subspace of $H^2(X;\mathbb{R})$.

gained mathematically from an interpretation of the Donaldson polynomials in terms of a path integral. The problem is that the path integral of a four-dimensional interacting quantum field theory is regarded by the mathematical community as a mythological being. And even for physicists willing to believe in its existence, doing effective computations looked like they were out of reach.

However, since the theory is topological, the path integral Z_{DW} does not depend on the metric on X. But if "we scale up the metric on X" meaning we replace:

$$g_{\mu\nu} \to t g_{\mu\nu} \tag{1.12}$$

and study the limit $t \to +\infty$ then all length scales go to infinity. In physics, length scales and energy scales are related by the uncertainty principle:

$$L \sim \frac{\hbar}{E} \tag{1.13}$$

so scaling lengths to infinity is the same as studying the "far infrared" - the behavior of the theory under processes that differ only infinitesimally from the vacuum. This is precisely the situation in which one can give an entirely different description of the theory using what is known as a *Low Energy Effective Theory* or LEET.

A good example of the use of LEET is QCD: At short distance (relative to a length scale at which the interactions become strong) the strong interactions are described by quarks and gluons, but at long distances they are described merely by pions. The short-distance theory is a nonabelian Yang-Mills theory with gauge group $SU(3)$ minimally coupled to Dirac spinors in the direct sum of N_f copies of the fundamental representation. The long distance theory is a nonlinear sigma model whose target space is $SU(N_f)$. (In nature $N_f = 2$ or $N_f = 3$ depending on how accurate the LEET should be.)

For our purposes, with a sufficiently good understanding of the LEET of N=2 $SU(2)$ SYM one can hope to recast the Donaldson-Witten path integral in a new form which might yield new insights.

This is precisely what happened. In the spring of 1994 Seiberg and Witten understood the LEET of N=2 SU(2) SYM [47, 46]. This was sufficient information for Witten to give a stunning reformulation of the Donaldson polynomials in terms of a new set of (more tractable) four-manifold invariants known as "Seiberg-Witten invariants" [55]. The formula he wrote is known as Witten's conjecture, and added a key piece of information to a structure theorem for Donaldson invariants that had been discovered previously by Kronheimer and Mrowka [21].

♣Introductory remarks took 36 minutes. ♣

The goal of these lectures is to:

1. Explain Witten's formal field theory interpretation of Donaldson's polynomials in terms of N=2 SU(2) SYM.

2. Explain how Seiberg and Witten's physical insights into the low energy dynamics of $N = 2$ SYM lead to a compelling (to me) derivation of the Witten conjecture.

3. Explain how the physical viewpoint explains many of the results on Donaldson polynomials and even has led to some mathematical predictions.

Some important references for the physical approach to Donaldson invariants are:

1. Les Houches lectures of Cordes, Moore, Ramgoolam [4]
2. Textbook of Labastida and Marino [24].
3. Two IAS Volumes: P. Degligne, et. al., *Quantum Fields and Strings: A Course for Mathematicians*, 2 volumes. AMS 1999. Of special relevance are the lectures by E. Witten in volume 2: Lectures 1-3, 8,12, and especially Lectures 16- 19.

2. Plan For The Rest Of The Lectures

1. Formal structure of cohomological TFT: Mathai-Quillen form of the path integral and localization.
2. How Donaldson Theory fits into the MQ framework: Topologically twisted N=2 SYM.
3. SW solution and structure of the vacuum: Mapping observables from UV to IR.
4. General form of the Higgs branch contribution.
5. Evaluation of the Coulomb branch: The u-plane integral.
6. Deriving the relation of Donaldson to SW invariants.
7. Simple type
8. Applications of the physical viewpoint: (Example: Superconformal simple type and the generalized Noether inequality.)
9. Possible future directions.

2.1 Acknowledgements

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3. A Brief Review Of Cohomological TFT Path Integrals

3.1 A Nice Integral

Today we are going to talk about some complicated and fancy integrals, some of them very complex but well-defined and finite dimensional, some of them infinite-dimensional integrals over functions spaces - the notorious path integrals of QFT. So it is good to start with some simple integrals we can all immediately appreciate and understand.

Let x be a real number and consider a function $s(\mathrm{x})$ whose graph is transverse to the x axis such that $|s(\mathrm{x})| \to \infty$ for $|\mathrm{x}| \to \infty$, as in

FIGURE

Consider the Gaussian integral:

$$Z = \int_{-\infty}^{+\infty} \frac{d\mathrm{x}}{\sqrt{2\pi}} s'(\mathrm{x}) e^{-\frac{1}{2}s(\mathrm{x})^2} \tag{3.1}$$

It is easy to do the integral by change of variable, and you find:

$$Z = \sum_{\mathcal{Z}(s)} \frac{s'(\mathrm{x}_\ell)}{|s'(\mathrm{x}_\ell)|} \tag{3.2}$$

where

$$\mathcal{Z}(s) = \{\mathrm{x}_\ell : s(\mathrm{x}_\ell) = 0\}. \tag{3.3}$$

It is easy to generalize to n-dimensions. Now $s : \mathbb{R}^n \to \mathbb{R}^n$ and:

$$Z = \int_{\mathbb{R}^n} \prod_{i=1}^n \frac{d\mathrm{x}_i}{\sqrt{2\pi}} \det(\frac{\partial s^i}{\partial \mathrm{x}^j}) e^{-\frac{1}{2}s(\mathrm{x})^2} = \sum_{\mathcal{Z}(s)} \mathrm{sign}(\det \frac{\partial s^i}{\partial \mathrm{x}^j}) \tag{3.4}$$

where we use the Euclidean metric on $\mathbb{R}^n$ to define $s(\mathrm{x})^2$.

Let us make a few remarks about these integrals

1. The answer is a sum of integers. It is a *signed* sum over solutions of the n real equations in n unknowns:
$$s(\mathrm{x}) = 0 \tag{3.5}$$
Our integral Z is counting solutions to equations with signs.

2. In fact, this integer has topological significance. It is the *degree* of the (proper) map $s : \mathbb{R}^n \to \mathbb{R}^n$. Another topological interpretation is that it is the oriented intersection number of the graph of s with the graph of $s = 0$. That intersection number is a topological invariant *provided we do not change the asymptotic behavior of $s(\mathrm{x})$ at $\mathrm{x} \to \infty$.* For example, in the one variable case, if $s(\mathrm{x})$ is a polynomial we cannot expect invariance under change of the sign of the leading coefficient.

3. Finally, note that we could put in a parameter $\hbar$ and equally well say:

$$Z = \int_{\mathbb{R}^n} \prod_{i=1}^n \frac{d\mathrm{x}_i}{\sqrt{2\pi\hbar}} \det(\frac{\partial s^i}{\partial \mathrm{x}^j}) e^{-\frac{1}{2\hbar}(s(\mathrm{x}),s(\mathrm{x}))} = \sum_{s(\mathrm{x})=0} \mathrm{sign}(\det \frac{\partial s^i}{\partial \mathrm{x}^j}) = \deg(s) \tag{3.6}$$

The answer is independent of $\hbar$. On the other hand, we could take $\hbar \to 0$ and clearly the measure *localizes* to the zero set

$$\mathcal{Z}(s) := \{\mathrm{x} : s(\mathrm{x}) = 0\} \tag{3.7}$$

Moreover, the saddle-point approximation gives the exact answer.

Now we are going to rewrite this integral as an integral over a superspace so that all these nice properties can be explained by the existence of a nilpotent "supersymmetry operator" Q. Our integrals over superspace will be finite dimensional models for topologically twisted path integrals in supersymmetric field theories.

3.2 Supersymmetric Representation Of The Nice Integral

3.2.1 Superspace

We recall a basic construction in supergeometry: [4] Suppose $E \to M$ is a vector bundle over a manifold. We let ΠE denote the superspace where the fibers are considered odd. [5] If we apply this to $E = TM$ then the associated *superspace* is denoted $\widehat{M}$, so:

$$\widehat{M} = \Pi TM \tag{3.8}$$

If x^i are local coordinates in a patch $\mathcal{U} \subset M$ then ψ^i are corresponding odd fiber coordinates.

We now have the key isomorphism between the complex of smooth superfields and the complex of smooth differential forms:

$$\mathcal{C}^\infty(\widehat{M}) \cong \Omega^*(M) \tag{3.9}$$

The basic idea is that a superfield is locally of the form

$$\Phi(\mathrm{x}, \psi) = \sum_k \sum_{i_1 < \cdots < i_k} \phi_{i_1 \cdots i_k}(x) \psi^{i_1} \cdots \psi^{i_k} \tag{3.10}$$

and the $\phi_{i_1 \cdots i_k}(x)$ transform across patches just like the coefficients of the differential form

$$\omega_\Phi = \sum_k \sum_{i_1 < \cdots < i_k} \phi_{i_1 \cdots i_k}(x) d\mathrm{x}^{i_1} \cdots d\mathrm{x}^{i_k} \tag{3.11}$$

The reason is that ψ^i are the linear functions on the fibers of TM:

$$\psi^i\left(\frac{\partial}{\partial \mathrm{x}^j}\right) = \delta^i{}_j \tag{3.12}$$

and are odd - so they can be identified with $d\mathrm{x}^i$. We will write $\mathcal{O}_\omega$ for the superfield corresponding to the differential form ω. Conversely $\omega_{\mathcal{O}}$ is the differential form associated to the superfield $\mathcal{O}$.

[4] For some background on supergeometry see [6][25][58].

[5] A supermanifold is defined by its "algebra of functions." Locally the functions form a sheaf of superalgebras. If $E \to M$ is a vector bundle and $\mathcal{U} \subset M$ is a neighborhood on which we trivialize $E|_{\mathcal{U}} \cong \mathcal{U} \times V$ for a vector space V then the corresponding Grassmann algebra is $\mathcal{C}^\infty(\mathcal{U}) \otimes \Lambda^* V$. Elements of this algebra are called superfields.

1. There is an integral grading so that "*ghost number*" corresponds to degree of the differential form:

$$\mathrm{gh}\#(\mathcal{O}_\omega) = \deg(\omega) \tag{3.13}$$

2. There is a degree one derivation which squares to zero:

$$Q\mathcal{O}_\omega \leftrightarrow d\omega \tag{3.14}$$

   Note particularly that in terms of local coordinates:

$$Q\mathrm{x}^i = \psi^i \qquad Q\psi^i = 0 \tag{3.15}$$

   We are particularly interested in the Q-cohomology H_Q^* and in integrals over superspace of Q-closed "operators" that only depend on the cohomology class.

3. To define these integrals we need to choose an orientation on the reduced manifold M. In general, superfields on a superspace can be integrated once one has chosen a section of the Berezinian of the cotangent bundle of the superspace, i.e. a section of $\mathrm{Ber}(\Omega^1(\hat{M}))$. In our case there is a canonical section that we will write as $\mathrm{ber}(\mathrm{x}|\psi)$. It has the property that

$$\int_{\widehat{M}} \mathrm{ber}(\mathrm{x}|\psi)\mathcal{O}_\omega = \int_M \omega \tag{3.16}$$

   where:

$$\mathrm{ber}(\mathrm{x}|\psi) = d\mathrm{x}^1 \cdots d\mathrm{x}^n [d\psi^1 \cdots d\psi^n] \tag{3.17}$$

   ♣Maybe include factors of (2π) in this definition? ♣

   where compatibility with the Fubini theorem forces:

$$\int [d\psi^1 \cdots d\psi^n]\psi^n \cdots \psi^1 = +1 \tag{3.18}$$

   we will also write this loosely at $\prod_{i=1}^n d\mathrm{x}^i d\psi^i$.

3.2.2 Rewriting The Integral

Now we introduce anticommuting variables χ_a, where, for the moment, $a = 1, \ldots, n$ and rewrite the integral as

$$Z = \xi i^n \int_{\widehat{\mathbb{R}^n}} \prod_{i=1}^n \frac{d\mathrm{x}^i d\psi^i}{\sqrt{2\pi\hbar}} \int_{\Pi(\mathbb{R}^n)^*} \prod_{a=1}^n d\chi_a e^{-\frac{1}{2\hbar}s^a(\mathrm{x})s^a(\mathrm{x}) + i\chi_a \frac{ds^a}{d\mathrm{x}^j}\psi^j} \tag{3.19}$$

Now, we want to write this in a manifestly Q-invariant way, so we need commuting partners H_a for the χ_a. We simply introduce them via a Gaussian integral:

$$Z = \xi(2\pi \mathrm{i})^{-n} \int_{\widehat{E}} \mathrm{ber}(\mathrm{x}, H|\psi, \chi) e^{-\frac{\hbar}{2}H_a H_a - iH_a s^a + i\chi_a \frac{ds^a}{d\mathrm{x}^j}\psi^j} \tag{3.20}$$

where $\xi = \pm 1$ and $\widehat{E}$ is a superspace so that the functions

♣With our conventions there is a definite sign. Work it out. ♣

$$\mathcal{C}^\infty(\widehat{E}) \tag{3.21}$$

are $\mathcal{C}^\infty$ functions of the bosonic variables x^i and H_a, both of ghost number zero and are polynomials in the fermionic variables ψ^i of ghost number one and χ_a of ghost number -1:

$$\mathrm{gh}\#(\chi_a) = -1 \qquad\qquad \mathrm{gh}\#(H_a) = 0. \tag{3.22}$$

The Berezinian measure is, concretely,

$$\mathrm{ber}(\mathrm{x}, H|\psi, \chi) = \prod_i d\mathrm{x}^i \prod_a dH_a [d\psi^1 \cdots d\psi^n d\chi_1 \cdots d\chi_n] \tag{3.23}$$

Now we have Z in a very useful form. Note:

1. If we extend Q so that
$$Q\chi_a = H_a \qquad QH_a = 0 \tag{3.24}$$
then we can write the "action" S (so the path integral contains e^S) as:
$$S = Q(\Psi) \tag{3.25}$$
$$\Psi = -\frac{\hbar}{2}\chi_a H_a - i\chi_a s^a \tag{3.26}$$

2. **Crucial Point**: Now note that since $Q \leftrightarrow d$ if we change the action by $Q(\Delta\Psi)$ so that the integral over the boundary (in this case, the integral at infinity) vanishes, then the integral is unchanged. That is

   Small Q-exact perturbations of the action do not change the result of the integral.

   The technical meaning of "small" should mean: Do not introduce singularities and do not "drastically" change the asymptotic behavior of the action at infinity in field space. A good example of a deformation which is not allowed is to consider $n = 1$ and take $s(x)$ to be a polynomial with leading order term of degree N:
$$s(x) = \varepsilon x^N + \cdots . \tag{3.27}$$
   A variation of ε does not change the answer unless the sign of ε changes. The latter is a "drastic" deformation. Note that in a continuous family where ε changes sign the order of the polynomial must drop below N. It would be desirable to give a useful technical definition of "drastically," especially in the case of path integrals.

3. In cohomological field theory, (χ_a, H_a) is called the *anti-ghost multiplet*. Note that χ_a has ghost number -1, so H_a has ghost number 0. The fields H_a are also called *auxiliary fields*. By definition, auxiliary fields are fields that can be eliminated from action by doing a Gaussian integral (or, more generally, by solving a purely algebraic - as opposed to differential - equation).

4. **Q-Fixed Points**: A very useful heuristic viewpoint on the localization of the integral Z is provided by the idea of Q-fixed points. We noted that for $\hbar \to 0$ the integral localizes on the zero set $\mathcal{Z}(s)$. Observe that this is the same as the space of Q-fixed points:
$$Q(Fields) = 0 \tag{3.28}$$

We solve this by putting the anticommuting fields to zero (<u>after</u> evaluation with Q and after setting the auxiliary fields to their stationary values in the Gaussian integral). So we need only examine $Q(\psi)$ and $Q(\chi)$. The equation $Q(\chi) = 0$ puts $H = 0$ but the Gaussian elimination set $H_a = -is^a/\hbar$ so that the Q-fixed point locus coincides with $\mathcal{Z}(s)$. There is a nice intuitive way of understanding this: We think of the integral as an integral on a space with an odd vector field. As long as the vector field acts freely then, since $\int d\theta = 0$ for a Grassmann integral, that "part of superspace," where the integrand does not depend on θ cannot contribute to the integral. (This is, of course, an extremely heuristic step.) Therefore the supersymmetric integral localizes to Q-fixed points. While it is very heuristic, this viewpoint is also extremely useful because of the meta-theorem:

All natural geometrical PDE's are Q-fixed point equations of some supersymmetric field theory.

3.2.3 A Generalization: The Case Of "Nonzero Index"

We can generalize a little by letting $s : \mathbb{R}^n \to V$ where $V \cong \mathbb{R}^m$ has dimension m not necessarily equal to n. Now we let the index on the anti-ghost multiplet (χ_a, H_a) run over $a = 1, \ldots, m$.

Define a superfield on $\widehat{M}$ by integrating out the antighost multiplet:

$$\widehat{\mathrm{Eul}}_s := \int_{\widetilde{V}} \prod_{a=1}^{m} \frac{d\chi_a dH_a}{(2\pi i)} e^{Q(\Psi)} \tag{3.29}$$

where $\widetilde{V}$ is a superspace whose functions are functions of degree zero bosonic variables H_a and degree -1 variables χ_a. This is BRST closed superfield in x^i and ψ^i and hence represents a closed differential form Eul_s via the correspondence (3.9). Importantly, it has ghost number m. This is obvious: χ_a has ghost number -1 so each $[d\chi_a]$ has ghost number $+1$ so $[d\chi_1 \cdots d\chi_m]$ has ghost number m.

Now, if $\mathcal{O}(\mathrm{x}, \psi)$ is another Q-closed observable, then we can consider the more general integral

$$\langle \mathcal{O} \rangle := \int_{\widehat{\mathbb{R}^n}} \mathrm{ber}(\mathrm{x}|\psi) \;\; \mathcal{O} \;\; \widehat{\mathrm{Eul}}_s \tag{3.30}$$

Note that

1. We are assuming there are more variables than equations, so $n > m$ and, at least along $\mathcal{Z}(s)$, the linear transformation ds is surjective.

2. The integral can only be nonzero when the ghost number of $\mathcal{O}$ is $n - m$.

3. The integral only depends on the cohomology class of $\mathcal{O}$ in H^*_Q.

Moreover, if $\mathcal{O}_\omega$ is a Q-closed superfield corresponding to a closed from ω is closed then we can generalize what we said above:

$$\langle \mathcal{O}_\omega \rangle = \int_{\widehat{\mathbb{R}^n}} \mathrm{ber}(\mathrm{x}|\psi) \; \mathcal{O}_\omega \; \widehat{\mathrm{Eul}}_s = \int_{\mathbb{R}^n} \omega \wedge \mathrm{Eul}_s = \int_{\mathcal{Z}(s)} \iota^*(\omega) \tag{3.31}$$

localizes on the zero set $\mathcal{Z}(s)$. Here $\iota : \mathcal{Z}(s) \hookrightarrow \mathbb{R}^n$ is the inclusion map and $\deg(\omega) = n-m$.

In the QFT application n and m will both be infinite, but $n-m$ will be the index of a Fredholm operator and hence the integral on the RHS is over a finite-dimensional space. We will repeat this remark below.

3.3 Thom Isomorphism Theorem

The localization identity (3.31) is very reminiscent of the Thom isomorphism theorem. We first recall that theorem and then use it to generalize (3.31).

Recall the Thom isomorphism theorem: Let $\pi : E \to M$ be a real oriented vector bundle over an n-dimensional manifold, where E has rank m. Then the theorem says there is an isomorphism

$$H^i(M) \cong H^{i+m}_{v-cpt}(E) \tag{3.32}$$

$$\omega \to \pi^*(\omega)\Phi(E) \tag{3.33}$$

Moreover if $s : M \to E$ is a generic section then $s^*\Phi(E)$ is the *Euler class of* E. If M is compact the Euler class is Poincaré dual to the zero set of s:

$$\int_M \omega s^*(\Phi(E)) = \int_{\mathcal{Z}(s)} \iota^*(\omega) \tag{3.34}$$

In topology the condition of compact vertical support is very natural, but it is not so natural in physics. To make contact with physics we endow $E \to M$ with a a Riemannian metric on the fibers of E. Then we can replace compact vertical support by rapid decrease (so the relevant forms are in a Schwarz-space along the fibers). We can then replace cohomology with compact vertical supports by cohomology with rapid decrease along fibers: $H^*_{v-cpt}(E) \cong H^*_{rd}(E)$.

The Thom theorem with $H^*_{rd}(E)$ motivates us to generalize $\widehat{\mathrm{Eul}}_s$ above to the case where we replace $\mathbb{R}^n \times V$ by an oriented real vector bundle $E \to M$. To do this we must give E a connection ∇ compatible with the fiber metric. We will denote [6] the local one-form relative to an ON basis by Θ^{ab}_j. To covariantize the action we must add a third term to Ψ:

$$\Psi = -\frac{\hbar}{2}\chi_a H_a - i\chi_a s^a + \frac{\hbar}{2}\chi_a \Theta^{ab}_i \psi^i \chi_b \tag{3.35}$$

Working out $Q\Psi$ the third term covariantizes the derivative of s, and integrating out the auxiliary fields H_a we find

$$S = -\frac{1}{2\hbar} s^a s^a + i\chi_a (\nabla_j s)^a \psi^j + \frac{\hbar}{4}\chi_a \chi_b F^{ab}_{ij}\psi^i\psi^j \tag{3.36}$$

3.4 The Localization Formula

The general localization formula associated with a real oriented vector bundle over a manifold, $\pi : E \to M$, together with a generic section s, is the following:

[6] Thus, if e^a is a local ON basis of sections of E and ∇ is the connection then $\nabla e^a = dx^j \Theta^{ab}_j e^b$ with a sum over j and b.

Define the superfield on $\widehat{M}$:

$$\widehat{\mathrm{Eul}}_s(E,\nabla) := \int \prod_{a=1}^{m} \frac{d\chi_a dH_a}{(2\pi\mathrm{i})} e^S \tag{3.37}$$

where $S = Q(\Psi)$ is given by (3.36) and (3.35).

The connection ∇ on E defines a linear operator:

$$\nabla s : T_p M \to E_p \tag{3.38}$$

We will assume that s is sufficiently generic so that the fibers of $\mathrm{Cok}\nabla s$ defined by the exact sequence

$$0 \to \mathrm{Im}\nabla s \to E \to \mathrm{Cok}\nabla s \to 0 \tag{3.39}$$

have finite rank and define a smooth vector bundle. Given orientations on M and E the bundle $\mathrm{Cok}\nabla s$ is canonically oriented and we claim that for Q-closed functions $\mathcal{O}$ on superspace:

$$\int_{\hat{E}} \mathrm{ber}(\mathrm{x}, H|\psi,\chi) e^S \mathcal{O} = \int_{\widehat{M}} \mathrm{ber}(\mathrm{x}|\psi) \widehat{\mathrm{Eul}}_s(E,\nabla)\mathcal{O} = \int_{\mathcal{Z}(s)} \iota^*(\omega_{\mathcal{O}}) \wedge \mathrm{Eul}(\mathrm{Cok}\nabla s) \tag{3.40}$$

where $\hat{E}$ is the total superspace corresponding to the bundle E. It is the natural generalization of (3.21).

Remarks:

1. The proof of (3.40) is straightforward. See [4].

2. It is possible to view $\widehat{\mathrm{Eul}}_s(E,\nabla)$ as the pullback by s of a representative of a Thom class $\hat{\Phi}(E)$. This particular representative of the Thom class in $H^m_{rd}(E)$ is due to Mathai and Quillen. Note that it has rapid decrease along the fibers rather than compact support. For a full explanation see [4].

♣Give section number, and also refer to Witten as in the CMR reference. ♣

3. When we discuss the equivariant case below we will need to add a term to Ψ. Then we will refer to Ψ in (3.35) as Ψ_{loc}.

4. It might be disturbing that (3.35) involves the non-gauge-invariant expression Θ_i^{ab}. There is no claim that Ψ is globally well-defined on field-space. Rather writing the action this way is meant to make clear that the action is Q-closed and small perturbations of the action are Q-exact (i.e. Q applied to a globally well-defined expression). The situation is quite analogous to writing a Chern-Weil representative of a characteristic class as the exterior derivative of a secondary class. (For example, writing $\mathrm{Tr}F \wedge F = d\mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right)$.)

♣43 minutes from beginning of section 3. ♣

3.5 Equivariance

For applications to gauge theories we need one more formal development in order to take into account gauge invariance. Unfortunately, this gets a little involved so we will skip most of the technical details and try to summarize the story as briefly as possible. For details see [4] for an extended and leisurely discussion and [37] for a lightning summary.

Suppose that M is a principal $\mathcal{G}$-bundle:

$$\pi : M \to \bar{M} \tag{3.41}$$

for some Lie group $\mathcal{G}$ and that $E \to M$ is a $\mathcal{G}$-equivariant vector bundle with $\mathcal{G}$-equivariant connection. Suppose moreover that the section s is also equivariant so that $\mathcal{G}$ acts on the zero-set $\mathcal{Z}(s)$. We would like to write integrals of closed forms on the *moduli space of solutions to the equation* $s(\mathrm{x}) = 0$

$$\mathcal{M} := \mathcal{Z}(s)/\mathcal{G} \hookrightarrow \bar{M} \tag{3.42}$$

in terms of integrals over $\widehat{E}$.

In order to do this one needs a *projection form* $\tilde{\mathcal{P}}(M \to \bar{M})$ that has the property that

$$\int_M \pi^*(\bar{\omega})\tilde{\mathcal{P}}(M \to \bar{M}) = \int_{\bar{M}} \bar{\omega} \tag{3.43}$$

It turns out that the most convenient way to write the projection from is in terms of the equivariant cohomology of $\mathcal{C}^\infty(\widehat{E})$, where we replace the LHS by an integral over equivariant differential forms:

$$\int_M \pi^*(\bar{\omega})\tilde{\mathcal{P}}(M \to \bar{M}) = \int_{\mathrm{Lie}(\mathcal{G})} [d\phi] \int_M \Omega_{\bar{\omega}} \;\; \mathcal{P}(M \to \bar{M}) \tag{3.44}$$

where we have introduced new (degree two) bosonic fields $\phi \in \mathrm{Lie}(\mathcal{G})$, [7] Ω_ω is an invariant and equivariantly closed differential corresponding to $\bar{\omega}$, and $\mathcal{P}(M \to \bar{M})$ is an equivariantly closed form. ♣say how ♣

To write the projection form $\mathcal{P}(M \to \bar{M})$ one can use either the Cartan or the Weil model of equivariant cohomology. In terms of the Cartan model our complex is

$$\mathrm{Sym}^\bullet(\mathrm{Lie}(\mathcal{G})) \;\widehat{\otimes}\; \mathcal{C}^\infty(\widehat{E}) \;\widehat{\otimes}\; \mathcal{W}\,(\mathrm{Lie}(\mathcal{G})))^\vee \tag{3.45}$$

where the first factor is generated by ϕ, the second by $\mathrm{x}, \psi, \chi, H$ and the last factor is the Weil algebra but with opposite sign degrees (this is what is meant by the dual). The generators of this dual Weil algebra are denoted $\bar{\phi}, \eta \in \mathrm{Lie}(\mathcal{G}))$ and they are, respectively,

[7] at this point violating our convention thus far of using Latin letter for commuting and Greek letters for anticommuting fields

bosonic and fermionic, with ghost numbers -2 and -1. The differential Q now acts by:

$$
\begin{aligned}
Q\mathrm{x} &= \psi \\
Q\psi &= -\mathcal{L}_\phi \mathrm{x} \\
Q\chi &= H \\
QH &= -\mathcal{L}_\phi \chi \\
Q\bar{\phi} &= \eta \\
Q\eta &= -\mathcal{L}_\phi \bar{\phi}
\end{aligned}
\tag{3.46}
$$

Here $\mathcal{L}_\phi$ is the Lie derivative using the vector field corresponding to ϕ and we have introduced two new fields: An even $\bar{\phi} \in \mathrm{Lie}(\mathcal{G})$ of degree -2 and an odd $\eta \in \mathrm{Lie}(\mathcal{G})$ of degree -1 generating superfields on $\widehat{\mathrm{Lie}(\mathcal{G})}$. Then we will have an equation like:

$$
\mathcal{P}(M \to \bar{M}) = \int_{\mathcal{W}(\mathrm{Lie}(\mathcal{G})))^\vee} d\bar{\phi} d\eta e^{Q(\Psi_{\mathrm{proj}})} \tag{3.47}
$$

To write Ψ_{proj} we put an invariant metric on $\mathrm{Lie}(\mathcal{G})$ (giving the measure $[d\phi]$ used above) as well as an invariant measure on the fibers of $\pi : M \to \bar{M}$. The vertical vector fields define a canonical map:

$$
\mathcal{V}_m : \mathrm{Lie}(\mathcal{G}) \to T_m M \tag{3.48}
$$

and since we have $\mathcal{G}$-invariant metrics we have a $\mathcal{G}$-covariant one-form: [8]

$$
\mathcal{V}_m^\dagger : T_m M \to \mathrm{Lie}(\mathcal{G}) \tag{3.50}
$$

Using the $\mathrm{Lie}(\mathcal{G})$-valued 1-form we have

$$
\Psi_{\mathrm{proj}} = \mathrm{i}(\bar{\phi}, \mathcal{V}^\dagger) + (\bar{\phi}, [\phi, \eta]). \tag{3.51}
$$

Here the form $\mathcal{V}^\dagger$ should be interpreted as a $\mathrm{Lie}(\mathcal{G})$-valued superfield. The second term reflects the fact that the fields $\bar{\phi}, \eta$ themselves transform under "gauge transformations." [9]aaa

One can now check that $\mathcal{P}(M \to \bar{M})$ is equivariantly closed. Since s is $\mathcal{G}$-equivariant we learn that

$$
0 \to \mathrm{Lie}(\mathcal{G}) \xrightarrow{\mathcal{V}} TM \xrightarrow{\nabla s} E \to \mathrm{cok}(\nabla s) \to 0 \tag{3.52}
$$

is a complex. We can role it up to define

$$
\mathbb{F} = \nabla s \oplus \mathcal{V}^\dagger : TM \to E \oplus \mathrm{Lie}(\mathcal{G}). \tag{3.53}
$$

[8] It might help here to note that the canonical connection given by orthogonal complements to the vertical vector fields is given by:

$$
\Theta = \frac{1}{\mathcal{V}^\dagger \mathcal{V}} \mathcal{V}^\dagger. \tag{3.49}
$$

[9] The second term has always been a bit of an annoyance in this subject, going back to Witten's original paper [53].

Then we have the most general localization formula we will consider:

$$\boxed{\begin{aligned} Z &:= \int_{\mathrm{Lie}(\mathcal{G})} [d\phi] \int_{\widehat{E}\times\mathcal{W}(\mathrm{Lie}(\mathcal{G})))^\vee} \mathrm{ber}(\mathrm{x}, H, \bar{\phi}|\psi, \chi, \eta) \quad \widehat{\mathcal{O}} e^{Q\Psi_s} \\ &= \int_{\mathcal{M}} \bar{\omega}_{\widehat{\mathcal{O}}} \wedge \mathrm{Eul}(\mathrm{cok}(\mathbb{F})) \end{aligned}} \tag{3.54}$$

where

$$\Psi_s = \Psi_{\mathrm{loc},s} + \Psi_{\mathrm{proj}} \tag{3.55}$$

and Ψ_{loc} is given in equation (3.35) and finally the formula assumes there is an isomorphism

$$(H_Q^*)^{\mathcal{G}} \cong H^*(\mathcal{M}) \tag{3.56}$$

from $\mathcal{G}$-invariant and equivariantly closed superfields $\widehat{\mathcal{O}}$ to forms $\omega_{\widehat{\mathcal{O}}}$ on $\mathcal{M}$:

$$\widehat{\mathcal{O}} \leftrightarrow \bar{\omega}_{\widehat{\mathcal{O}}} \tag{3.57}$$

Note particularly that the integral can only be nonvanishing for

♣Comment on how $\widehat{\mathcal{O}} = \phi$ corresponds to ω given by the curvature of Θ. ♣

$$\mathrm{gh}\#(\widehat{\mathcal{O}}) = \mathrm{Index}(\mathbb{F}). \tag{3.58}$$

3.6 The Fields, Equations, Symmetries Paradigm

The above development should be susceptible to a rigorous presentation (ours is not) in the finite-dimensional setting. We now make a giant leap and assume that the formalism generalizes to infinite-dimensional path integrals. The main point of all the above formal development is this:

All topologically twisted quantum field theories fit in the above paradigm.

Quite generally, to specify a topological field theory in what we will call the "Mathai-Quillen" form one needs to specify

1. *Fields*: The primary fields of the problem are represented in the above by the x^i. The equations and symmetries of the problem then dictate the rest of the field content: $H, \phi, \bar{\phi}, \psi, \chi, \eta$.

2. *Equations*: We are interested in some equations on the fields $s(\mathrm{x}^i)$. They are generally interesting partial differential equations. We view then as the zero locus of a canonical section of a *bundle of equations* denoted above by E.

3. *Symmetries*: Typically the equations have gauge symmetry. The group of gauge transformations is denoted $\mathcal{G}$.

The main statement, as above, is that the path integral localizes to the moduli space

$$\mathcal{M} := \{\mathrm{x} : s(\mathrm{x}) = 0\}/\mathcal{G} \tag{3.59}$$

and, if we include operator insertions, the path integral computes integrals of cohomology classes over this moduli space.

The linear operator $\mathbb{F} = \nabla s \oplus \mathcal{V}^\dagger$ we encountered above will be a Fredholm operator, typically associated with an elliptic complex related to the equation. Its index is the generalization of $n - m$. Now n and m are infinite, but the difference can be given a sensible definition by virtue of index theory. When the index is nonzero we will need to insert operators with the appropriate ghost number in order to get a nonzero path integral.

The basic paradigm here is due to Witten [Cite:ICTP Lectures]. The reference [4] works out in detail the MQ formalism for many of the popular cohomological topological field theories.

Examples

1. Donaldson-Witten theory: The x^i should be interpreted as connections $A \in \mathrm{Conn}(P)$ of a principal G-bundle over an oriented Riemannian four-manifold X. The equations are $s(A) = F + *F$. The group $\mathcal{G} = \mathrm{Aut}(P)$ is the group of gauge transformations.

2. A-type topological sigma model: Now x^i are maps from a Riemann surface to a target space with a symplectic manifold with a compatible almost complex structure. The equation is the pseudo-holomorphic map equation and the group of gauge transformations is trivial.

3. B-type topological sigma model: Now x^i are maps from a Riemann surface to a target space with complex structure. ♣Check ♣

4. Other examples include supersymmetric quantum mechanics (as used in the path integral proofs of the index theorem [CITE: ALVAREZ-GAUME; FRIEDAN-WINDEY], topological gravity, topological string theory, 2d large N YM as a string theroy, Vafa-Witten theory, Kapustin-Witten theory, instantons for special holonomy manifolds, ...

♣20 minutes from beginning of section 3.5. ♣

4. Twisted N=2 SYM In Mathai-Quillen Form

4.1 Fields, Equations, Symmetries

The basic data:

1. A compact Lie group G with Lie algebra $\mathfrak{g}$.

2. A closed, oriented, Riemannian 4-manifold $(X, g_{\mu\nu})$.

3. A principal bundle $P \to X$

Our space of fields M with coordinates x^i will be replaced by the space of connections $A \in \mathcal{A} := \mathrm{Conn}(P)$.

Our bundle $E \to M$ of equations will be replaced by

$$\mathcal{E} = \mathcal{A} \times \Omega^{2,+}(X, \mathrm{ad}P) \tag{4.1}$$

and our section will be

$$s(A) := F^+ := F + *F \tag{4.2}$$

The bundle of equations and section is equivariant for the group of symmetries:

$$\mathcal{G} = \mathrm{Aut}(P) \tag{4.3}$$

So G is the gauge group of the theory and $\mathcal{G}$ is the group of gauge transformations. Locally we can think of elements of $\mathcal{G}$ as gauge transformations given by maps $X \to G$.

We can now run the machine of cohomological field theory reviewed in section 3. According to the formal structure there the path integral over $(A, H, \phi, \bar{\phi}; \psi, \chi, \eta)$ localizes to the moduli space of instantons $\mathcal{M}(P, g)$. We now explain how this is related to the physical supersymmetric Yang-Mills theory.

4.2 Relation To Twisted $N = 2$ Field Theories

We now explain in detail what is meant by topologically twisted $N = 2$ QFT in four dimensions.

In physics, an $N = 2$ QFT is a QFT with a unitary representation of the $N = 2$ super-Poincaré algebra in $3+1$ dimensions on its Hilbert space. After Wick rotation to Euclidean signature this superalgebra has an even part

$$\mathfrak{S}\mathfrak{P}^0 = \mathbb{R}^4 \rtimes (\mathfrak{su}(2)_- \oplus \mathfrak{su}(2)_+) \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R \tag{4.4}$$

and an odd part

$$\mathfrak{S}\mathfrak{P}^1 = (1;2;2)^1 \oplus (2;1;2)^{-1} \tag{4.5}$$

The first two summand of $\mathfrak{S}\mathfrak{P}^0$ are the rotation generators, split according to left- and right- action of unit quaternions. The subscript "R" on the next two summands means they are "R-symmetries." In general, an R-symmetry is a global symmetry that does not commute with the supercharges.

We have written the odd part as a representation of $\mathfrak{S}\mathfrak{P}^0$. The superscript refers to the $\mathfrak{u}(1)_R$-charge. There is an equivariant map

$$\mathrm{Sym}^2(\mathfrak{S}\mathfrak{P}^1) \to \mathbb{R}^4 \tag{4.6}$$

satisfying the super-Jacobi relation.

Physicists usually denote odd generators by the supercharges $\bar{Q}^A_{\dot{\alpha}}$ in $(1;2;2)^{+1}$ and Q^A_α in $(2;1;1)^{-1}$. The indices $\alpha, \dot{\alpha}, A$ all run from 1 to 2 and indicate a doublet in the relevant factors. In physics the map (4.6) is almost always written as

$$\begin{aligned} \{Q^A_\alpha, \bar{Q}^B_{\dot{\beta}}\} &= 2\epsilon^{AB}\sigma^\mu_{\alpha\dot{\beta}}P_\mu \\ \{Q^A_\alpha, Q^B_\beta\} &= 0 \end{aligned} \tag{4.7}$$

Moreover, when the $Q, \bar{Q}$ are operators acting on a physical Hilbert space there is a reality condition $(Q^A_\alpha)^\dagger = \epsilon_{AB}\bar{Q}^B_{\dot{\alpha}}$. [10]

[10] In the topologically twisted theory this will be violated. In fact, one can show there is no unitary TFT that reproduces Donaldson invariants [13].

The fact that there is an $\mathfrak{su}(2)_R$ symmetry is the origin of the name "$N = 2$" - it means there are two sets of $N = 1$ supersymmetry operators.

In an $N = 2$ field theory the fields will transform in representations of the supersymmetry algebra. There will be two kinds of field representations that are very important to our story: The "vectormultiplets" and the "hypermultiplets."

4.2.1 $N = 2$ Vectormultiplets

The fields of a vectormultiplet are listed in the following table.

	$su(2)_- \oplus su(2)_+ \oplus su(2)_R$	$u(1)_R$
A_μ	$(2,2,1)$	0
$\bar{\psi}^A_{\dot{\alpha}}$	$(1,2,2)$	-1
ψ^A_α	$(2,1,2)$	1
ϕ	$(1,1,1)$	2
$\bar{\phi}$	$(1,1,1)$	-2
D	$(1,1,3)$	0

1. I have used mathematicians' notation for representations of $su(2)$, denoting them by their *dimension*.

2. The $u(1)_R$ quantum number will correspond to "ghost number," which in turn will correspond to differential form degree on the moduli space of instantons.

3. In the physical theory the fields satisfy a reality condition, and they are all valued in the adjoint representation of $\mathfrak{g}$. The fields ϕ and $\bar{\phi}$ are complex conjugates of each other so we could write $\phi = B_1 + \mathrm{i}B_2$ and $\bar{\phi} = B_1 - \mathrm{i}B_2$ with B_1, B_2 scalar fields valued in $\mathfrak{g}$. In the topologically twisted theory ϕ and $\bar{\phi}$ play very different roles, and a relation of complex conjugation is never used. The path integral over B_1 and B_2 is replaced by a "contour integral" over ϕ and $\bar{\phi}$.

4. The above quantum numbers under the local Lorentz group tell us how to interpret the fields of a vectormultiplet for a general principal G bundle P over a general four-manifold X. In order to do this we <u>must</u> introduce the extra data of an *R-symmetry bundle*

$$P_R \to X \tag{4.8}$$

P_R is a principal $SU(2)$ bundle. It is extra data needed to put the vectormultiplet on a general four-manifold X. Then:

1. A is a connection,

2. ϕ and $\bar{\phi}$ are sections of $\mathrm{ad}P \otimes \mathbb{C}$

3. D is a section of $\mathrm{ad}P \otimes W$ where $W \to X$ is a real rank 3 vectorbundle associated to P_R.

4. ψ and $\bar{\psi}$ are sections of $S^\pm \otimes S_R \otimes \mathrm{ad}(P)$ where $S^\pm$ are the spin bundles of X and S_R is a spin representation associated to an

5. In this subject it is quite important to consider four-manifolds X which are not spin, (since "most" four-manifolds are not spin). [11] When X is not spin the bundles $S^{\pm}$ do not exist. When this is the case we should take the R-symmetry bundle to be a principal $SO(3)$ bundle and we must have

$$w_2(X) = w_2(P_R) \tag{4.9}$$

Then the bundles $S^{\pm} \otimes S_R$ will exist, even if the separate factors do not. This is the UV analog of the need to introduce spin-c structures in the IR. In fact, we will take the R-symmetry bundle with structure group $SO(3)$ to be isomorphic to the principal $SO(3)$ bundle P^+ over X associated to the bundle of self-dual forms. That is, we choose an isomorphism

$$P_R \cong P^+ \tag{4.10}$$

♣Is there any interesting dependence on the choice of isomorphism? ♣

4.2.2 Topological Twisting

One of the key innovations of Witten's 1988 paper was the concept of topological twisting. There are two useful ways of thinking about this topological twisting:

As we saw, in order to interpret the VM fields for a general principal G bundle $P \to X$ over a general four-manifold X we must also introduce an $SO(3)$ R-symmetry bundle P_R with $w_2(X) = w_2(P_R)$. In order to write the action and hence the path integral we introduce a Riemannian metric $g_{\mu\nu}$ on X and we must also introduce a connection ω_R on P_R. The Riemannian metric on X defines a Levi-Civita connection on TM and splitting the connection one-form into (anti-)self-dual components $\omega^{\pm}$ we see that the path integral will be a function of the three external fields:

$$Z(\omega^-, \omega^+, \omega_R) = \int [dA d\psi \cdots] e^{S_{\rm phys}} \tag{4.11}$$

where

$$S_{\rm phys} = -\int_X \frac{1}{g_0^2} {\rm tr} \left(F \wedge *F + D\phi \wedge *D\phi^* - \frac{1}{4}[\phi, \phi^*]^2 {\rm vol} \right) + \frac{\theta_0}{8\pi^2} \int_X {\rm tr}\, F \wedge F + S_{\rm phys,Fermi} \tag{4.12}$$

where ${\rm tr}\,(XY) = -\frac{1}{2h^\vee} {\rm tr}_{\mathfrak{g}} {\rm ad}(X) {\rm ad}(Y)$ is a positive form on the Lie algebra $\mathfrak{g}$, vol is the volume form induced by the metric $g_{\mu\nu}$, while g_0^2 is the bare Yang-Mills coupling, and $S_{\rm phys,Fermi}$ are the terms involving Fermions. In fact, the dependence on the connection ω_R on P_R only enters in the kinetic energy of the fermionic terms.

Now choose an isomorphism (4.10) as above and restrict the partition function (4.11) to external fields so that

$$\omega^+ \cong \omega_R \tag{4.13}$$

under the isomorphism. Then something amazing happens:

The dependence on both ω^- and ω^+ drops out!

[11] See Appendix A for a crash course on spin structures and related matters.

At least, that is what happens at the formal level. To see why it is convenient to review a second viewpoint on topological twisting (which was the original viewpoint in Witten's paper [53]).

By setting $\omega_R = \omega^+$ we are changing the coupling of the theory to the gravitational field $\omega^\pm$. The coupling to gravity is determined by the Lorentz quantum numbers of the fields, and this change of coupling to gravity is described by choosing an isomorphic Lorentz group with Lie algebra

$$\mathfrak{su}(2)_- \oplus \mathfrak{su}(2)'_+ \tag{4.14}$$

where

$$\mathfrak{su}(2)'_+ := \mathrm{Diag} \subset \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_R \tag{4.15}$$

This has the practical consequence that we read off the geometrical interpretation of the fields simply by taking the tensor product of the appropriate representations of $\mathfrak{su}(2)_-$ and $\mathfrak{su}(2)_R$ in the above table, viewing them as a representation of a common $\mathfrak{su}(2)$.

The key motivation for this topological twisting is that we want a supersymmetry operator which can function as a BRST operator. That means:

1. It must square to zero.
2. It must be a scalar, and so it can be defined on arbitrary Riemannian 4-folds.

This motivates the twisting defined above, and we take

$$Q = \delta^{\dot\alpha}_{A} \bar{Q}^{A}_{\dot\alpha}. \tag{4.16}$$

Now, given the topological twisting we can recognize the fields in the paradigm of CohTFT described in section 3:

1. A_μ remains a connection. But from the field ψ^A_α we get an odd 1-form ψ_μ, and $QA_\mu = \psi_\mu$. These correspond to the multiplets (x^i, ψ^i) of section 3.

2. From $\bar\psi^A_{\dot\alpha}$ the symmetric product of $\mathbf{2} \otimes \mathbf{2}$ gives an odd self-dual form $\chi_{\mu\nu}$ of ghost number -1. This is the same χ_a we had above.

3. From $D_{\mu\nu}$ we get an even self-dual form of ghost number 0. This is - essentially - the field H_a we had above. We said "essentially" because after Gaussian elimination the standard physical auxiliary field D is set to zero (in the absence of matter hypermultiplets) whereas in the general CohTFT formalism H is proportional to the equations.

4. The field $\bar\psi^A_{\dot\alpha}$ also gives an odd zeroform η of ghost number -1 from the anti-symmetric product of $\mathbf{2} \otimes \mathbf{2}$. This pairs with $\bar\phi$ to give the "projection multiplet" needed to construct Ψ_{proj}.

5. What remains is ϕ, playing the role of the degree two generator of symmetric algebra of the Lie algebra of $\mathcal{G}$, as used in the Cartan model of equivariant cohomology.

6. One can also take into account BRST gauge fixing, including the usual ghost fields b, c. This corresponds to using a different model for the projection gauge fermion that involves the Weil model of equivariant cohomology rather than the Cartan model. For details see [4].

Now having transcribed the fields in this way we find [12] that when we compare the action produced by the CohTFT formalism with $s(A) = F^+$ with the physical action we get: ♣NEED TO MODIFY THE FOOTNOTE NOW! ♣

$$S_{\text{phys}} = Q(\Psi) + 2\pi \mathrm{i} \int \tau_0 \mathrm{tr}\,(F \wedge F) \tag{4.17}$$

where

$$\tau_0 = \frac{\theta_0}{2\pi} + \frac{4\pi \mathrm{i}}{g_0^2} \tag{4.18}$$

and the trace is normalized so that $\int_X \mathrm{tr}\,(F \wedge F)$ is an integer on every X, and all integral values are obtained for a suitable principal G bundle P over X. Note that in deriving (4.17) one uses the identity

$$\int_X \mathrm{tr}\,(F^{\pm})^2 = 2 \int_X (\mathrm{tr}\, F * F \pm \mathrm{tr}\, FF) \tag{4.19}$$

which the student should verify!

Now we can give a formal proof of the claim that $Z(\omega^-, \omega^+, \omega_R)$ is metric-independent when $\omega_R = \omega^+$: On this locus we have a Q-symmetry of the QFT and moreover the action is Q-exact except for a metric-independent term as in (4.17). Therefore changes in the metric $g_{\mu\nu}$ lead to Q-exact perturbations of the action and should not change the path integral Z nor the correlation functions of Q-closed operators.

Put differently: In physics the response to a change in metric is measured by the energy-momentum tensor, but in this case

$$T_{\mu\nu} := \frac{1}{\sqrt{\det g_{\mu\nu}}} \frac{\delta}{\delta g^{\mu\nu}} S_{phys} = \{Q, \Lambda_{\mu\nu}\}. \tag{4.20}$$

is Q-exact. So correlation functions of $T_{\mu\nu}$ with Q-closed operators should vanish. We will need to re-examine this argument when we consider wall-crossing of the correlation functions of operators on manifolds with $b_2^+(X) = 1$. ♣34 minutes from beginning of section 4 ♣

4.3 A Little Bit About The Moduli Space Of Instantons

If we accept that the general localization formula (3.54) also applies to infinite-dimensional path integrals, then it asserts that the path integral of the SYM for a vectormultiplet with gauge group G will localize to an integral over the moduli space of the zeroes of $s(A)$, that is, the moduli space of ASD connections, aka *instantons*:

$$\mathcal{M}(P, g) = \{A \in \mathcal{A} | F^+ = 0\}/\mathcal{G} \tag{4.21}$$

So let us discuss this moduli space a little.

[12] Section 8.11.3 of [4] noted an extra shift of Ψ relative to $\Psi_{\text{loc}} + \Psi_{\text{proj}}$. This corresponds to a shift of the one-form $\mathcal{V}^\dagger$ to take account of the gauge action on the Cartan complex.

4.3.1 Deformation Complex And Virtual Dimension

We begin by considering the tangent space to the moduli space. The tangent space to the space of all connections is $T_A\mathcal{A} \cong \Omega^1(\mathrm{ad}P)$. That is, any path of connections through A is, to first order in t, of the form $A + t\alpha$ where α is a globally defined one-form valued in the adjoint bundle associated to P. A simple computation shows that

$$F^+(A + t\alpha) = F^+(A) + t\nabla_A^+\alpha + \mathcal{O}(t^2) \tag{4.22}$$

This defines an operator $\nabla_A^+ : \Omega^1(\mathrm{ad}P) \to \Omega^{2,+}(\mathrm{ad}P)$. Now, if we consider a path of connections in the moduli space then a tangent vector α must be in the kernel of this operator. On the other hand, before dividing by the gauge group we have trivial tangent vectors to the space of solutions to $F^+(A) = 0$ in $\mathcal{A}$ obtained by gauge transformation.

The above considerations lead one to define the first order deformation complex of the instanton equation:

$$0 \to \Omega^0(\mathrm{ad}P) \overset{\nabla_A}{\to} \Omega^1(\mathrm{ad}P) \overset{\nabla_A^+}{\to} \Omega^2(\mathrm{ad}P)^+ \tag{4.23}$$

At least formally, the cohomology of this complex should define the tangent space to the moduli space of instantons. Indeed, this complex was studied by Atiyah, Hitchin, and Singer in [1] for exactly this reason. They show it is elliptic, and in fact, if we role up the complex we just get that of the chiral Dirac operator:

$$D^+ : \Gamma(S^- \otimes \mathfrak{V}) \to \Gamma(S^+ \otimes \mathfrak{V}) \tag{4.24}$$

coupled to the complex vector bundle:

$$\mathfrak{V} = \mathrm{ad}P \otimes S^+ \tag{4.25}$$

This shows that the moduli space $\mathcal{M}(P, g)$ of ASD instantons has virtual dimension

$$\begin{aligned} \mathrm{vdim}\mathcal{M}(P, g) &= 4h^\vee k - \dim G(b_2^+ - b_1 + 1) \\ &= 4h^\vee k - \dim G\frac{\chi + \sigma}{2} \end{aligned} \tag{4.26}$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ and

$$k := \frac{1}{16\pi^2 h^\vee}\int_X \mathrm{Tr}_{\mathfrak{g}} F^2 = p_1(\mathrm{ad}P)/4h^\vee \tag{4.27}$$

is an integer (and the second equality holds when $\mathfrak{g}$ is a simple Lie algebra). We have chosen the normalization of k so that when $X = S^4$ all integral values are obtained for a suitable bundle P. The sign is such that for an anti-self-dual instanton $k > 0$. In particular, for $SU(2)$, ♣Need to check sign on the second equality. ♣

$$\mathrm{vdim}\mathcal{M}(P, g) = 8k - 3(b_2^+ - b_1 + 1) \tag{4.28}$$

with

$$k = -\frac{1}{8\pi^2}\int_X \mathrm{Tr}_2 F^2 \tag{4.29}$$

4.3.2 The Virtual Dimension Is The Ghost Number Anomaly

Now, it is interesting to compare this discussion with CohTFT. The complex (4.23) is a special case of the sequence (3.52) in the general discussion. In particular, the operator $\mathbb{F}$ of that discussion is indeed Fredholm.

Moreover, the fermions in the vectormultiplet have $\mathfrak{u}(1)_R$ charge ± 1. Therefore, after topological twisting, where we identify $S_R \cong S^+$ the Dirac operator, whose index computes the anomaly for the $\mathfrak{u}(1)_R$ symmetry, is precisely the same as that used to compute the index of the AHS complex (4.24). Now, recall that $\mathfrak{u}(1)_R$ charge is ghost number - formally equivalent to the degree of differential forms on $\mathcal{M}(P, g)$. Thus:

The index $\mathrm{Ind}(\mathbb{F})$ *coincides with the anomaly in* $\mathfrak{u}(1)_R$ *symmetry and is also the same as the ghost number anomaly.*

4.3.3 Some Essential Mathematical Properties Of Instanton Moduli Space

1. The moduli space $\mathcal{M}(P, g)$ is in general singular and noncompact. Singularities occur when H^0 of (4.23) is nonzero and the connection is reducible or when H^2 is nonzero, and there are obstructions to the deformations of the instanton being true deformations. However, the *generic metrics theorem* [15] states that for $k > 0$ and $G = SU(2)$ or $G = SO(3)$ then for $b_2^+(X) > 0$ and for generic metrics the moduli space will be smooth. Therefore, in the general localization formula of CohTFT, equation (3.54) above, we will have $\mathrm{cok}(\mathbb{F}) = 0$ and we don't need to include the Euler character. In [22] Kronheimer has generalized some aspects of the generic metrics theorem to the case of $PSU(N)$ bundles. . ♣SAY MORE ♣

2. At smooth points,
$$\mathrm{vdim}\mathcal{M}(P, g) = \dim \mathcal{M}(P, g) \tag{4.30}$$
will be the actual dimension of the moduli space $\mathcal{M}(P, g)$.

3. The moduli space $\mathcal{M}(P, g)$ is also noncompact because instantons can degenerate to point instantons where $\mathrm{Tr} F \wedge F$ is a smooth measure plus a sum of Dirac measures.

4.3.4 Instanton Moduli Space As The Q-Fixed-Point Locus

Since we will consider generalizations of the pure VM theory below it is worth looking at this as a Q-fixed point equation. We have:

$$Q\chi_{\mu\nu} = i(F^+_{\mu\nu} - D^+_{\mu\nu}) \tag{4.31}$$

Eliminating the auxiliary field through Gaussian integration gives $D^+_{\mu\nu} = 0$. The equation $D^+ = 0$ will change in a very important way when we include hypermultiplets below.

It is interesting to note that the Q-fixed point equations for the other fermions in the theory signal where there should be trouble: Setting $Q\psi_\mu = 0$ gives

$$D_A \phi = 0 \tag{4.32}$$

and $Q\eta = 0$ gives:

$$[\bar{\phi}, \phi] = 0 \tag{4.33}$$

Equation (4.32) will only have the solution $\phi = 0$ when the connection is irreducible. At reducible connections we should expect trouble for many reasons. The equation (4.33) plays an important role in the physical analysis of the LEET as we'll explain below.

4.4 Observables

To get an interesting path integral we will need to insert Q-closed and gauge invariant observables. Recall that in the Cartan model

$$Q\phi = 0. \tag{4.34}$$

Now, ϕ is adjoint-valued, so not gauge invariant. However, given any invariant polynomial $\mathcal{P}$ on $\mathfrak{g}$ we can make a Q-closed and gauge invariant observable $\mathcal{P}(\phi)$. We will call these 0-observables, because they are local operators defined at *points*. They form a ring, generated by the Casimirs of $\mathfrak{g}$. For example, for $SU(N)$ we have generating observables:

$$\mathcal{O}_s^{(0)}(\wp) := \mathrm{Tr}\phi^s(\wp) \qquad s = 2, \ldots, N \tag{4.35}$$

Given any zero-observable $\mathcal{O}^{(0)}$ one can define a hierarchy of nonlocal observables using the *descent formalism*. The new observables can be canonically constructed by noting that under topological twisting $Q_{\alpha\dot{\alpha}} \to K_\mu$ with

$$\{Q, K_\mu\} = \partial_\mu \tag{4.36}$$

Therefore if we let $K := dx^\mu K_\mu$ be a one-form-valued supersymmetry operator we can define $\mathcal{O}^{(1)} := K\mathcal{O}^{(0)}$ to get a 1-form-valued operator. Now, for any 1-chain γ

$$Q \int_\gamma \mathcal{O}^{(1)} = \mathcal{O}^{(0)}|_{\partial\gamma} \tag{4.37}$$

This implies:

1. A change of location of the point $\wp$ in $\mathcal{O}^{(0)}(\wp)$ is Q-exact: If $\partial\gamma = \wp_1 - \wp_2$ then $\mathcal{O}^{(0)}(\wp_1) = \mathcal{O}^{(0)}(\wp_2) + Q\int_\gamma \mathcal{O}^{(1)}$.

2. If γ is a closed cycle then $\mathcal{O}(\gamma) := \int_\gamma \mathcal{O}^{(1)}$ is BRST closed.

Similarly, $\mathcal{O}^{(j)} := K^j\mathcal{O}^{(0)}$ define j-forms on X and if Σ_j is a closed j-cycle then

$$\mathcal{O}(\Sigma_j) := \int_{\Sigma_j} \mathcal{O}^{(j)} \tag{4.38}$$

is a Q-closed and gauge-invariant observable which only depends on the homology class of Σ_j. We call these the "j-observables."

In our computations below we will mostly be concentrating on the rank one groups $SU(2)$ and $SO(3)$. In this case all the 0-observables are generated from the ghost-number $= 4$ observable:

$$\mathcal{O}_2^{(0)}(\wp) := \frac{1}{8\pi^2}\mathrm{Tr}_2\phi^2(\wp). \tag{4.39}$$

At this point we adopt the following policy: To keep equations readable we will suppress real coefficients in some equations. They are typically (fractional) powers of 2 and π. When we do this we use the symbol $\sim$. When I write "=" I really mean "equals." The full expressions with correct coefficients can be found in [38].

Of particular importance in the rank one topologically twisted SYM are the two-observables, which work out to be

$$\mathcal{O}_2^{(2)}(\Sigma) \sim \int_\Sigma \mathrm{Tr}(\phi F + \psi \wedge \psi) \tag{4.40}$$

To lighten the notation we henceforth write:

$$\mathcal{O}_2^{(0)}(\wp) \to \mathcal{O} \tag{4.41}$$

Note that since the dependence on the point $\wp$ is Q-exact we can drop the position $\wp$ from the notation. Similarly, we just write:

$$\mathcal{O}_2^{(2)}(\Sigma) \to \mathcal{O}(\Sigma) \tag{4.42}$$

Now we consider correlation functions of these operators in the topologically twisted theory - which we regard as the physical theory with $\omega_R = \omega^+$. Our notation then is that - for any expression $\mathcal{F}$ in the fields (typically a product of operators) we write

$$\left\langle \prod_i \mathcal{F}_i \right\rangle_{\mathcal{T}} := \int [dAd\psi \cdots] e^{S_{\mathcal{T}}} \mathcal{F} \tag{4.43}$$

for the path integral of the $N = 2$ field theory $\mathcal{T}$ evaluated for $\omega_R = \omega^+$.

Let us now define "Witten polynomials" to be polynomial functions on the homology $H_0(X) \oplus H_2(X)$ with real coefficients by

$$P_W(\wp^\ell \Sigma^r) := \left\langle \mathcal{O}^\ell \mathcal{O}(\Sigma)^r \right\rangle_{\mathcal{T}} \tag{4.44}$$

We observe that

1. They are complex-valued polynomials on the homology that are formally independent of the metric and hence depend only on the smooth structure.

2. Because of the ghost number anomaly $P_W(\wp^\ell \Sigma^r)$ can only be nonvanishing for

$$4\ell + 2r = \mathrm{vdim}\mathcal{M}(P, g) \tag{4.45}$$

Remarks:

1. Note well that neither η nor χ appears in the observables. This will play a very important role in the evaluation of the u-plane integral below.

2. We stress that even with topological twisting, i.e. setting $\omega_R = \omega^+$, the correlation functions of the theory on X of almost all operators will depend on the metric (i.e. on the spin connections $\omega^{\pm}$). For example if we were to consider correlators of $\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}$ or $Tr(D_\mu F^{\mu\nu})^2$ and so on there would be metric dependence. Physicists speak of a "topological sector of the theory" - meaning that one only considers Q-closed observables such as those described above.

4.5 The Donaldson Polynomials

We first briefly recall the Donaldson polynomials: We begin with Donaldson's μ map:

$$H_*(X) \to H^*(\mathcal{M}) \tag{4.46}$$

from the homology of X to the cohomology of $\mathcal{M}$. We briefly recall Donaldson's construction.

Let us recall Donaldson's formulation: We consider the principal G bundle and its classifying map f:

$$\begin{tikzcd} P \times \mathcal{A}/\mathcal{G} \arrow[d] & \\ (P \times \mathcal{A})/(G \times \mathcal{G}) \arrow[r, "f"] & BG \end{tikzcd} \tag{4.47}$$

If we choose any invariant polynomial $\mathcal{P}$ on $\mathfrak{g}$ of degree d, thus producing a class $\varpi \in H^{2d}(BG)$ then we can define the *slant product*:

$$\mu_D(\Sigma_j) := \int_{\Sigma_j} f^*(\varpi) \in H^{2d-j}(\mathcal{A}/\mathcal{G}) \tag{4.48}$$

This is a cohomology class which can be restricted to $\mathcal{M} \subset \mathcal{A}/\mathcal{G}$, and it only depends on the homology class of Σ_j.

For $SU(2)$ we choose ϖ to be a generator of $H^4(BSU(2);\mathbb{Z}) \cong \mathbb{Z}$ and thereby define forms:

♣Probably better to use Σ instead of S for surfaces. ♣

$$\wp \to \mu_D(\wp) \in H^4(\mathcal{M}) \tag{4.49}$$

$$\Sigma \to \mu_D(\Sigma) \in H^2(\mathcal{M}) \tag{4.50}$$

Now we recall that Donaldson defines his polynomials on $H_0(X) \oplus H_2(X)$ by showing one can choose compactly supported representatives of the cohomology classes and defining the value on the monomial $\wp^\ell \Sigma^r$ as

$$P_D(\wp^\ell \Sigma^r) := \int_{\mathcal{M}} \mu_D(\wp)^\ell \mu_D(\Sigma)^r \tag{4.51}$$

That is, the coefficients are given by intersection numbers on moduli space.

From the rigorous mathematical analysis we know that:

1. The $P_D(\wp^\ell \Sigma^r)$ are rational number and independent of the metric, except for X such that $b_2^+(X) \leq 1$. When $b_2^+(X) > 1$ the moduli space $\mathcal{M}(P,g)$ is smooth for generic metrics. This is known as the generic metrics theorem of Freed and Uhlenbeck [15]. Thus, in those cases, they define smooth invariants of X. When $b_2^+(X) = 1$ the polynomials are only piecewise constant in the space of metrics and jump across real codimension one walls. [13]

Remark: One can give a precise argument relating $\mathcal{O}$ and $\mathcal{O}(\Sigma)$ to $\mu_D(\wp)$ and $\mu_D(\Sigma)$ following a discussion of Baulieu and Singer. It uses a model for the $\mathcal{G}$-equivariant cohomology of $\mathcal{A}$ and the "universal connection" on

$$P \times \mathcal{A} \to X \times \mathcal{A}/\mathcal{G} \tag{4.52}$$

See [4], §8.8 for details. Baulieu and Singer argue that one can identify

$$\mathcal{F} = F + \psi + \phi \tag{4.53}$$

with the curvature of the universal connection, decomposed according to form degree along X and $\mathcal{A}/\mathcal{G}$, respectively. Then the slant product on

$$\mathrm{Tr}\mathcal{F}^2 \tag{4.54}$$

links Donaldson's map $H_*(X) \to H^*(\mathcal{M}(P,g))$ to the descent formalism map from $H_*(X)$ to the Q-cohomology H_Q^*.

Certainly it is straightforward to show that when computing quantum correlation functions of ϕ we can identify it with the components of the curvature of the universal connection along $\mathcal{A}/\mathcal{G}$.

In the final statement of the localization formula (3.54) above we had a map from gauge-invariant equivariantly closed classes $\widehat{\mathcal{O}}$ to differential forms $\omega_{\widehat{\mathcal{O}}}$ on $\mathcal{M} = Z(s)/\mathcal{G}$, putting all these together we have a commutative diagram for each invariant polynomial of degree d on $\mathfrak{g}$:

$$\begin{tikzcd} H_*(X) \arrow[r, "\text{descent}"] \arrow[dr, "\mu"'] & H_Q^{2d-*} \arrow[d, "\widehat{\mathcal{O}} \to \omega_{\widehat{\mathcal{O}}}"] \\ & H^{2d-*}(\mathcal{M}) \end{tikzcd} \tag{4.55}$$

Thus, we are naturally led to Witten's main claim in his 1988 paper:

Under the above correspondence, the physical correlation functions of Q-closed operators coincide with Donaldson polynomials - up to an overall constant. That was the goal of Witten's 1988 paper: The 0- and 2 - observables precisely correspond to Donaldson's forms $\omega_D(\wp)$ and $\omega_D(\Sigma)$.

[13]When Q_X is definite we can choose orientation so that $b_2^+(X) = 0$. In that case the moduli space need not be smooth. In particular, taking $k = 1$ it is a 5-dimensional bordism between X and a set of singular points corresponding to the reducible connections where the moduli space is locally a cone on $\mathbb{CP}^2$ or $\overline{\mathbb{CP}^2}$. This is precisely how Donaldson proved that if Q_X is definite then it must be diagonalizable over the integers.

4.6 The Donaldson-Witten Partition Function

In physics it is extremely natural to assemble a collection of correlation functions into a single generating function corresponding to perturbing the action of the theory by the observables in question. We call this generating function the *Donaldson-Witten Partition function*:

$$Z_{DW}(p,\Sigma) := \left\langle e^{p\mathcal{O}+\mathcal{O}(\Sigma)} \right\rangle_{\mathcal{T}} \tag{4.56}$$

We may expand this as a formal series to write:

$$Z_{DW}(p,\Sigma) = \sum_{\ell,r\geq 0} \frac{p^\ell}{\ell! r!} \left\langle \mathcal{O}^\ell \mathcal{O}(\Sigma)^r \right\rangle_{\mathcal{T}} \tag{4.57}$$

Each correlation function at fixed ℓ, r is a sum over instanton sectors: Thus we sum over isomorphism classes of principal $SU(2)$ bundles P. However, due to the selection rule from the $\mathfrak{u}(1)_R$ anomaly we know that only the instanton sector with

$$4\ell + 2r = \mathrm{vdim}\mathcal{M}(P,g) \tag{4.58}$$

can possibly contribute. We can thus sharpen Witten's main claim as follows: For $\mathcal{T}$ given by the $SU(2)$ SYM theory,

$$Z_{DW}(p,s) := \left\langle e^{p\mathcal{O}+\mathcal{O}(\Sigma)} \right\rangle_{\mathcal{T}} = \frac{1}{2}\Lambda^{-\frac{3}{4}(\chi+\sigma)} \sum_{\ell,r\geq 0} \frac{p^\ell s^r}{\ell! r!} \Lambda^{2\ell+|r|} P_D(\wp^\ell \Sigma^r) \tag{4.59}$$

Here we must make several remarks:

1. We have introduced a parameter Λ which is implicit in the path integral of the quantum field theory. It is needed in order to define the quantum theory. We will comment more on it below. In most discussions of the physical approach to Donaldson theory it is set to one, but this is, in fact, misleading.

2. The path integral in (4.59) is a generating functional for the correlation functions of $\mathcal{O}$ and $\mathcal{O}(\Sigma)$. As with all path integrals in an $SU(2)$ gauge theory we must sum over bundles with connection. The sum over bundles is also known as the sum over instanton sectors. For a given insertion of $\mathcal{O}^\ell \mathcal{O}(\Sigma)^r$ only one term in the sum over instanton sectors will contribute, namely the one satisfying the ghost-number selection rule (4.45).

3. Mathematical sticklers will insist that a polynomial on a vector space is valued in the symmetric algebra of the dual of that vector space. Indeed, we should think of p as dual to the homology class $[\wp] \in H_0(X)$. In this spirit, $\mathcal{O}(\Sigma)$ should be interpreted as follows: Choose a basis $[\Sigma_\alpha]$ for $H_2(X;\mathbb{Z})$ and a dual basis s^α. Then

$$\mathcal{O}(\Sigma) = s^\alpha \mathcal{O}(\Sigma_\alpha) \tag{4.60}$$

and the generating function $Z_{DW}(p,\Sigma)$ becomes a formal power series in p and s^α. The sum over $r \geq 0$ is a sum over a multi-index $\vec{r} = (r_1, \ldots, r_{b_2})$ and $|r| = \sum r_\alpha$ so our shorthand means:

$$\sum_{r\geq 0} \Lambda^{|r|} \frac{s^r}{r!} \mathcal{O}(\Sigma)^r := \sum_{r_\alpha \geq 0} \prod_\alpha \frac{(\Lambda s_\alpha)^{r_\alpha}}{r_\alpha!} \mathcal{O}(\Sigma_\alpha)^{r_\alpha} \tag{4.61}$$

The coefficient of any term in the series is well-defined, but there is no claim that the full sum makes sense as a function on the homology of $H_*(X)$ rather than as a generating series of polynomials on $H_*(X)$. We will often be quite sloppy about this point and just speak of $Z_{DW}(p,\Sigma)$.

4. We are working with an $SU(2)$ gauge theory, but the fields are in the adjoint representation so it makes sense to define the path integral for a "twisted $SU(2)$ bundle," that is, an $SO(3)$ bundle which does not lift to an $SU(2)$ bundle. A principal $SO(3)$ bundle over X has two characteristic classes, $\xi = w_2(P) \in H^2(X;\mathbb{Z}_2)$, which we will refer to as the 't Hooft flux, and the instanton number $k \in H^4(X;\mathbb{Z}) \cong \mathbb{Z}$. In the generating function we sum over the instanton number k, as appropriate for a path integral of an $SU(2)$ gauge theory, but there is no difficulty extending the path integral to fields valued in $\mathrm{ad}P$ where there is a nonzero 't Hooft flux. We will denote the corresponding generating function

$$Z_{DW}(p,\Sigma) \to Z^\xi_{DW}(p,\Sigma) \tag{4.62}$$

We stress that we are doing $SU(2)$ gauge theory: We do sum over instanton number, but not over ξ. In $SO(3)$ gauge theory we would also be obliged to sum over ξ.

5. The generating function is closely related to the generating function introduced by Kronheimer and Mrowka in [21]:

$$Z^\xi_{KM}(p,s) := \sum_{\ell,r\geq 0} \frac{p^\ell s^r}{\ell! r!} P_D(\wp^\ell \Sigma^r) \tag{4.63}$$

The relative overall factor of $\frac{1}{2}$ is due to the fact that physicists divide by the order of the center of $SU(2)$, which does not act effectively on the fields.

6. As a simple example, for $X = K3$ the usual K3-surface we can take $\xi = 0$ and

$$Z_{KM}(p,s) = \sinh\left(\frac{1}{2}s^2 + 2p\right) \tag{4.64}$$

♣Proper reference? Kotschick-Morgan ♣

7. Of course, when X is not simply connected we can extend this to include 1-observables.

♣End this section: 52 minutes from section 4.3. ♣

4.7 Generalizations: Lagrangian N=2 Theories

The N=2 SYM theory with gauge group $SU(2)$ is just one of a much larger class of $N=2$ QFT's. It will be quite fruitful to look at some of the generalizations.

The most general Lagrangian N=2 QFT has two kinds of field representations of the $N=2$ superPoincare algebra: Vectormultiplets and *Hypermultiplets*.

The theories with VM's and HM's can again be topologically twisted - and again the resulting theory fits in the general framework of cohomological TFT described in section 3.

4.7.1 Hypermultiplets

There is another field multiplet - the *hypermultiplet*

Now, in N=2 SYM theory it is possible to include another kind of field multiplet, known as a *hypermultiplet*. A hypermultiplet in a theory with gauge group G is defined by choosing a quaternionic representation W of G. For simplicity, and with some loss of generality, we assume that the representation can be written in the form

$$W = R \oplus R^* \tag{4.65}$$

where R is a complex representation of G. The field multiplet can be viewed as an N=1 chiral superfield transforming in the representation $R \oplus R^*$. An $N=1$ chiral superfield has a leading component which is just a complex scalar field, so we have complex scalars $q \oplus \tilde{q}$ transforming in the $R \oplus R^*$ of G. In a hypermultiplet if we consider instead

$$M := q \oplus \tilde{q}^* \tag{4.66}$$

then we get a doublet of scalars under $SU(2)_R$ transforming in the representation R. (Note that $SU(2)_R$ does not commute with the supercharges, so M is <u>not</u> a bottom component of an $N=1$ chiral superfield.)

Now, when we topologically twist the theory as mentioned above a doublet under $SU(2)_R$ becomes a spinor under $SU(2)'_+$. Therefore, in the topologically twisted theory on X the scalar fields in a hypermultiplet become sections:

$$M \in \Gamma(S^+ \otimes \mathfrak{R}) \qquad \bar{M} \in \Gamma(S^+ \otimes \mathfrak{R}^*) \tag{4.67}$$

where $\mathfrak{R} \to X$ is now a vector bundle associated to $P \to X$ by the representation R. (The fermions in a hypermultiplet are invariant under $SU(2)_R$ and therefore are also sections of a spin bundle times $\mathfrak{R}$.)

This raises again the important issue that it is important to be able to formulate the theory X is not spin, $w_2(X) \neq 0$. The resolution is the same as it was for putting the vectormultiplet on a general four-manifold: We regard $S^\pm$ as "twisted" bundles (in the sense of twisted K-theory) and multiply them by "twisted" bundles $\mathfrak{R}$ so that $S^+ \otimes \mathfrak{R}$ exists as an honest bundle. So we must take $w_2(\mathfrak{R}) = w_2(X)$ which might require us to choose a certain 't Hooft flux ξ for P.

Auxiliary fields D are valued in $\mathfrak{g} \otimes \mathbb{R}^3$ and form an $SU(2)_R$ triplet. When we consider the Lagrangian of VM + HM with the HM in a quaternionic representation W Gaussian elimination of the D-field identifies it with the quaternionic moment map $\mu(M) \in \mathfrak{g}^\vee \otimes \mathbb{R}^3$ defined by the action of G on W. Choosing a complex structure so that $W \cong R \oplus \bar{R}$ with $M = q \oplus \tilde{q}$ the quaternionic moment map is given by the quadratic expression:

$$\begin{aligned} T \cdot \mu_r(M) &= \langle q, T \cdot q \rangle - \langle \tilde{q}, T \cdot \tilde{q} \rangle \\ T \cdot \mu_c(M) &= \tilde{q} \cdot Tq \end{aligned} \tag{4.68}$$

On the LHS we use the natural pairing of $\mathfrak{g}$ with $\mathfrak{g}^\vee$ and on the RHS we use the Hermitian structure on R and the corresponding identification of $\bar{R} \cong R^\vee$. Since the action of the QFT uses an invariant metric we can identify $\mathfrak{g} \cong \mathfrak{g}^\vee$ and this identification is used in doing the Gaussian integral on auxiliary fields to set $D = \mu$.

It now follows that in the topologically twisted theory the D-field defines a map

$$\mu : \Gamma(S^+ \otimes \mathfrak{R}) \to \Gamma(\Lambda^+ \otimes \mathrm{ad}P) \tag{4.69}$$

from hypermultiplet scalar fields to the self-dual two-forms on X valued in $\mathfrak{g}$. Now recall that $\delta\chi = i(F^+ - D^+)$. Together with the variation of fermions in the HM, the Q-fixed point equations work out to be

$$\begin{aligned} F^+ &= \mu(M) \\ \not{D} M &= 0 \end{aligned} \tag{4.70}$$

where $\not{D} : \Gamma(S^+ \otimes \mathfrak{R}) \to \Gamma(S^- \otimes \mathfrak{R})$ is the "spin-c Dirac operator coupled to $\mathfrak{R}$." These are known as the *generalized monopole equations.* ♣End this section: 18 minutes from 4.7 . ♣

4.7.2 Charge One Hypermultiplets In A $G = U(1)$ Theory: Spin-c Structures And The Seiberg-Witten Equations

A special case of the generalized monopole equations has proven to be of fundamental importance in four-manifold (and three-manifold) theory. We take the gauge group $G = U(1)$ and the representation $R \cong \mathbb{C}$ to be the defining representation of charge $+1$.

As discussed above, when X is not spin we must make sense of $S^+ \otimes \mathfrak{R}$. In this case we must choose a Spin-c structure.

The way the physicists say this is that we choose a line bundle L^2 which only has a square-root locally, but L does not exist globally. We require that the first Chern class satisfy

$$w_2(X) = c_1(L^2) \mathrm{mod} 2 \tag{4.71}$$

and then take $M \in \Gamma(S^+ \otimes L)$. Neither S^+ nor L exist globally because of -1-signs in the cocycle relation for the transition functions, but the product does exist as an honest vector bundle. Mathematicians usually think of L^2 as the determinant of a rank two bundle which we would write as:

$$\mathrm{Det}(S^+ \otimes L) \cong L^2 \tag{4.72}$$

Note that since the line bundle L need not exist we <u>cannot</u> write $c_1(L^2) = 2c_1(L)$, as would be the case for ordinary line bundles, and hence $c_1(L^2) \in H^2(X; \mathbb{Z})$ is <u>not necessarily</u> divisible by two! When it is divisible by two, we can find a spin structure.

The presentation we have just given makes many a mathematicians gag. In Appendix A.3 we give a mathematically more respectable description of a $Spin^c$ structure.

Let us write out the generalized monopole equations in this case. For the $U(1)$ case, in indices if R is the charge q representation of $U(1)$ then after topological twisting the D-term is the self-dual form:

$$\mu(M) = q\bar{M}_{(\dot\alpha} M_{\dot\beta)} \tag{4.73}$$

where we identify self-dual forms with symmetric tensors $Sym^2(W^+)$. In this case the generalized monopole equations are the famous *Seiberg-Witten equations* [14] associated to a spin-c structure:

$$\begin{aligned} F^+_{\dot\alpha\dot\beta} &= \bar{M}_{(\dot\alpha} M_{\dot\beta)} \\ D_{\alpha\dot\beta} M^{\dot\beta} &= 0 \end{aligned} \tag{4.74}$$

As we have noted, every orientable four-manifold X is spin-c and therefore the Seiberg-Witten equations can be written on any *smooth* X.

Remark: In our u-plane computations below we will encounter sums over spin-c structures, in particular we will encounter theta functions that are associated with the torsor of spin-c structures. We will be a little sloppy and ignore torsion classes and simply identify a spin-c structure with an element

$$\lambda \in \Gamma_w := \frac{1}{2}\bar{w}_2(X) + \bar{H}^2(X) \tag{4.75}$$

where $\bar{w}_2(X)$ is a fixed choice of integral lift of $w_2(X)$. In terms of the above discussion $2\lambda = c_1(L^2) = c_1(\det W^{\pm})$.

4.7.3 Generalizing The Donaldson-Witten Partition Function

♣Say more here. Explain some of what Kronheimer did. ♣

The general classical N=2 field theory is defined by

1. Choosing a compact Lie group G. We take the VM for this group.
2. Choosing a quaternionic representation $\mathcal{W}$ of G.
3. Choosing "mass parameters".

One can then write a classical action. The theory can be topologically twisted (subject to restrictions on the bundles discussed above for non-spin-c manifolds) and the topologically twisted theories have actions that fit in the cohomological field theory paradigm of section 3. [15]

Given the above remarks, it would seem that there is a vast generalization of Donaldson theory. However, physics puts a strong constraint on the data for which we can expect to find reasonable answers: The underlying physical QFT should be well defined. The quantum field theory is thought to be a perfectly well-defined QFT so long as the beta

---

[14] They are also sometimes called the "monopole equations."

[15] The "mass parameters" are defined by a choice of element of the Lie algebra of the subgroup of the orthogonal group of $\mathcal{W}$ that commutes with the action of the gauge group. After topological twisting these parameters can be interpreted as equivariant parameters for the action of global symmetries on the moduli space of Q-fixed points. This is discussed in [Labastida-Marino], [26][24]

function for all the gauge couplings is not positive. This will never happen if the Lie algebra of G has abelian summands, and so we should only attempt to define generalizations for G which is semi-simple.

If G is simple and $W \cong R \oplus \bar{R}$ then the beta function is proportional to

$$\beta = -2h^{\vee} + C_2(R) \tag{4.76}$$

where $C_2(R)$ is the quadratic Casimir of R, normalized so that $C_2(\mathfrak{g} \otimes \mathbb{C}) = 2h^{\vee}$.

The simplest generalization to consider is the pure VM theory for a simple Lie group G. In this case the path integral should localize to the moduli space of ASD instantons for group G. There are more observables generated by the Casimirs on the Lie algebra. At least at the formal level we fully expect the obvious generalization of Witten's basic identity (4.59) to hold.

On the mathematical side, this generalization appears to be highly nontrivial: The ASD moduli spaces have additional singularities. Even for $SU(N)$ with no hypermultiplets there is no known analog of the "generic metrics theorem" of Freed-Uhlenbeck. Nevertheless, in [22] P. Kronheimer gave a definition of the $SU(N)$ invariants for all N. Further rigorous mathematical treatments of higher rank invariants can be found in [5]. [16] Physics suggests it can be done for any simple Lie group. The "answer" - worked out using the IR methods described below for the $SU(N)$ theory has been given in [31]. There is a generalization of the "Witten conjecture" and one may expect it to generalize further for arbitrary compact simple Lie group, although this has not yet been carried out.

♣Or is it known to be false...? ♣

Even more generally, one should be able to include hypermultiplets so long as the beta function (4.76) is nonpositive. Formally, the correlators of Q-closed observables of the twisted theory are computing intersection numbers on the moduli spaces of solutions to the generalized monopole equations (4.70). For the special case of $G = SU(2)$ and $R = \mathbf{2}$ is the fundamental representation of $SU(2)$ the moduil space has been discussed extensively by P. Feehan and Leness [14]. The mathematical technicalities they have encountered are formidable. Nevertheless, the "answer" in terms of the IR theory has in fact been worked out for $G = SU(2)$ and $R = N_f\mathbf{2}$ for $N_f \leq 4$ in [38]. [17] It might be illuminating to work out other examples of the IR theory for UV theories with nontrivial hypermultiplet representations. We will return to this and other generalizations suggested by physics at the end of the notes.

4.8 So, What Good Is It?

Witten's 1988 paper introduced the idea of a topological field theory and in particular the idea of a topological twisting. This led to a beautiful quantum-field-theory interpretation of Donaldson's polynomials.

[16] For the case when X is an algebraic surface higher rank invariants have also been discussed from the point of view of algebraic geometry in [36].

[17] The reason Feehan and Leness study this moduli space is not so much to do intersection theory on it, but rather to use it as a bordism between the moduli space of $SU(2)$ ASD connection and the Seiberg-Witten moduli space. This is one approach one might take to giving a mathematically rigorous proof of the relation between Donaldson and Seiberg-Witten invariants.

With 20-20 hindsight the interpretation naturally suggests the Seiberg-Witten equations as a natural outcome of the QFT approach, simply because it is natural to couple the VM to HM's. In fact, some physicists DID consider the topologically twisted theory coupled to hypermultiplets. (See p 77 of [24] and references 52,53,54 cited there.) The generalized monopole equations can, generously interpreted, be considered to be implicitly defined in those papers, but they were not written out explicitly, nor was the geometrical content of the equations elucidated. Nobody before Witten seems to have realized the great power of the $U(1)$ version of the equations.

♣Also Karlhede-Rocek. ♣

♣Last sentence out of place? ♣

In the years following 1988 people asked: "But does the interpretation actually lead to an effective way of evaluating the Donaldson polynomials?" This was not at all clear and several naysayers took a negative attitude. There was certainly a certain amount of skepticism, if not outright hostility, until the fall of 1994....

♣End this section: 25 minutes from 4.7.2 . ♣

5. Mapping The UV Theory To The IR Theory

5.1 Motivation For Studying Vacuum Structure

For topological invariant correlation functions the partition function $Z^{\xi}_{DW}(p,\Sigma)$ - which is defined by the UV path integral - should be computable in terms of a low energy effective action:

$$Z^{\xi}_{DW}(p,\Sigma) := \left\langle e^{p\mathcal{O}(\wp)+\mathcal{O}(\Sigma)} \right\rangle_{UV} = Z^{\xi}_{IR}(p,\Sigma) := \left\langle e^{p\mathcal{O}_{IR}(\wp)+\mathcal{O}_{IR}(\Sigma)+\cdots} \right\rangle_{IR} \tag{5.1}$$

The reason is that we can scale up the metric: We replace:

$$g_{\mu\nu} \to t g_{\mu\nu} \tag{5.2}$$

and we take the limit $t \to +\infty$.

On the one hand, changing t is a Q-exact change in the path integral: It cannot change the integral.

On the other hand, from the physical point of view, we are stretching lengths to infinity, and correspondingly scaling energies to zero. That is, we are studying dynamics infinitesimally above the vacuum. Therefore, it must be possible to evaluate the partition function in the low energy effective theory. (By *definition* of a LEET!!)

Our goal is going to be to make (5.1) as explicit as possible. We are going to see it is a generalization of both the Kronheimer-Mrowka structure theorem [21] and Witten's conjecture [55].

Remarks:

1. We will find that N=2 theories on $\mathbb{R}^3 \times \mathbb{R}$ do not have a unique quantum vacuum but rather a <u>moduli space of quantum vacua</u>. When evaluating the path integral of the theory on a <u>compact</u> manifold X one must integrate over *all* the vacua. The reason is that quantum fluctuations lead to tunneling between these vacua on a compact manifold. The tunneling amplitude is exponentially suppressed in the volume of X and goes to zero when X is noncompact. In that case one fixes a particular vacuum at infinity as part of the boundary conditions.

♣Mathematicians find this a very difficult point to grasp. Further explanation would be good here. ♣

2. The LEET involves a nonlinear sigma model whose target space is the moduli space of quantum vacua just alluded to. Thus, integration over fields in the LEET involves integration over these quantum vacua.

3. In general a LEET is not renormalizable, and the LEET describing N=2 field theories is no exception. Thus, a great deal needs to be said about the meaning of such a path integral. Fortunately, in the topological field theory one needs to work at most at one-loop order in perturbation theory about a vacuum, and hence we can in fact evaluate the path integrals of these non-renormalizable field theories.

4. We will also need to have a precise correspondence of Q-closed operators

$$\mathcal{O}(\wp) \to \mathcal{O}_{IR}(\wp) \tag{5.3}$$

$$\mathcal{O}(\Sigma) \to \mathcal{O}_{IR}(\Sigma) \tag{5.4}$$

between the UV and IR theories. Roughly speaking, this is the mapping of operators under RG flow to the IR. Although the Q-closed operators form a (graded)commutative ring the RG map is <u>not</u> a ring homomorphism. This is the origin of the $+\cdots$ in the exponential on the RHS of (5.1) and will be filled in below. It has to do with a subject known as "contact terms."

5.2 The Classical Vacua And Spontaneous Symmetry Breaking

We now study the theory on $\mathbb{M}^{1,3}$ with Minkowski metric.

The "vacua" are the minimal energy states in the Hilbert space. In a relativistic theory, they will be relativistically invariant and thus provide points around which we can do perturbation theory in the Wick-rotated Euclidean theory.

The Hamiltonian of the general classical Lagrangian with Lie group G and HM representation $W = R \oplus \bar{R}$ is a sum of nonnegative terms. The classical vacua of the theory all have zero energy. Putting various derivative terms to zero puts the gauge fields to zero and sets the scalar fields to be constant as a function of position. We will write the constant values as $\phi, q, \tilde{q}$. They are not to be confused with the fields of the same name, which are functions on $\mathbb{M}^{1,3}$. (These constant values should be thought of as "classical vacuum expectation values.")

Setting the potential energy terms to zero we find [18]

$$[\phi, \phi^*] = 0 \tag{5.5}$$

$$\mu(M) = 0 \tag{5.6}$$

$$\phi \cdot q = 0 \qquad \phi^* \cdot \tilde{q}^* = 0 \tag{5.7}$$

The solution set $\mathcal{V}$ of pairs (ϕ, M) that satisfy (5.5)-(5.7) is a G space and the moduli space of classical vacua is, by definition:

$$\mathcal{M}^{\mathrm{Classical}} := \mathcal{V}/G \tag{5.8}$$

[18] We have put mass parameters to zero.

Let us describe the space $\mathcal{M}^{\rm Classical}$ a little. Equation (5.5) says that ϕ is semisimple. It can therefore be conjugated to a maximal torus $\mathfrak{t}\otimes\mathbb{C}$. In general its vev spontaneously breaks the gauge symmetry:

$$G \to {\rm Stab}(\phi) \tag{5.9}$$

When ϕ is both semisimple and regular the stabilizer is just the normalizer of the Cartan torus and the unbroken gauge group is just the maximal torus T whose Lie algebra is $\mathfrak{t}$. In this case equations (5.6) and (5.7) set $M = 0$. Thus a subset of $\mathcal{M}^{\rm Classical}$ can be identified with

$$\mathcal{M}^{\rm classical}_{\rm Coulomb} = (\mathfrak{t}\otimes\mathbb{C} - \Delta)/W \tag{5.10}$$

where W is the Weyl group Δ is the subset of semisimple but non-regular elements. This is known as the *Coulomb branch* of vacua. It is called the "Coulomb branch" because the unbroken abelian gauge symmetry means that there are massless abelian gauge fields in the LEET and they will exert electromagnetic forces on charged particles in the theory. In particular, two stationary electrically charged particles will feel a long-range Coulomb force.

When $\phi \in \Delta$ there are still solutions of the vacuum equations, but some of the non-abelian gauge symmetry is restored, and it is also possible have nonzero values of M. The extreme case of this is the point $\phi = 0$. If there are no HM's then the gauge symmetry is fully restored. If there are HM's then there is a branch of vacua which is just the hyperkahler quotient of W. This is known as the *Classical Higgs branch*:

$$\mathcal{M}^{\rm classical}_{\rm Higgs} = W////G \tag{5.11}$$

It is called the Higgs branch because if M is generic then the solution to (5.7) forces $\phi = 0$ and completely breaks the continuous gauge symmetry. Thus the gauge symmetry has been Higgsed.

In general $\mathcal{M}^{\rm Classical}$ is a complicated stratified space

$$\mathcal{M}^{\rm Classical} = \amalg_\alpha \mathcal{M}^{\rm Classical}_\alpha = \mathcal{M}^{\rm Classical}_{\rm Coulomb} \amalg \cdots \amalg \mathcal{M}^{\rm Classical}_{\rm Higgs} \tag{5.12}$$

with the closures of the various components intersecting in complicated ways. The other components have $\phi \in \Delta - \{0\}$ so it is semisimple but not regular and M is also nonzero. These are called "hybrid branches" and are not much less well investigated in the literature.

Example: Our main example is $G = SU(2)$ with no matter hypermultiplets. In the classical theory the vacuum energy is $V = {\rm Tr}([\phi^*,\phi])^2$. It is minimized by normal matrices so we can gauge ϕ to the form

$$\phi = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \tag{5.13}$$

uniquely up to Weyl transformation which takes $a \to -a$. The classical vacua are parametrized by the gauge invariant parameter

$$u^{\rm classical} = {\rm Tr}\phi^2 = 2a^2 \tag{5.14}$$

$u^{\text{classical}}$ can take any value in the complex plane, and conversely, a choice of v uniquely determines a classical vacuum of the theory on $\mathbb{R}^4$. At every point on the $u^{\text{classical}}$-plane with $a \neq 0$ the gauge group is spontaneously broken:

$$SU(2) \to U(1) \tag{5.15}$$

and the $W^{\pm}$ bosons (i.e. the gauge fields associated with the off-diagonal generators) have mass proportional to $|a|$. Classically, at $u^{\text{classical}} = 0$, these gauge bosons become massless, the stabilizer jumps to $\text{Stab}(\phi) = SU(2)$ and the full $SU(2)$ symmetry is restored. Thus for $SU(2)$:

$$\mathcal{M}^{\text{classical}}_{\text{Coulomb}} = \mathbb{C} - \{0\} \tag{5.16}$$

FIGURE: COMPLEX $u^{\text{classical}}$-PLANE. POINT AT ORIGIN HAS ENHANCED GAUGE SYMMETRY. Caption: The classical Coulomb branch for $G = SU(2)$ with no hypermultiplets is $\mathbb{C} - \{0\}$ shown here at the complex $u^{\text{classical}}$-plane. At $u^{\text{classical}} = 0$ the nonabelian $SU(2)$ gauge symmetry is unbroken.

♣End this section: 21 minutes from 5 . ♣

5.3 Quantum Vacua And The LEET

A key point is that:

In the weakly coupled quantum theory the classical vacuum degeneracy is not lifted: But quantum effects can change the description of low energy fluctuations around that vacuum.

Moreover, purely quantum-mechanical branches of vacua can emerge due to strong quantum effects.

The argument that the classical vacuum degeneracy is not lifted goes like this: Choose a classical vacuum and define the theory in perturbation theory by expansion around that vacuum. Nonperturbative effects are defined by defining the theory in perturbation theory around instanton solutions. This quantum theory has a LEET description. Since the N=2 supercharges are integrals of conserved currents the supersymmetry will not be broken by RG flow to the IR so long as there is no supersymmetry anomaly - and there is absolutely no evidence for a supersymmetry anomaly. Therefore, we assume there is in fact no supersymmetry anomaly in the quantum theory. Therefore the LEET will be described by an N=2 supersymmetric effective action.

However: N=2 supersymmetric actions do not allow for the possibility of spontaneous supersymmetry breaking $N = 2 \to N = 0$, so the original vacuum must have been an exact quantum vacuum.

(It is crucial that we have $N = 2$ supersymmetry here. In $N = 1$ supersymmetric actions one can induce a superpotential and/or an FI parameter, either of which can break supersymmetry.)

How do we describe the LEET for the theory associated with a classical vacuum state?

As long as no new quantum states become massless as functions of the vacuum parameters, we can describe the LEET using the low energy degrees of freedom obtained by expanding around the classical vacua.

In order to implement this let us assume for simplicity that we are working with a single VM with compact simple group G. Then in the usual quantization of the theory one must introduce a mass scale Λ at which the coupling becomes strong. One introduces a bare coupling constant

♣Need to give normalization of YM term for this to be meaningful. ♣

$$\tau_0 = \frac{4\pi \mathrm{i}}{g_0^2} + \frac{\theta_0}{2\pi} \tag{5.17}$$

and a cutoff on perturbative Feynman diagrams, Λ_0, and defines a quantum theory with parameter Λ by holding the combination:

$$\Lambda^{2h^\vee} := \Lambda_0^{2h^\vee} e^{2\pi \mathrm{i}\tau_0} \tag{5.18}$$

fixed as $\Lambda_0 \to \infty$ and simultaneously $g_0 \to 0$. [19]

Now, with a scale Λ in the problem the theory is effectively weakly coupled when u is far from the nonregular locus - in the natural metric. For $SU(2)$ this is the criterion $|u| \gg |\Lambda^2|$ and one can reliably compute the low energy spectrum of fluctuations around a vacuum on the classical Coulomb branch.

Applying this reasoning to our main example of $G = SU(2)$ with no HM's we can say that, at least for $|u| \gg |\Lambda^2|$, we can parametrize the quantum vacua $|\Omega(u)\rangle$ by

$$2u := \langle\Omega(u)|\mathcal{O}|\Omega(u)\rangle = \langle\Omega(u)|\frac{\mathrm{Tr}_{\mathbf{2}}\phi^2}{8\pi^2}|\Omega(u)\rangle \tag{5.19}$$

at least so long as we know there are no extra massless states in the quantum theory. The normalization factor of two on the left-hand-side of the equation is meant to facilitate comparison with the mathematical results.

In their breakthrough paper in the spring of 1994 Seiberg and Witten proposed - based on physical reasoning such as semiclassical monodromy and the mass gap and confinement of the $N = 1$ SYM obtained by perturbing with a superpotential $W = m\mathrm{Tr}\phi^2$ - that in the quantum theory the Coulomb branch vacua is not $\mathbb{C} - \{0\}$ but rather the entire u-plane. That is, for *every* point on the u plane the gauge group is spontaneously broken $SU(2) \to U(1)$. However, something special does happen at $u = \pm\Lambda^2$. At these points, indeed, certain quantum states associated with "BPS particles" become massless and new *"quantum Higgs branches"* emerge. For the vacua on $\mathbb{R}^4$ this is just the HK quotient of $W = \mathbb{C}\oplus\mathbb{C}$ and is a single point. On a general four-manifold X we will get vacua associated with the solutions of the Seiberg-Witten equations. [20]

The vacua of the $SU(2)$ theory may thus be pictured as follows:

FIGURE: COMPLEX u-PLANE. TWO SINGULAR POINTS. Caption: Two special points at which there are new massless HM's are at $u = \pm\Lambda^2$. In the classical limit $\Lambda^2 \to 0$ these points coalesce and we recover the classical picture of the vacua. In the quantum theory there is an important order-of-limits issue: At any point $u \neq \pm\Lambda^2$ on the u-plane the LEET is described by a $U(1)$ N=2 abelian gauge theory. However, at a small but nonzero energy scale the region in a neighborhood of $u = \pm\Lambda^2$ is only accurately described by a $U(1)$ N=2 abelian gauge theory coupled to a charge one hypermultiplet.

♣11 minutes from section 5.3. ♣

[19] This needs to be modified in the presence of matter. We replace $-2h^\vee$ by β defined in *** above.

[20] On a general four-manifold we are abusing the term "vacua." However, on $X = \Sigma_3 \times \mathbb{R}$ we do have a notion of vacua. These would be related to the moduli space of solutions to the Seiberg-Witten equations on Σ_3. In the quantum theory we would consider wavefunctions on this space. The resulting Hilbert space $\mathcal{H}(\Sigma_3)$ should be related to one of the flavors of Seiberg-Witten-Floer homology.

5.3.1 The General N=2 Supersymmetric LEET Describing Fluctuations On The Coulomb Branch For Theory On $\mathbb{R}^4$

The general N=2 LEET with only abelian vector multiplets in an abelian gauge group T with Lie algebra $\mathfrak{t}$ is described by a nonlinear sigma model whose target space is

$$\mathcal{B} = (\mathfrak{t} \otimes \mathbb{C})\,/W \tag{5.20}$$

The description will be valid on the complement $\mathcal{B}^*$ of some codimension one subvarieties where new massless states beyond those described by the abelian vectormultiplets appear in the spectrum.

An important theorem of $N = 2$ supersymmetry is that the general action of the LEET for fields mapping into $\mathcal{B}^*$ is defined by the data of a family of abelian varieties over $\mathcal{B}^*$

$$\pi : \mathfrak{A} \to \mathcal{B}^* \tag{5.21}$$

together with a holomorphically varying $N = 2$ *central charge function* Z. It should be viewed as a holomorphic function on the total space:

$$Z : \Gamma \to \mathbb{C} \tag{5.22}$$

which is linear on the fibers. Here Γ is a local system of lattices

$$\Gamma \to \mathcal{B}^* \tag{5.23}$$

where the fibers $\Gamma_u := H_1(\mathfrak{A}_u; \mathbb{Z})$. The lattices have an anti-symmetric integral-valued "Dirac-Schwinger-Zwanziger" pairing. The local system typically has nontrivial monodromy.

In any monodromy-free neighborhood $\mathcal{U}$ of a point $u \in \mathcal{B}^*$ we may choose a maximal Lagrangian decomposition of the fibers of Γ:

$$\Gamma_u = \Gamma_u^{\text{electric}} \oplus \Gamma_u^{\text{magnetic}} \qquad\qquad u \in \mathcal{U} \tag{5.24}$$

Such a choice is known as a *choice of duality frame.*

The reason for the labels "electric" and "magnetic" on the sub-lattices of Γ_u in equation (5.24) is that the lattice Γ_u has the physical interpretation of being the lattice of electric and magnetic charges under the unbroken abelian gauge symmetry. The integral anti-symmetric pairing has the interpretation of being the Dirac pairing of electric and magnetic charges. This follows from the Lagrangian described below together with abelian S-duality.

Associated to any choice of duality frame is a system of local coordinates

$$a^I = Z(\alpha^I) \tag{5.25}$$

where I runs over the rank of Γ_u. Moreover, $N = 2$ supersymmetry provides a locally-defined holomorphic function $\mathcal{F}$, known as the *prepotential* such that

$$\frac{\partial \mathcal{F}}{\partial a^I} = Z(\beta_I) \tag{5.26}$$

These quantities are usually defined to be $a_{D,I}$. Finally, the period matrix of the abelian variety $\mathfrak{A}_u$ with respect to the duality frame is

$$\tau_{IJ} := \frac{\partial a_{D,I}}{\partial a^J} = \frac{\partial^2 \mathcal{F}}{\partial a^I \partial a^J} \tag{5.27}$$

Now, in terms of all this data the action of the LEET is given by

$$S_{IR} = \int \mathrm{i}(\tau_{IJ} F^{-,I} F^{-,J} + \bar{\tau}_{IJ} F^{+,I} F^{+,J}) + \mathrm{Im}\tau_{IJ} da^I * d\bar{a}^J + \cdots \tag{5.28}$$

where all terms in the $+\cdots$ involve fermions and nonlinear interactions defined by τ_{IJ} and its derivatives.

The general result of defining the quantum theory of the theory of a vector-multiplet for compact simple group G around a weakly-coupled vacuum on the Coulomb branch is that there is a distinguished family of duality frames with special coordinates $a \in \mathfrak{t} \otimes \mathbb{C}$ and the "prepotential" is

$$\mathcal{F} = \mathcal{F}^{\text{classical}} + \mathcal{F}^{1-\text{loop}} + \mathcal{F}^{\text{instanton}} \tag{5.29}$$

$$\mathcal{F}^{\text{classical}} + \mathcal{F}^{1-\text{loop}} = \frac{\mathrm{i}}{4\pi} \sum_{\alpha \in \Delta^+} \langle \alpha, a \rangle^2 \log \frac{\langle \alpha, a \rangle^2}{2\Lambda^2} \tag{5.30}$$

where Δ^+ is a set of positive roots of $\mathfrak{g}$. To separate this into classical and 1-loop contributions we use equation (5.18) and get:

$$\mathcal{F}^{\text{classical}} = const.\tau_0(a,a) \tag{5.31}$$

♣Work out constant. ♣

The nonperturbative corrections have the form:

$$\mathcal{F}^{\text{instanton}} = \sum_{k=1}^{\infty} \mathcal{F}_k(\langle \alpha, a \rangle^{-1}) \Lambda^{2kh^\vee} \tag{5.32}$$

where $\mathcal{F}_k$ is a homogeneous polynomial on $|\Delta^+|$ variables of degree $2kh^\vee - 2$.

Remarks:

1. Using techniques developed in [39, 40], Nekrasov and collaborators have computed these homogeneous polynomials $\mathcal{F}_k$ from first principles for all the classical groups, as well as for extensions with various kinds of semisimple groups with matter [43, 44, 45].

2. It should be stressed that when it is used as a LEET for a UV theory the action (5.28) is only the leading term in an derivative expansion - that is an expansion of E/Λ where E is a typical energy scale. However, for the topological questions we address here only the leading terms in the derivative expansion are of importance. The higher terms are all Q-exact, and would not matter in the low energy limit in any case.

5.4 Seiberg And Witten's LEET For $G = SU(2)$

Returning to the case of $G = SU(2)$ with no HM's original analyzed by Seiberg and Witten the LEET is described as follows:

When expanding around any vacuum away from $u = \pm\Lambda^2$ the fields in the LEET comprise a single $U(1)$ vectormultiplet:

$$a, A, D, \eta, \chi, \psi \tag{5.33}$$

valued in the Lie algebra $\mathfrak{u}(1)$. Thus a is a complex valued scalar field on $\mathbb{R}^4$. Conceptually it should be thought of as defining a nonlinear sigma model of maps $\mathbb{R}^4 \to \mathcal{M}_{\text{Coulomb}}^{\text{Quantum}}$. A is an abelian gauge field and so forth.

Remarks:

1. Physicists always write a to describe both the quantum VEV about which we are expanding as well as the quantum field describing long-wavelength fluctuations of that VEV.

2. As stressed above, in order to write the LEET we must make a choice of duality frame. The description in other duality frames is obtained by using electic-magnetic duality on the $U(1)$ gauge theory - something which can be carried out explicitly in path integrals [46, 56]. Although there is monodromy of the local system at large u it is well-understood in terms of the Witten effect. So from physics we know that a generator acts by

$$\begin{pmatrix} a \\ a_d \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} a \\ a_d \end{pmatrix} \tag{5.34}$$

   In particular, a is well-defined up to sign. (In fact one can derive this immediately from the expression for the prepotential given above.)

♣Check factor of 2 ♣

The Lagrangian is:

$$\begin{aligned} \mathcal{L}_{IR,vm} \sim\ & i(\bar{\tau} F^+ F^+ + \tau F^- F^-) \\ & + \mathrm{Im}\tau da * d\bar{a} + \mathrm{Im}\tau D * D \\ & + \tau\psi * d\eta + \bar{\tau}\eta d * \psi + \tau\psi d\chi - \bar{\tau}\chi(d\psi) \\ & + i\frac{d\bar{\tau}}{d\bar{a}}\eta\chi(D + F^+) + \cdots \end{aligned} \tag{5.35}$$

1. Here we have given it in the topologically twisted form we need and the $+\cdots$ contain other complicated interaction terms we will not need (but they would be relevant on non-simply connected manifolds).

2. As far as the constraint of $N = 2$ supersymmetry is concerned $\tau(a)$ can be an arbitrary holomorphic function of a. To give $\tau(a)$ is to specify the Lagrangian. Therefore, to specify the low energy theory we we need to:

   a.) Compute $\tau(a)$

   b.) Explain how $\tau(a)$ is related to u.

Seiberg-Witten's solution to this problem is the following: (We will not try to justify the solution. We merely state the result in a succinct way tailored to our purpose here.)

1. Consider the family of elliptic curves: [21]

$$E_u : \qquad y^2 = (x-u)(x-\Lambda^2)(x+\Lambda^2) \tag{5.36}$$

   Note that the discriminant locus is clearly at $u = \pm\Lambda^2$.

2. We equip these curves with a meromorphic one-form

$$\lambda_{SW} := \frac{dx}{y}(x-u) \tag{5.37}$$

   and then the central charge function is given by taking periods of λ_{SW}. Note that this differential has a pole at $x = \infty$. The holomorphic differential is $\omega = dx/y$.

3. The local system associated to the family (5.36) has indeed a monodromy given by the transpose of the generator in (5.34). we *choose* a Lagrangian homology basis A, B of $H_1(E_u)$ so that A is invariant under twice the generator and *define*

$$a = \oint_A \lambda_{SW}. \tag{5.38}$$

4. Then τ is the period of E_u with respect to this homology basis - this tells us the function $\tau(a)$, *and* we have -at least implicitly - explained how the (vev of) a is related to the vacuum u. The relation is given by (5.38). Note that, as is standard in discussions of LEETs, we are using the same notation, a, for a field, such as $a(x^\mu)$ on $\mathbb{R}^4$ and its vacuum expectation value on $\mathbb{R}^4$.

5. Of course, we have made a *choice* of homology basis, but the effective theory does not depend on this choice because a change of Lagrangian homology basis corresponds to an electromagnetic duality transformation on the abelian theory. In certain regions of the u-plane there is a preferred choice: In the domain $u \to \infty$ we should choose a homology basis so that

$$\mathrm{Im}\tau(u) \to \infty \tag{5.39}$$

   because $\tau = \frac{4\pi i}{e^2} + \frac{\theta}{2\pi}$.

This solution of the vacuum structure leads to a very notable phenomenon: The local system $H_1(E_u;\mathbb{Z})$ has monodromy around the discriminant locus $u = \pm\Lambda^2$ where the fibration $E \to \mathbb{C}$ becomes singular. At these points a period of λ_{SW} goes to zero. This will be connected with some very important physics.

♣DESCRIBE LOCAL SYSTEM BETTER! CHOOSE CUTS, BASIS and GIVE MATRICES OVER 3 explicit loops. ♣

FIGURE: SHOW TORI OVER u-PLANE DEGENERATING AT 2 POINTS.

[21] This is actually not the most convenient presentation of the family for computational purposes. See section 5.4.1 below.

5.4.1 An Alternative Description

There can be many ways of presenting the family of Seiberg-Witten curves. The above description is a family of curves over the modular domain for $\Gamma(2)$. An equivalent description in terms of curves defined over the modular domain for $\Gamma^0(4)$ is

$$y^2 = x^2(x-u) + \frac{\Lambda^4}{4}x \tag{5.40}$$

This is a little better because it generalizes more readily to the curves for the inclusion of HM representations. The discriminant locus is still at $u = \pm\Lambda^2$.

Using τ relative to the preferred basis at infinity it turns out that the relation of u and τ can be expressed very nicely in terms of modular functions:

$$u = \frac{1}{2}\frac{\vartheta_2^4 + \vartheta_3^4}{(\vartheta_2\vartheta_3)^2} = \frac{1}{8q^{1/4}}\left(1 + 20q^{1/2} - 62q + 216q^{3/2} + \cdots\right) \tag{5.41}$$

The duality frame was not quite fixed because the choice of B-cycle was not unique. The monodromy takes $\tau \to \tau + 4$, and leaves u invariant, as it should.

Similarly, $a(u)$ and $a_d(u)$ are also very explicitly expressed in terms of modular functions. For example,

$$a(u) = \frac{1}{6}\left(\frac{2E_2 + \vartheta_2^4 + \vartheta_3^4}{\vartheta_2\vartheta_3}\right) = \frac{1}{4}q^{-1/8}\left(1 + \mathcal{O}(q^{1/2})\right) \tag{5.42}$$

The u-plane can be identified with the fundamental domain for $\Gamma^0(4)$ with the two cusps at $\tau = 0$ and $\tau = 2$ corresponding to the monopole and dyon point, respectively.

FUNDAMENTAL DOMAIN FOR $\Gamma^0(4)$:

SEMICLASSICAL REGION: $\mathcal{F} \cup T\cdot\mathcal{F} \cup T^2\cdot\mathcal{F} \cup T^3\cdot\mathcal{F}$

MONOPOLE REGION: $S\cdot\mathcal{F}$

DYON REGION: $T^2S\cdot\mathcal{F}$

In general, if $F(\tau)$ is modular for $\Gamma(2)$ then $G(\tau) := F(\tau/2)$ is modular for $\Gamma^0(4)$. Thus the two families of elliptic curves are isogenous

Remarks:

1. Except for the case of pure $SU(2)$ gauge theory we do not have the luxury of identifying the u-plane with a modular curve. So it is better to try to use arguments that generalize to arbitrary Coulomb branches. However, we will make use of some of the simplifications of identifying the u-plane with the modular domain for $\Gamma^0(4)$.

2. In fact, the modern point of view based on theories of class S gives a rather different (and generally superior) way of writing down Seiberg-Witten curves and differentials. But we are using the older formulation here because that is the language in which the u-plane integrals have been done.

5.4.2 An Important Symmetry

In the case of an N=2 theory with a single VM for compact simple group G there is an important global symmetry of the quantum theory. It is a finite subgroup of the $U(1)_R$ symmetry group. In general, instanton effects explicitly break this $U(1)_R$ to a finite group $\mathbb{Z}_{2h^\vee}$. Then the vacuum expectation values on the Coulomb branch further break

$$\mathbb{Z}_{2h^\vee} \to \mathbb{Z}_2 \tag{5.43}$$

The $\mathbb{Z}_2$ is unbroken because it is equivalent to a 2π rotation in space.

In terms of ϕ we can choose a generator of $\mathbb{Z}_{2h^\vee}$ to act by

$$\phi \to \omega\phi \tag{5.44}$$

where $\omega = e^{2\pi \mathrm{i}/2h^\vee}$. It follows that the gauge invariant parameters on the Coulomb branch

$$u_k \to \omega^k u_k \tag{5.45}$$

for 0-observalbes $u_k \sim \mathrm{Tr}\phi^k$.

In particular, for $G = SU(2)$ we have a symmetry under $u \to -u$. This maps the singularity at $u = +\Lambda^2$ to the singularity at $u = -\Lambda^2$. That is very useful and cuts down our work when evaluating the Higgs-branch contribution to Z^{ξ}_{DW} by half.

5.4.3 Seiberg-Witten Curves In General

The LEET for the $SU(2)$ theory, with "quark" matter hypermultiplets in the representations $R = \mathbf{2}^{\oplus N_f}$, $0 \leq N_f \leq 4$, (to use the notation of equation (4.65)) was solved by Seiberg and Witten [47, 46]. Their discussion has been generalized to a large number of theories by many other physicists. For a useful review see [51].

As just noted the Seiberg-Witten curve has been derived from instanton calculus for a large number of theories by Nekrasov et. al. There is also a large class of theories - the "theories of class S" - which are closely related to Hitchin systems. For these, the SW curve is the spectral curve over the Hitchin base and can be readily written down.

However, it should be stressed that it is still not known how to write the SW family of curves and the SW differential for an arbitrary N=2 field theory.

5.5 BPS States

The LEET (5.35) becomes singular at $u = \pm\Lambda^2$. This is particularly obvious if we think of the u-plane in terms of the fundamental domain for $\Gamma^0(4)$: At the two cusps $\tau = 0, 2$ we have $\mathrm{Im}\tau \to 0$, a strong coupling limit. Many terms in the action will blow up. This is a signal that the LEET with action (5.35) is *WRONG* in a small neighborhood of $u = \pm\Lambda^2$.

The reason the LEET breaks down is that it is obtained by integrating out massive modes. Quite generally, suppose the spectrum of the theory, as a function of the vacuum parameter u, is such that some particles become massless at some point u_* on the u-plane. Then, to have a nonsingular LEET, the fields corresponding to these particles must be "integrated in." That is, one needs to find a new LEET that includes both the $U(1)$ VM describing fluctuations on the u-plane together with the new light fields such that, if we "integrate out" these light fields we recover the original LEET (at energy scales below the mass of those light fields). What "integrate out" means is that we do the path integral over those light fields, but not over the $U(1)$ VM fields and keep the leading order terms in a low-energy expansion. When one tries to integrate out massless particles, Feynman diagrams are singular, and singularities in the purported LEET will result.

We conclude that there must be some kind of massless particle in the spectrum of the theory at $u = \pm\Lambda^2$ in addition to the massless $N = 2$ Maxwell field multiplet describing low energy fluctuations around the Coulomb vacuum.

Note the order of limits problem here: For any fixed $u \neq \pm\Lambda$ the theory is well-described by (5.35) so long as "typical" field configurations do not extend over $u = \pm\Lambda^2$. However, if one wants a description valid for field configurations that are valued in some open neighborhood $\mathcal{U}_{\Lambda^2}$ that includes, say, $u = \Lambda^2$ one must "integrate in" the fields which become massless at $u = \pm\Lambda^2$.

In general, finding the spectrum of the Hamiltonian, as a function of u, even in the relatively amenable d=4 N=2 theories is out of the question. Happily, there is a subspace of the Hilbert space of the theory where the spectrum of the Hamiltonian is known exactly. This is known as the *BPS spectrum*. In brief, we can define the BPS spectrum as follows: The Hilbert space of the d=4 N=2 QFT on $\mathbb{R}^3$ with vacuum u at $|\vec{x}| \to \infty$, denoted $\mathcal{H}_u$ is graded by the lattice of electro-magnetic charges Γ_u under the unbroken Abelian gauge symmetry:

$$\mathcal{H}_u = \oplus_{\gamma \in \Gamma_u} \mathcal{H}_{u,\gamma} \tag{5.46}$$

In the super-selection sector $\mathcal{H}_{u,\gamma}$ the $N = 2$ central charge operator Z in the $N = 2$ super-Poincaré algebra acts as a c-number $Z_\gamma(u)$. Moreover, by considering the Hermitian squares of suitable linear combinations of supercharges and using the supersymmetry algebra one can show that the Hamiltonian, restricted to $\mathcal{H}_{\gamma,u}$ is bounded below by $|Z_\gamma(u)|$. This bound is known as the "Bogomolnyi bound." The BPS spectrum then, is, by definition the sum over γ of the BPS states of charge γ, defined by

$$\mathcal{H}^{\rm BPS}_{u,\gamma} := \{\Psi \in \mathcal{H}_{u,\gamma} | H\Psi = |Z_\gamma(u)|\Psi\}. \tag{5.47}$$

To find the BPS spectrum one must solve two problems: First one must find explicit formulae for the $N = 2$ central charges $Z_\gamma(u)$, and secondly one must actually identify what states in the Hilbert space actually saturate the Bogomolnyi bound. The first step follows from the Seiberg-Witten curve since $Z_\gamma(u)$ are given by the periods around cycles of the curve of a meromorphic differential form known as the Seiberg-Witten differential.

For the pure $SU(2)$ theory an exact BPS spectrum was proposed in the original Seiberg-Witten paper. [22] Since we know the periods of λ we have an exact mass formula:

$$M = const. |\oint_\gamma \lambda_{SW}| \tag{5.48}$$

[22] Nowadays one can prove it by solving for the semiclassical spectrum and using the Kontsevich-Soibelman wall-crossing formula.

for BPS particles of electromagnetic charge γ.

Now recall that the electromagnetic charge lattice is identified with the fibers Γ_u. That is, an electromagnetic charge is identified with a homology class $\gamma \in H_1(E_u;\mathbb{Z})$. In particular, the A-cycle corresponds to electric charge and a dual B-cycle corresponds to magnetic charge.

In the weak coupling region $|u| \ll |\Lambda^2|$ one can prove the existence of BPS particles with magnetic charge by using collective coordinate quantization of the moduli space of magnetic monopoles. This is an interesting story which, mathematically, involves the study of the L^2-kernel of Dirac-like operators on monopole moduli space. [23] In particular, magnetic monopoles of magnetic charge 1 have a charge γ that is a B-cycle and moreover this B-cycle has the property that the period

$$a_d := \oint_B \lambda_{SW} \tag{5.49}$$

vanishes at $u = \Lambda^2$. This is the famous "massless magnetic monopole" of Seiberg and Witten.

In the neighborhood $\mathcal{U}_{\Lambda^2}$, where the monopole hypermultiplet is becoming light it should be included in the LEET as a hypermultiplet field. Since the particle is magnetically charged, the corresponding field M couples to the $U(1)$ photon via

$$\int |(d + A_d)M|^2 + \cdots \tag{5.50}$$

where A_d is the magnetically dual $U(1)$ photon. It is related to the photon field A we use at $|u| \ll |\Lambda^2|$ by

$$F(A) = *F(A_d) \tag{5.51}$$

Thus, in order to write a local LEET we must perform a suitable change of duality frame. The supersymmetric partner of A_d is the period

$$a_d = \oint_B \lambda_{SW} \tag{5.52}$$

The electric and magnetic charge lattices get swapped so that

$$\tau_d = -1/\tau \tag{5.53}$$

(The description in terms of the fundamental domain for $\Gamma^0(4)$ is very useful here.)

Now recall the symmetry of the theory taking $u \to -u$. There must be a similar particle that becomes massless at $u = -\Lambda^2$. It is a 'dyon" corresponding to cycle $B+A$. In general, "dyons" are particles with both electric and magnetic charge. The semiclassical analysis shows that, in addition to the magnetic monopole, there is also a dyon with magnetic charge one and electric charge one. This corresponds to the cycle $B + A$ and becomes massless at $u = -\Lambda^2$.

[23] This is a long story with a long history with contributions from many authors. For a recent discussion see [41] and references therein.

[24]

Remarks

1. In general $d = 4, N = 2$ field theory there are actually *two* things we expect to be able to solve for exactly in $N = 2$ field theories. In addition to the LEET we expect to be able to solve for the "BPS spectrum." These are the *lightest stable particles* in a fixed charge sector (of the low energy abelian gauge theory). Again, SW found the BPS spectrum for the pure $SU(2)$ theory and there has been much progress in the meantime in understanding that spectrum for many other theories, but again the general solution has not been achieved. This fascinating topic requires a course all by itself...

5.6 The Low Energy Theory Near $u = \pm\Lambda^2$

To summarize the previous section: Although the description (5.35) breaks down at $u = \pm\Lambda^2$ it is clear how to correct the low energy description in a small neighborhood $\mathcal{U}_{\pm\Lambda^2}$ around these points. (The size of the neighborhood depends on the energy cutoff we use to describe the LEET.) We need to "integrate in" fields corresponding to the particles that become massless at $u = \pm\Lambda^2$. These particles turn out to be BPS particles in hypermultiplet representations.

Thus, for example, the particle that becomes light at $u = \Lambda^2$ a single hypermultiplet which, at weak coupling is a monopole with magnetic charge 1 and electric charge 0. The field that describes this particle is, in a small neighborhood around $u = \Lambda^2$ just a hypermultiplet which has charge +1 under the electromagnetic dual of the U(1) VM (a, η, χ, ψ, A).

Thus the LEET in a neighborhood of $u = +\Lambda^2$ is described by a pair of field multiplets:

1. A $U(1)$ VM with scalar field a_d and vector field A_d, the electromagnetic dual of A.
2. A HM defined by the charge +1 representation of $U(1)_D$, the dual electromagnetic gauge group.

♣The following paragraph has already been discussed above. Delete it? ♣

One way to see that we need to use the dual gauge field is that the monopole scalar fields M are have magnetic charge +1 in the duality frame at large $|u|$ and hence when we include them as fields in the Lagrangian the kinetic term must be of the form

$$|DM|^2 = |dM + A_d M|^2 \tag{5.54}$$

where A_d is the electromagnetic dual field. Since A and A_d are nonlocally related, and we wish to work with a local Lagrangian, when writing the contribution of the vectormultiplets to the action we should use the dual vectormultiplet fields $(a_d, A_d, \dots)$.

[24]More precisely, because of the monodromy of the charge lattice at infinity, a dyon with magnetic charge one and even electric charge can become massless at $u = \Lambda^2$, while a dyon with magnetic charge one and odd electric charge can become massless at $u = -\Lambda^2$.

The action of this LEET will be denoted $S_{IR,\mathcal{U}_{\Lambda^2}}$ to emphasize that it is to be applied in a small region around $u = \Lambda^2$. To obtain it one takes the action (5.35) in the duality frame defined by (5.53), adds the standard HM Lagrangian and topologically twists. The details can be found in [24]. The LEET accordingly has an action of the form:

$$S_{IR,\mathcal{U}_{\Lambda^2}} = S_{IR,VM}(a_d, A_d, \eta_d, \chi_d, \psi_d) + S_{HM}(M = q \oplus \tilde{q}^*) \tag{5.55}$$

Upon topological twisting, that is, coupling to a background $SU(2)_R$ symmetry connection and setting $\omega^+ = \omega_R$, the above action is in the canonical form for cohomological field theory, localizing on the SW equations.

A very similar story happens at $u = -\Lambda^2$. Here, the period of the cycle $B+A$ vanishes. The relevant modular transformation is

♣NEED TO SAY HOW YOU TRIVIALIZE THE LOCAL SYSTEM FOR THAT TO MAKE SENSE!! ♣

$$\tau_d = -\frac{1}{\tau} + 2 \tag{5.56}$$

and it is the dyon with magnetic charge 1 and electric charge 1 (in units where the $W^\pm$ bosons have charges ± 2) becomes massless. The resulting theory has action denoted $S_{IR,\mathcal{U}_{-\Lambda^2}}$

In practice we can use the useful symmetry of section 5.4.2 to obtain the answer once we have worked things out at $u = +\Lambda^2$.

♣33 minutes from section 5.3.1 ♣

5.7 Putting The Twisted LEET On A General Four-Manifold X

Now that we have reviewed the LEET for the theory on $\mathbb{R}^4$ we consider what happens when we formulate the topologically twisted theory on a general connected, compact, oriented, Riemannian four-manifold X.

As we have mentioned above, because X has finite volume we must integrate over the vacua. To motivate this we should think about path integrals on $Y \times \mathbb{R}$ or $Y \times S^1$. The crucial issue is whether there will be tunneling between different vacua. When the volume of Y is infinite tunneling amplitudes are zero and we choose a vacuum at infinity when defining the path integral. [25] On the other hand, when the volume of Y is finite nonzero tunneling amplitudes will mix all vacua and we must sum over all of them in the path integral on $Y \times S^1$. The only natural generalization to compact X is to continue to sum over all the vacuum configurations. The sum over vacua is in part achieved by evaluating the partition function for the LEET describing the fluctuations on the u-plane.

However, in a small neighborhood of $u = \pm\Lambda^2$ we must use instead the LEET $S_{IR,\mathcal{U}_{\Lambda^2}}$ given in (5.55) or its analog in $\mathcal{U}_{-\Lambda^2}$.

When topologically twisted, this is again in standard form for CohTFT. [26] Therefore we should study the Q-fixed point equations of this combined theory. These are nothing other than the famous Seiberg-Witten equations (4.74) we discussed above:

$$\begin{aligned} F(A_d)^+ &= \mu(M) \\ \not{D} M &= 0 \end{aligned} \tag{5.57}$$

[25] It is important that we are working with a field theory in $d > 3$ spacetime dimensions. In lower dimensions there are issues with infrared divergences that require modifications of this statement.

[26] Once one uses the proper off-shell formulation of the hypermultiplets. See [24] for details.

We must stress that the gauge field used in (5.57) is *electro-magnetically dual* to the gauge field used elsewhere on the Coulomb branch.

So, on a general four-manifold *there is a new branch of quantum "vacua"*, which we will call the quantum Higgs branch: It is the moduli space of solutions to the Seiberg-Witten equations. [27] Consequently, the IR evaluation of $Z^{\xi}_{DW}(p,s)$ involves a sum of two terms:

$$\boxed{Z^{\xi}_{IR}(p,s) = Z_{Coulomb} + Z_{Higgs}} \tag{5.58}$$

Of course, the term Z_{Higgs} is itself a sum of two path integrals, one for the contribution at $u = \Lambda^2$ and one for the contribution at $u = -\Lambda^2$. We will often abbreviate $Z_{\text{Coulomb}} = Z_u$ since it reduces to an integral over the u-plane, as we will see.

Now, in order to work out Z_{Coulomb} and Z_{Higgs} explicitly we need to complete the description of the action for the theory on a general Riemannian manifold X. Seiberg and Witten determined their effective action on $\mathbb{R}^4$. When coupling to a gravitational field there are new terms which must be taken into account.

5.7.1 Gravitational Couplings On The Coulomb Branch

Let us first discuss the couplings in the effective action on the Coulomb branch.

Because of topological invariance we can only have coupling to the metric via:

$$\Delta_{\text{grv}} S = \int_X e(u)\text{Tr} R \wedge R^* + p(u)\text{Tr} R \wedge R + \frac{i}{4}\int_X F \wedge w_2(X) \tag{5.59}$$

where F is the fieldstrength of the low energy $U(1)$ abelian gauge theory.

The first two terms exponentiate to functions in the integral over the u-plane:

$$E(u)^{\chi} P(u)^{\sigma} \tag{5.60}$$

Now, one can derive from physical arguments that:

$$E(u) = \alpha \left(\frac{du}{da}\right)^{1/2} \tag{5.61}$$

$$P(u) = \beta \Delta^{1/8} \tag{5.62}$$

up to constants α, β (which can depend on Λ, but not u).

1. Here Δ is the discriminant of the curve:

   $$\Delta \sim \prod_i (u - u_i) \tag{5.63}$$

   where u_i are the points where BPS states become massless. [28] For pure SYM it is normalized so that as $u \to \infty$

   $$\Delta = \Lambda^{\kappa} \prod_{\alpha} Z_{\alpha}^2 \tag{5.64}$$

♣The power κ is determined by dimensional analysis ♣

[27] These are not vacua in the usual sense of the word: They are not quantum vacua of the theory on $\mathbb{R}^4$. But the path integral clearly must localize to this locus.

[28] As emphasized in [49] This is, in general, <u>not</u> the discriminant of the elliptic curve describing the rank one theories. NEED TO CLARIFY IF IT IS THE BRANCH LOCUS OF THE SPECTRAL CURVE FORM - I THINK IT SHOULD BE.

2. The derivations given in [38, 56] rely on consistency of the integrand on the u-plane, the known behavior at $u \to \infty$ and especially the behavior as a function or R-symmetry transformations.

3. We have written E and P in a form which generalizes when we couple the $SU(2)$ vm to hypermultiplets (that will be important when we come to geography, later.) In the pure $SU(2)$ theory $\Delta = u^2 - \Lambda^4$.

4. Here α, β are numerical constants. Ultimately they will be the *only* unknowns in the full computation, and will be "fit to the experimental data." The can be provided by

   a. Explicit computations of the Donaldson polynomials in two special cases.

   b. The wall-crossing-formula.

   c. The blowup formula.

   Of course all three methods lead to the same answer! In these notes we will use method (b) - matching to the mathematically derived wall-crossing-formula. The result will be

$$\alpha = \left(\frac{2}{\pi^2}\right)^{1/8} \qquad \beta = \left(\frac{2^7}{\pi^2}\right)^{1/8} \tag{5.65}$$

♣Differs from Labastida-Marino eq. 8.17. Why? ♣

The third term in (5.59) is an important phase in the sum over flux sectors for the abelian gauge field A in the LEET $U(1)$ VM:

$$Z_{Coulomb} = \sum_{\lambda} \langle \cdots \rangle_{[F]=4\pi\lambda} \tag{5.66}$$

♣We should change notation from $w_2(E)$ to $w_2(P)$ where P is an $SO(3)$ bundle. It makes more sense, and E is overused anyway. ♣

To state this phase properly recall that in the UV we have an $SO(3)$ bundle P with 't Hooft flux $\xi \in H^2(X, \mathbb{Z}/2\mathbb{Z})$. We choose an integral lift $\overline{\xi}$ of the characteristic class ξ of the $SO(3)$ bundle P, (we assume such a lift exists), embed it in $H^2(X, \mathbb{R})$ and divide its image in $H^2(X; \mathbb{R})$ by two to define

$$2\lambda_0 = \overline{\xi} \tag{5.67}$$

Then the line bundles which arise in the low energy abelian gauge theory have "first Chern class" in the torsor [29]

$$\lambda \in \Gamma_\xi := \lambda_0 + \bar{H}^2(X) \tag{5.68}$$

One way to think about λ_0 is the following: When we have an $SU(2)$ gauge bundle P there is an associated rank two complex vector bundle Q using the spinor representation. After spontaneous symmetry breaking the gauge symmetry is broken to $U(1)$ and $Q \cong L \oplus L^{-1}$, where L is a line bundle. The sum over instanton sectors becomes a sum over the topological classes of the line bundles L. Now, when P is an $SO(3)$ bundle with nontrivial $w_2(P)$, (i.e. nontrivial 't Hooft flux) and hence does not lift to an $SU(2)$ bundle, then Q and L do not exist. The obstruction to their existence can be seen in the ± 1's that

[29] Again, we are being sloppy about torsion here.

spoil the cocycle conditions on triple overlaps of charts. These obstructions disappear if we consider the symmetric square

$$\mathrm{Sym}^2(L \oplus L^{-1}) = L^2 \oplus \mathcal{O} \oplus L^{-2} \tag{5.69}$$

and we are writing $2\lambda = c_1(L^2)$ so that

$$c_1(L^2) = 2\lambda = \bar{\xi} \mathrm{mod} 2\bar{H}^2(X) \tag{5.70}$$

When evaluating Z_{Coulomb} we need to sum over such line bundles, so the path integral is a sum over a torsor for $\bar{H}^2(X)$.

In [56] Witten argued that there is a relative phase of the path integral measure for the massive modes which are integrated out to produce the LEET. The only problem is in orienting the measure for the charged fermions. Remember that $\psi, \bar{\psi}$ are (locally) valued in $S^{\pm} \otimes S_R \otimes \mathfrak{g}$ and those in $\mathfrak{g} - \mathfrak{t}$ are charged under the unbroken gauge symmetry with Lie algebra $\mathfrak{t}$.

One can try to choose a standard orientation of the fermion measure in the UV theory by using, say, an almost complex structure [30] After the reduction of structure group, for each choice λ there is a different canonical orientation of the path integral measure for the same set of fermions. Witten argues the latter is positive and computes the relative orientation using the index of a Dirac operator. The net result is that the λ-dependence of integrating out the massive fermionic modes has a relative sign

$$(-1)^{w_2(X)\cdot(\lambda_1 - \lambda_2)} \tag{5.71}$$

between sectors λ_1 and λ_2. This still leaves an overall phase to fit. The evaluation of Z_{DW} below is real (for Λ real) if we take

$$e^{\pm 2\pi \mathrm{i} \lambda_0^2}(-1)^{(\lambda-\lambda_0)\cdot w_2(X)} \tag{5.72}$$

The choice of sign in the first exponential corresponds to a choice of orientation of the moduli space $\mathcal{M}(P, g)$. We will choose the $+$ sign:

$$e^{2\pi \mathrm{i} \lambda_0^2}(-1)^{(\lambda-\lambda_0)\cdot w_2(X)} \tag{5.73}$$

Finally, for later reference, note that if we shift the origin:

$$\lambda_0 \to \lambda_0 + \beta \tag{5.74}$$

thus shifting the lift $\bar{\xi} \to \bar{\xi} + 2\beta$ then the overall phase shifts by

$$(-1)^{\beta\cdot w_2(X)} = (-1)^{\beta^2} \tag{5.75}$$

since $w_2(X)$ is a characteristic vector.

Remark:

1. The above discussion is closely related to the discussion in Donaldson-Kronhmeimer [8], section 7.1.6. See especially the final paragraph on p.283.
2. There remains the issue of orienting the measure for the massless fermions in the LEET. This orientation will be λ-independent, and is deferred to section 7.1

♣25 minutes from section 5.7 . ♣

[30] An almost complex structure exists iff there is a $c \in \bar{H}^2(X)$ with $c^2 = 2\chi + 3\sigma$. We will see that DW and SW theory are empty if this condition is not satisfied.

5.7.2 Gravitational Couplings On The Higgs Branch

There is a similar story for the gravitational couplings on the Higgs branch:

The action (5.55) upon topological twisting is of the form

$$Q(\Psi) + \int_X c(u) F_d \wedge F_d \tag{5.76}$$

where $F_d = F(A_d)$ is the curvature of the *dual photon.* The gravitational couplings will be of the form:

$$\Delta_{\text{grv}} S_{Higgs} = \int_X e_h(u) \text{Tr} R \wedge R^* + p_h(u) \text{Tr} R \wedge R + \frac{i}{4} F_d \bar{\xi} \tag{5.77}$$

Of these four terms the last can be derived from the analogous term in (5.59). This follows from the electric-magnetic duality transformation that is necessary to go from the duality frame appropriate to the u-plane away from $u = \pm\Lambda^2$ to the duality frame appropriate to the monopole and dyon points. The detailed derivation is best postponed until we have written the u-plane integrand but the upshot will be that we have photon quantizations and couplings given by

Away from $u = \pm\Lambda^2$:

$$[F] = 4\pi\lambda \qquad \text{with} \qquad \lambda \in \frac{1}{2}\bar{\xi} + \bar{H}^2(X) \tag{5.78}$$

and interaction:

$$e^{\int_X \frac{\mathrm{i}}{4} F \wedge \overline{w_2(X)}} = e^{\mathrm{i}\pi\lambda \cdot \overline{w_2(X)}}. \tag{5.79}$$

Near $u = \pm\Lambda^2$:

$$[F_d] = 4\pi\lambda \qquad \text{with} \qquad \lambda \in \frac{1}{2}\overline{w_2(X)} + \bar{H}^2(X) \tag{5.80}$$

and interaction:

$$e^{\int_X \frac{\mathrm{i}}{4} F_d \wedge \bar{\xi}} = e^{\mathrm{i}\pi\lambda \cdot \bar{\xi}} \tag{5.81}$$

Note, especially, that in equation (5.80) the class 2λ is the characteristic class of a Spin-c structure.

The nature of the other terms is very different. These exponentiate to

$$e^{2\pi i(\lambda_0^2 + \lambda \cdot \lambda_0)} C(u)^{\lambda^2} P_h(u)^{\sigma} E_h(u)^{\chi} \tag{5.82}$$

where now λ is a spin-c structure and $2\lambda_0$ is the integral lift of $w_2(P)$ we used before.

A key point is that the three functions C, P, E are thus far undetermined and not easily derived from first principles.

However, the functions C, P, E are universal - in the sense that they are independent of the 4-fold X - and this, together with the wall-crossing phenomenon of the u-plane integral will allow us to determine them explicitly.

♣Actually, $C(u)$ should easily be derivable from integrating out the monopole field and matching to the Coulomb branch action! ♣

♣6 minutes from section 5.7.2. ♣

5.8 Mapping Operators From UV To IR

Now we must understand how to express the operators $\mathcal{O}$ and $\mathcal{O}(\Sigma)$ in the low energy effective theory.

The secret is to understand how the 0-operator maps under the RG and then to realize that the K operator of (4.36) is RG invariant: The supersymmetry operators do not evolve with scale.

5.8.1 Mapping Operators On The Coulomb Branch

The operator $\mathcal{O}$ is the same as $2u$, by definition. This is true both at low and high energy. The expression for u in terms of fields will be very different in the UV and IR theories.

In any case, in the IR theory we obtain $\mathcal{O}^{(1)}$ by acting with K on $2u$ using the fields and supersymmetry transformation laws in the low energy effective abelian theory. Then using standard supersymmetry transformations one finds:

$$\begin{aligned} Ka &\sim \psi \\ K\psi &\sim (F^- + D) \end{aligned} \tag{5.83}$$

and so forth. (Recall that D is a self-dual 2-form.)

Thus in the low energy theory $\mathcal{O}^{(1)} = Ku \sim \frac{\partial u}{\partial a}\psi$, and acting with K again we get

$$\begin{aligned} \mathcal{O}_{IR,c} &= 2u \\ \mathcal{O}_{IR,c}(\Sigma) &\sim \int_\Sigma \frac{\partial u}{\partial a}(F^- + D) + \frac{\partial^2 u}{\partial a^2}\psi^2 \end{aligned} \tag{5.84}$$

5.8.2 Mapping Operators On The Higgs Branch

Similarly, on the Higgs branch there is only one 0-operator with the right ghost charge, and it is a_d. The operator $\mathcal{O} = u$ is a known function of a_d, expressed in terms of modular functions. Therefore, by exactly the same strategy as we used on the Coulomb branch, we find the low energy operators

$$\begin{aligned} \mathcal{O}_{IR,h} &= 2u \\ \mathcal{O}_{IR,h}(\Sigma) &\sim \int_\Sigma \frac{du}{da_d}F(A_d) + \frac{d^2 u}{da_d^2}\psi^2 \end{aligned} \tag{5.85}$$

where A_d is the $U(1)$ gauge field in the duality frame in which the monopole is purely electrically charged.

5.8.3 Contact Terms

In evaluating correlation functions of the operators $\mathcal{O}(\Sigma)$ in the low energy effective theory there is an important subtlety. When Σ has self-intersections there will be singularities even in the topological field theory which must be accounted for. One is looking at contractions of the operators in a path integral:

$$\left\langle \cdots \int_{\Sigma_1} \mathrm{Tr}\left(\phi(x_1)F_{\mu\nu} + \cdots\right) dx_1^\mu dx_1^\nu \int_{\Sigma_2} \mathrm{Tr}\left(\phi(x_2)F_{\lambda\rho} + \cdots\right) dx_2^\mu dx_2^\nu \cdots \right\rangle \tag{5.86}$$

Because of topological invariance the contributions from contractions of fields with $x_1 \neq x_2$ will combine into total derivatives, and the result will only depend on the topology of Σ_1 and Σ_2. However, there will be contributions from these total derivatives at the intersection points of Σ_1 and Σ_2 where $x_1 = x_2$. Again, invoking topological invariance, one works with off-shell susy this must be a Q-closed operator associated with a point, and hence is just a holomorphic function of u. the result must be expressible in terms of some local Q-invariant operator at $\wp = x_1 = x_2$.

FIGURE: TWO INTERSECTING SURFACES

Put differently, the RG map $\mathcal{O}_{UV} \to \mathcal{O}_{IR}$ is <u>not</u> a ring homomorphism on surface operators. Rather

$$\mathcal{O}_{UV}(\Sigma_1)\mathcal{O}_{UV}(\Sigma_2) \to \mathcal{O}_{UV}(\Sigma_1)\mathcal{O}_{UV}(\Sigma_2) + \Sigma_1 \cdot \Sigma_2 T(u) + Q(*) \tag{5.87}$$

for some 0-observable $T(u)$. Note that we have used here that $\mathcal{O}(\wp_1) - \mathcal{O}(\wp_2) = Q(*)$. Since there are no such singularities between 0-observables we have a Wick theorem:

$$\left\langle e^{\mathcal{O}_{UV}(\Sigma)} \right\rangle_{UV} = \left\langle e^{\mathcal{O}_{IR}(\Sigma)+\Sigma^2 T(u)} \right\rangle_{IR} \tag{5.88}$$

The operator $T(u)$ can, in principle, depend on the branch of vacuum and so we have now:

$$\begin{aligned} Z^{\xi}_{DW}(p,\Sigma) &= \langle e^{p\mathcal{O}+\mathcal{O}(\Sigma)}\rangle_{UV} \\ &= \langle e^{p\mathcal{O}_{IR,c}+\mathcal{O}_{IR,c}(\Sigma)+\Sigma^2 T_c(u)}\rangle_{\text{Coulomb}} + \langle e^{p\mathcal{O}_{IR,h}+\mathcal{O}_{IR,h}(\Sigma)+\Sigma^2 T_h(u)}\rangle_{\text{Higgs}} \end{aligned} \tag{5.89}$$

The functions $T_c(u)$ and $T_h(u)$ can be determined by self-consistency arguments, as was done in [38]. A systematic theory of these contact terms was developed by Losev-Nekrasov-Shatashvili [26]. See also [29] for a simple derivation of the result:

♣Perhaps just leave $T(u)$ as another unknown function to be fixed later when we discuss making the u-plane integral well-defined. ♣

$$T_c(u) \sim \frac{\partial^2 \mathcal{F}}{\partial \tau_0^2} \qquad \tau_0 \sim \log \Lambda. \tag{5.90}$$

For pure $SU(2)$ theory this is a certain weight zero almost modular form under the duality group $\Gamma^0(4)$:

$$T_c(u) = -\frac{1}{24}\left(\left(\frac{du}{da}\right)^2 E_2 - 8u\right) \tag{5.91}$$

We also denote $\hat{T}_c$ where we replace E_2 by the standard modular (but nonholomorphic) object

$$\hat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi y} \tag{5.92}$$

where here, and below, we write the real and imaginary parts of τ as $\tau = x + iy$.

♣10 minutes from section 5.8. ♣

6. General Form Of The Higgs Branch Contribution

Let us sketch now how to evaluate $Z_{IR,Higgs}$ in terms of the unknown functions T_h, C, P, E.

As we said, $Z_{IR,Higgs}$ is a sum of two terms at $u = \Lambda^2$ and $u = -\Lambda^2$. The contribution at $u = -\Lambda^2$ is related by the symmetry described in section 5.4.2. It takes $u \to -u$ and $a \to \mathrm{i}a$. Therefore we focus on $u = \Lambda^2$. This term can be written as:

$$Z_{IR,Higgs,\Lambda^2} = \sum_{\lambda \in \Gamma_w} \left\langle e^{2pu + \frac{i}{4\pi}\int_\Sigma \frac{du}{da_d} F(A_d) + \Sigma^2 T_h(u)} \right\rangle_{u=\Lambda^2,\lambda} \tag{6.1}$$

where we sum over the first Chern class λ of spin-c structures as in (4.75). In writing the 2-observable we have simplified things a little by assuming here that $b_1 = 0$ so that we can drop the term involving ψ^2.

Now we evaluate the path integral in a fixed "flux sector" λ. The low energy effective action coupled to the light "monopole hypermultiplets" is in standard MQ form for localizing on the SW equations. Therefore the path integral in a fixed flux sector is:

$$\int_{\mathcal{M}(\lambda)} e^{2\pi i(\lambda_0^2 + \lambda\cdot\lambda_0)} e^{2pu + i\frac{du}{da_d}\Sigma\cdot\lambda + \Sigma^2 T_h(u)} C(u)^{\lambda^2} P(u)^\sigma E(u)^\chi. \tag{6.2}$$

Here $\mathcal{M}(\lambda)$ is the moduli space of solutions to the Seiberg-Witten equations based on spin-c structure λ. It is known to be *smooth, compact, orientable* and of dimension [31]

$$\mathrm{vdim}\mathcal{M}(\lambda) = \frac{(2\lambda)^2 - (2\chi + 3\sigma)}{4} := 2n(\lambda) \qquad\qquad n(\lambda) \in \mathbb{Z} \tag{6.3}$$

The moduli space $\mathcal{M}(\lambda)$ is a submanifold of an infinite-dimensional manifold

$$\left(\mathcal{A} \times \Gamma(S^+ \otimes L)\right)/\mathcal{G} \tag{6.4}$$

where $\mathcal{A}$ is now the space of spin-c connections: It is a torsor for imaginary globally well-defined one-forms on X. The group of gauge transformations is simply $\mathcal{G} = \mathrm{Map}(X, U(1))$. It acts on a spin-c connection $\nabla \to \nabla + u^{-1}du$ and on monopole fields $M \to uM$. In ordinary abelian gauge theory the isomorphism classes of $U(1)$ connections $\mathcal{A}/\mathcal{G}$ is the differential cohomology group $\check{H}^2(X)$. It is an infinite-dimensional abelain group which can be written (noncanonically) as $T^{b_1} \times V \times H^2(X;\mathbb{Z})$ where T^{b_1} is the torus of harmonic one forms modulo those with integer periods, $H^2(X;\mathbb{Z})$ is the first Chern class and V is an infinite-dimensional vector space. In our case we are working with spin-c connections but the space is almost the same: The space of spin-c connections is an affine space modeled on $\Omega^1(X)$ and the quotient space is $T^{b_1} \times V$ with V an infinite-dimensional vector space. Global $U(1)$ gauge transformations do not act on the spin-c connection, but they do act on the spinor field and hence topologically, the space of gauge inequivalent configurations of (∇, M) can be written as $T^{b_1} \times V \times \Gamma(S^+ \otimes L)/U(1)$ where the last factor is the space of spinor sections modulo global gauge transformations. This is an infinite-dimensional Hilbert space modded out by a single $U(1)$ and is therefore a cone on $\mathbb{CP}^\infty$. The tip of the cone only intersects the moduli space $\mathcal{M}(\lambda)$ if there are abelian instantons $F^+ = 0$. Otherwise $\mathbb{CP}^\infty$ has a cohomology ring which is just $\mathbb{Z}[x]$ generated by x of degree two.

[31] The quantity $2\chi + 3\sigma$ shows up very frequently in SW theory. It is the first Pontryagin class of the real bundle Λ^+ of self-dual two-forms. For any complex structure there is a canonical class and $K^2 = 2\chi + 3\sigma$.

All these assertions are proved in the textbooks [17, 48, 42].

In the topological field theory the cohomology class x is - up to normalization, and again invoking the correspondence (3.9) - the field a_d of ghost number 2. [32] The way we should interpret the integral (6.2) is that we expand

$$u = \Lambda^2 - 2i\Lambda a_d + \mathcal{O}(a_d^2) \tag{6.5}$$

This exact expansion is completely known in terms of elliptic functions. Then we define the *Seiberg-Witten invariant* to be:

$$\mathrm{SW}(\lambda) := \int_{\mathcal{M}(\lambda)} a_d^{n(\lambda)} \tag{6.6}$$

This is an integer, and thus we can express the contribution of a spin-c structure λ to $Z_{IR,Higgs,\Lambda^2}$ as

$$\mathrm{SW}(\lambda)\mathrm{Res}_{a_d=0}\frac{da_d}{a_d^{1+n(\lambda)}}\left(e^{2pu+\cdots}C^{\lambda^2}P_h^{\sigma}E_h^{\chi}\right) \tag{6.7}$$

This is as far as we can go without an explicit knowledge of T_h, C, P_h, E_h.

7. The Coulomb Branch Contribution aka The u-Plane integral

Now we will evaluate $Z_{IR,Coul}$.

We have described above all the ingredients that go into doing the u-plane integral.

An important scaling argument [38] shows that when X has $b_2^+ > 0$ then the result is determined by the *tree-level path integral*: We can forget about one-loop determinants and nontrivial Feynman graphs.

Thus, we are immediately left with a finite-dimensional integral

$$\int (dad\bar{a})(d\eta d\chi d\psi)(dAdD)e^{S+\Delta_{\mathrm{grv}}S}e^{2pu+\mathcal{O}_{IR,c}(\Sigma)+\Sigma^2 T_c(u)} \tag{7.1}$$

REMARK ON $b_2^+ = 0$. JUST ONE-LOOP CONTRIBUTES AND COULD BE INTERESTING.

7.1 The Integral Over Fermions

Let us first consider the fermionic integral:

1. The space $H^0(X;\mathbb{R})$ of η zeromodes is one-dimensional.
2. The space $H^1(X;\mathbb{R})$ of ψ zeromodes is b_1-dimensional.
3. The space $H^{2,+}(X;\mathbb{R})$ of χ zeromodes is b_2^+-dimensional.

There are three remarks to make about the fermionic integral:

[32] The cohomology classes associated with the factor T^{b_1} are of course the IR versions of the 1-observables which can be defined on non-simply-connected four-manifolds.

1. Note that a choice of orientation is needed to define the Grassmann integral $d\eta d\psi d\chi$. As with the dependence on the choice of lift $2\lambda_0$ of $w_2(P)$, this orientation question is beautifully mirrored in Donaldson theory where a choice of orientation of $H^0 \oplus H^1 \oplus H^{2,+}$ determines an orientation of the moduli space of instantons $\mathcal{M}$. See [8] Proposition 7.1.39, p. 282. It is reassuring to see these "fine structure details" mirrored in the physical approach.

2. For simplicity we will assume X is simply connected. This means we can drop the ψ-integral, and moreover we can drop the ψ-dependence in the action (5.35) and in the observables. This simplifies the equations a lot. The equations with $b_1 \neq 0$ have been worked out in [38] and [30].

3. Now, a glance at the action (5.35) reveals that the χ zeromode always appears together with the η zeromode in the combination $\eta\chi$. Since there is only a one-dimensional space of η zeromodes it follows that for the u-plane integral to be nonzero we must have $b_2^+(X) = 1$. This might seem discouraging, but we will press on. Note that integrating out η and χ then brings down a factor of

$$\frac{d\bar{\tau}}{d\bar{a}}(D+F)_+ \tag{7.2}$$

A useful remark at this point is to note that when $b_2^+ = 1$, the cohomology space $H^2(X;\mathbb{R})$ is a vector space with a Lorentzian metric. Once we choose an orientation of $H^{2,+}$ there is a unique self-dual class ω so that $\omega \cdot \omega = 1$, so ω lies on the "mass-shell."

FIGURE: HYPERBOLOID

Therefore, when $b_2^+ = 1$ we will adopt the convention that 2-forms can be written as

$$F = \omega F_+ + F^- \tag{7.3}$$

where F_+ is a <u>scalar</u>. This was implicitly used when writing (7.2).

7.2 The Photon Path Integral

The integral over the gauge field is straightforward. As we have discussed, we sum over flux sectors labeled by $\lambda \in \Gamma_\xi$. The result is that we replace

$$F \to 4\pi\lambda \tag{7.4}$$

and get

$$\frac{e^{2\pi i\lambda_0^2}}{\sqrt{\mathrm{Im}\tau}} \sum_{\lambda\in\Gamma_\xi} e^{-i\pi\bar{\tau}\lambda_+^2 - i\pi\tau\lambda_-^2 - i\frac{du}{da}(\Sigma,\lambda_-)}(-1)^{(\lambda-\lambda_0)\cdot w_2(X)}\left[4\pi\lambda_+ + D\right] \tag{7.5}$$

7.3 Final Expression For The u-plane Integral

Finally, we just do the 1-dimensional (because $b_2^+ = 1$) Gaussian integral over the auxiliary field D. The final expression is: $Z_{IR,Coul} = Z_u^\xi(p,\Sigma)$ with

$$Z_u^\xi(p,\Sigma) = \alpha^\chi\beta^\sigma \int da d\bar{a} \frac{d\bar{\tau}}{d\bar{a}} \left(\frac{du}{da}\right)^{\chi/2} \Delta^{\sigma/8} e^{2pu+\Sigma^2 T_c(u)}\Theta \tag{7.6}$$

$$\Theta = \frac{e^{\frac{\Sigma_+^2}{8\pi y}\left(\frac{du}{da}\right)^2}}{\sqrt{y}} e^{2\pi i \lambda_0^2} \sum e^{-i\pi\bar{\tau}\lambda_+^2 - i\pi\tau\lambda_-^2 - i\frac{du}{da}(\Sigma,\lambda_-)} (-1)^{(\lambda-\lambda_0)\cdot w_2(X)} \left[\lambda_+ + \frac{i}{4\pi y}\frac{du}{da}\Sigma_+\right] \tag{7.7}$$

where we define $y = \mathrm{Im}\tau$ and the explicit terms involving y arise from the Gaussian integral on D.

Let us make a number of comments about this result for the u-plane integral

1. The expression has been written in a form valid for the inclusion of hypermultiplets (in the rank one case). That will be useful later.

2. We can rewrite the integral as

$$\int da d\bar{a} (\cdots) = \int_{\mathbb{C}} du d\bar{u} \; |\frac{da}{du}|^2 \; (\cdots) \tag{7.8}$$

However, notice that the integrand makes (extensive!) use of a duality frame. It is not at all obvious that the expression is in fact a well-defined measure on the u-plane, but this can be checked using the transformation properties of the various terms under change of duality frame. In this discussion it is important that Θ is essentially a theta-function and has nice duality transformation properties. Once one has checked the measure is single-valued one must next check that the integral is actually well-defined. This turns out to be subtle and is discussed below.

3. In pure $SU(2)$ theory it is better to write

$$\int da d\bar{a} (\cdots) = \int_{\mathcal{F}} d\tau d\bar{\tau} \; |\frac{da}{d\tau}|^2 \; (\cdots) \tag{7.9}$$

where the integration region $\mathcal{F}$ is isomorphic to the fundamental domain for $\Gamma^0(4)$ on the upper-half-plane:

FIGURE OF FUNDAMENTAL DOMAIN FOR $\Gamma^0(4)$

Then all the factors in the integrand can be written as modular functions of τ, although the relevant expansion in q is different near $\tau = i\infty$, $\tau = 0$ and $\tau = 2$.

4. Near the discriminant locus, and $u = \infty$, various terms in the integrand become singular. One must define the integral with care in these regions. As an example, let us examine in detail the nature of the integrand at $u = \infty$. Our integral can be written

$$Z_u^{\xi}(p, \Sigma) = \int_{\mathcal{F}} d\tau d\bar{\tau} H(\tau) \Theta \tag{7.10}$$

where $H(\tau)$ is a function of τ and not $\bar{\tau}$:

$$H(\tau) = \alpha^{\chi} \beta^{\sigma} \frac{da}{d\tau} \left(\frac{du}{da}\right)^{\chi/2} \Delta^{\sigma/8} e^{2pu + \Sigma^2 T_c(u)} \tag{7.11}$$

Now consider the region $\mathrm{Im}\tau \to \infty$. From expressions such as (5.41) and (5.42) we find the large y behavior

$$\begin{aligned} u &= \frac{1}{8} q^{-1/4} \left(1 + \mathcal{S}(q^{1/2})\right) \\ \frac{da}{du} &= q^{1/8} \left(1 + \mathcal{S}(q^{1/2})\right) \\ \frac{du}{da} &= q^{-1/8} \left(1 + \mathcal{S}(q^{1/2})\right) \\ \frac{da}{d\tau} &= \frac{da}{du}\frac{du}{d\tau} = -\frac{2\pi \mathrm{i}}{32} q^{-1/8} \left(1 + \mathcal{S}(q^{1/2})\right) \\ T_c(u) &= q^{1/4} \left(1 + \mathcal{S}(q^{1/2})\right) \end{aligned} \tag{7.12}$$

The notation here means the following: $\mathcal{S}(q^{1/2})$ is a series in positive powers of $q^{1/2}$ with integral coefficients and leading term of order $q^{1/2}$. The specific series appearing in the four lines above are all different - the notation $\mathcal{S}(q^{1/2})$ simply stands for a generic such series. Recall that $q = e^{2\pi \mathrm{i}\tau} = e^{2\pi \mathrm{i}x - 2\pi y}$ so we have separated out the exponentially growing terms as a function of $y \to \infty$ from the exponentially decaying terms. Both $H(\tau)$ and Θ have singularities at $y \to \infty$.

The net result is that the integral has the form:

$$\int_{\mathcal{F}} \frac{dxdy}{y^{1/2}} \sum_{\lambda} e^{-\mathrm{i}\pi\bar{\tau}\lambda_+^2 - \mathrm{i}\pi\tau\lambda_-^2} (-1)^{(\lambda-\lambda_0)\cdot w_2(X)} \sum_{\mu \in \frac{1}{8}\mathbb{Z}} q^{\mu} (c_\lambda^{(0)}(\mu) + y^{-1} c_\lambda^{(1)}(\mu) + \cdots) \tag{7.13}$$

There are a number of remarks to make about this expression: First although the series in q^μ involves both even and odd powers of $q^{1/8}$, nevertheless, the integrand is invariant under $\tau \to \tau + 4$ after changing $\lambda \to -\lambda$ and using the fact that $\overline{w_2(X)}$ is a characteristic vector. Next, the series is rather formal: Arbitrary positive and negative powers of $q^{1/8}$ appear. If we organize the integrand by polynomials of degree r in S and ℓ in p (with $r, \ell \geq 0$) then again there is an expansion in $1/y$ and the coefficient of $1/y^n$ for all $n \geq 0$ is a modular expression with q-expansion of the form

$$q^{-\frac{3+r+2\ell}{8}} (1 + \mathcal{S}(q^{1/2})) \tag{7.14}$$

(where now the series $\mathcal{S}(q^{1/2})$ has complex coefficients). So an instanton moduli space $\mathcal{M}$ has leading divergence $q^{-(6+\dim \mathcal{M})/16}$ plus higher order terms. Moreover, for a fixed degree r the series in $1/y^n$ terminates at $n = r$. If we fix a power of q^μ as in (7.13) then each $c_\lambda^{(n)}$ is a series in Σ and p with $r + 2\ell \geq -3 - 8\mu$. The leading order term in the $1/y$ expansion is $c_\lambda^{(0)}(\mu) = \lambda_+ c(\mu)$ where $c(\mu)$ is a series in $p/4, \Sigma$ whose coefficients are coefficients of modular forms for $\Gamma^0(4)$ with integral Fourier coefficients.

Now consider the definition of the integral near the cusp at $y \to \infty$. We first integrate over x over the range $0 \leq x \leq$ to produce a function of y. This projects the sum on λ so that only the zeroth Fourier coefficient contributes:

$$2\mu = \lambda^2 \tag{7.15}$$

leaving us with

$$\int_{y_{\rm cutoff}}^{\infty} \frac{dy}{y^{1/2}} \sum_{\lambda} e^{-2\pi y \lambda_+^2} (-1)^{(\lambda-\lambda_0)\cdot w_2(X)} \left(\lambda_+ c(\frac{1}{2}\lambda^2) + y^{-1} c_\lambda^{(1)}(\frac{1}{2}\lambda^2) + \cdots \right) \tag{7.16}$$

Now, the integral is absolutely convergent *so long as there are no terms with* $\lambda_+ = 0$. Physically, solutions with λ_+ correspond to abelian instantons, and we should expect trouble from extra bosonic zeromodes. Moreover, in the series in $1/y$ only the leading term can lead to a divergent integral, even if $\lambda_+ = 0$. This potential divergence is the source of wall-crossing discussed below.

5. While the integral is subtle and complicated, we must stress that the topology of X only enters through the classical cohomology ring, and therefore $Z_{IR,Coul}$ is only a function of the homotopy type of X.

6. Notice that although we are discussing topological field theory the integrand certainly has nontrivial metric dependence since it explicitly uses the projection of λ to its self-dual λ_+ and anti-self-dual λ_- parts. Since we are dealing with topological field theory we might hope that the result of the integral is metric independent. We next turn to a detailed study of this question.

7.4 Metric Dependence: Wall-crossing

The formalism of topological field theory guarantees that the variation of the path integral with respect to the metric will be a total derivative in field space:

$$\frac{\delta}{\delta g_{\mu\nu}} Z = \langle T_{\mu\nu} \rangle = \langle \{ Q, \Lambda_{\mu\nu} \} \rangle \tag{7.17}$$

however, in some situations that total derivative will not be zero. One example is the holomorphic anomaly of BCOV. The u-plane integral is another striking example of this.

The metric dependence enters the u-plane integrand entirely through the projections such as

$$\lambda = \lambda_+ \omega + \lambda_- \tag{7.18}$$

Therefore we can just consider a family $\omega(t)$ along the hyperboloid and take derivatives.

One can work out the explicit total derivative and reduce the variation of Z wrt to the metric to a boundary integral:

$$\frac{d}{dt} Z_u^\xi(p,\Sigma) = -i\alpha^\chi \beta^\sigma \sum_{u*=\pm\Lambda^2,\infty} \lim_{\epsilon\to 0} \oint_{S^1(\epsilon)} du \left(\frac{da}{du}\right)^{1-\frac{1}{2}\chi} \Delta^{\sigma/8} e^{2pu+\Sigma^2 T(u)} \Upsilon \tag{7.19}$$

where Υ is another theta function similar to Θ. Close analysis shows that this is a δ-function (except for $N_f = 4$).

The support of the δ-function is at certain walls of the form

$$W(\lambda) := \{ \omega : \omega \cdot \lambda = 0 \} \tag{7.20}$$

To understand this wall crossing in more detail we refer back to the regularization of the integral in (7.16). Near $u \to \infty$ the gauge coupling $\mathrm{Im}\tau \to \infty$, (in the almost-canonical duality basis), exponentially suppressing all terms in the theta function but one, namely the term associated with a vector λ so that $\lambda_+ \to 0$. From (7.16) we see that the integral behaves like:

$$c(\frac{1}{2}\lambda^2)\int^{\infty}\frac{dy}{y^{1/2}}e^{-2\pi\lambda_+^2 y}\lambda_+ \sim c(\frac{1}{2}\lambda^2)\mathrm{sign}(\lambda_+) \tag{7.21}$$

where $c(\mu)$ are formal series in $\wp$ and Σ and the coefficient of any term are the Fourier coefficients of a modular form for $\Gamma^0(4)$.

Physically, what happens at the $u = \infty$ walls is that the connection can become reducible and there is an abelian instanton, i.e. a connection on the line bundle L with $F^+ = 0$. This leads to an extra bosonic zeromode in the path integral leading to a δ-function divergence.

The walls are located at

$$\text{From } u = \infty \ : \qquad \lambda \in \Gamma_\xi = \frac{1}{2}\bar{w}_2(P) + \bar{H}^2(X) \tag{7.22}$$

$$\text{From } u = \pm\Lambda^2 \ : \qquad \lambda \in \Gamma_w = \frac{1}{2}\bar{w}_2(X) + \bar{H}^2(X) \tag{7.23}$$

The discontinuity across the walls $\Delta_{u_*,\lambda}Z_u^\xi(p,\Sigma)$ can be expressed as a residue of a *holomorphic* object: This is the Fourier coefficient of a modular form.

The walls divide up the forward light-cone into chambers. A correlator $\langle \mathcal{O}^\ell \mathcal{O}(\Sigma)^r \rangle$ for fixed ℓ, r will only change across a finite number of chambers.

FIGURE: CHAMBERS

The metric dependence of any correlator is then piecewise constant. The *wall-crossing formula* across the walls will involve Fourier coefficients of modular forms.

The WCF for the walls coming from $u = \infty$, $\Delta_{\infty,\lambda}Z_u^\xi(p,\Sigma)$ reproduce *precisely* the formula of L. Göttsche for the change of the Donaldson polynomials for $b_1 = 0$ and $b_2^+ = 1$ if we set:

$$\alpha^\chi \beta^\sigma = \frac{2^{(2+3\sigma)/4}}{\pi} \tag{7.24}$$

Now $\chi + \sigma = 4$, but $\sigma = 1 - b_2^-$ can vary, so this completely fixes α, β. (We have also scaled $\Lambda = 1$ in these equations.)

However, we also have a WCF across the walls coming from singularities at $u = \pm\Lambda^2$. Since we have already completely accounted for the change of the Donaldson polynomials from $\Delta_{\infty,\lambda}Z_u^\xi(p,\Sigma)$, these new discontinuities must *not* be discontinuities of the full partition function $Z_{DW}^\xi(p,\Sigma)$.

8. Derivation Of The Relation Between SW And Donaldson Invariants.

Let us recap the situation:

We have (always!)

$$Z_{DW}^\xi(p,\Sigma) = Z_u^\xi(p,\Sigma) + Z_{IR,Higgs} \tag{8.1}$$

When X has $b_2^+ = 1$ we know that $Z_u^\xi(p,\Sigma)$ has discontinuities as a function of $\omega \in H^2(X;\mathbb{R})$ across walls $W(\lambda)$ coming from the singularities $u = \infty$ and $u = \pm\Lambda^2$.

Moreover, $Z_{DW}^\xi(p,\Sigma)$ also has discontinuities, and these are perfectly accounted for by the discontinuities of $Z_u^\xi(p,\Sigma)$ coming from $u = \infty$.

Therefore, across all walls $W(\lambda)$ we must have

$$0 = \Delta_{u=\Lambda^2,\lambda} Z_{IR,Coul} + \Delta_{u=\Lambda^2,\lambda} Z_{IR,Higgs} \tag{8.2}$$

Indeed, mathematically, the SW invariant $\mathrm{SW}(\lambda)$ is known *not* to be an invariant when X has $b_2^+ = 1$ and changes across walls $W(\lambda)$ determined by spin-c structures. The WCF is particularly easy:

$$\mathrm{SW}(\lambda)|_{\omega\cdot\lambda=0^+} - \mathrm{SW}(\lambda)|_{\omega\cdot\lambda=0^-} = (-1)^{1+n(\lambda)} \tag{8.3}$$

Mathematically, at such walls there is a solution of the SW equations with $M = 0$, this is a reducible solution fixed under global $U(1)$ gauge transformations and the moduli space becomes singular. Physically, at these walls since $M = 0$ the Higgs and Coulomb branches can "mix."

Now, we can *compute* $\Delta_{u=\Lambda^2,\lambda} Z_{IR,Coul}$ since we have an explicit expression (7.19) for it and, given the general form (6.7) of the Higgs contribution and the SW WCF (8.3) we can *compute* the unknown couplings $C(u), P(u), E(u)$. For example, we find

$$\begin{aligned} C(u) &= \left(\frac{a_d}{q_d}\right)^{1/2} = 4e^{i\pi/4} + \mathcal{O}(a_d) \\ P(u) &= e^{i\pi/32} 2^{5/4} + \mathcal{O}(a_d) \\ E(u) &= e^{i\pi/8} 2^{3/4} + \mathcal{O}(a_d) \end{aligned} \tag{8.4}$$

where $q_d = e^{2\pi i \tau_d}$. These are completely explicitly known series determined by modular functions.

To summarize: we now have a completely explicit expression for $Z_{DW}^\xi(p,\Sigma)$, expressed in terms of the SW invariants and the classical cohomology ring. It is valid for all simply connected 4-folds with $b_2^+ > 0$, and can be easily generalized to include the non-simply-connected case.

♣51 minutes from section 6. ♣

9. Simple Type And Witten's Conjecture

A key property about the Seiberg-Witten invariants on a 4-fold X is that $\mathcal{M}(\lambda)$ is only nonempty for a *finite* set of λ. This follows from a Weitzenbock-type argument by taking the sum of squares of the Seiberg-Witten equations.

Now, let us define X to be of *Seiberg-Witten simple type* if $\mathcal{M}(\lambda) \neq \emptyset$ only for λ such that $n(\lambda) = 0$. In this case $\mathcal{M}(\lambda)$ is a finite union of oriented points. When evaluating $\mathrm{SW}(\lambda)$ we are literally counting solutions to equations, just as we began our lecture.

For $b_2^+ > 1$ the SW moduli space depends on the metric, but the cobordism type is invariant. The SW invariant $\mathrm{SW}(\lambda)$ is then metric independent. The *Seiberg-Witten basic classes* are the spin-c structures λ for which $\mathrm{SW}(\lambda) \neq 0$.

♣This is not the same as the set of λ for which $\mathcal{M}(\lambda) \neq \emptyset$. Clarify. ♣

It is a strange fact that all known simply connected X with $b_2^+ > 1$ are of Seiberg-Witten simple type, but there is no proof that all such X must be of simple type.

In any case, let us now suppose that X is of SW simple type, and moreover that $b_2^+ > 1$. In that case $Z_{IR,Coul} = 0$ and the integral is given entirely by the contributions at $u = \pm\Lambda^2$. Moreover, these are easily evaluated since $n(\lambda) = 0$. Putting it all together we obtain the key statement of [55], referred to as the "Witten conjecture" in the mathematics literature:

$$
\begin{aligned}
Z_{DW}^{\xi}(p,\Sigma) = 2^{\frac{1}{4}(7\chi+11\sigma)} \Bigg(& e^{\frac{1}{2}\Sigma^2+2p} \sum_{\lambda\in\Gamma_w} \mathrm{SW}(\lambda) e^{2\pi i(\lambda\cdot\lambda_0+\lambda_0^2)} e^{2\Sigma\cdot\lambda} \\
& + i^{\chi_h} e^{-\frac{1}{2}\Sigma^2-2p} \sum_{\lambda\in\Gamma_w} \mathrm{SW}(\lambda) e^{2\pi i(\lambda\cdot\lambda_0+\lambda_0^2)} e^{-i2\Sigma\cdot\lambda} \Bigg)
\end{aligned}
\tag{9.1}
$$

where $\chi_h := (\chi+\sigma)/4$.

Here we have set $\Lambda = 1$. The first sum comes from the monopole point, $u = \Lambda^2$ and the second sum comes from $u = -\Lambda^2$, the dyon point. Note that we have written the physical partition function. The generating function of Kronheimer-Mrowka is a factor of two larger.

Now, in their analysis of Donaldson polynomials Kronheimer and Mrowka introduced the idea of *simple type* - which we will call KM simple type. It says that the partition function Z_{DW} satisfies the simple differential equation:

$$
\left(\frac{\partial^2}{\partial p^2} - 4\right) Z_{DW}^{\xi}(p,\Sigma) = 0 \tag{9.2}
$$

We note that from our general physical expression it is an immediate consequence that for $b_2^+ > 1$, if X is of SW simple type then it is of KM simple type.

KM also introduced a notion of *generalized simple type*. This says that for some r

$$
\left(\frac{\partial^2}{\partial p^2} - 4\right)^r Z_{DW}^{\xi}(p,\Sigma) = 0 \tag{9.3}
$$

Note that we have a physical proof that *all* simply connected 4-folds of $b_2^+ > 1$ are of generalized KM simple type. This is a simple consequence of the fact that there are only a finite number of basic classes. Therefore, we can take

$$
r = 1 + \max_{\lambda} n(\lambda) \tag{9.4}
$$

Remark There is a beautiful interpretation of the localization of Z_{DW} in terms of localization to $N = 1$ vacua. The essential idea is that sometimes (e.g. on a Kahler manifold) one can add a mass term for the fermions breaking $N = 2$ to $N = 1$, but preserving a topological symmetry. See [54]

ADD MORE ABOUT THIS.

♣16 minutes from section 9. ♣

10. Applications Of The Physical Approach: Postdictions

The introduction of the Seiberg-Witten equations into the theory of four-manifolds had immediate and immense impact. They continue to exert a great influence on the subject. Having spent years building up an arsenal of techniques for dealing with the much more difficult nonabelian equations all of Donaldson's theorems were reproven with the SW equations in a matter of *weeks*. Moreover the SW equations led to the resolution of long-standing conjectures (such as the Thom conjecture), and allowed mathematicians to go beyond what had been achieved with Donaldson's theory.

For a lucid and masterful account see the review by S. Donaldson [9]. For popular accounts of what happened see [19, 20].

By and large, having gotten the hint that one could work with the technically much more tractable Seiberg-Witten equations of an abelian gauge theory instead of the non-abelian anti-self-dual equations the mathematicians have *not* really used the physical insights I have just explained. Nevertheless there have been a number of mathematical "postdictions" - that is, illuminating physical interpretations of known mathematical facts, as well as "predictions" - physically motivated new mathematical statements which are sufficiently well-defined that they should be susceptible to rigorous mathematical proof (or counterexample).

We list here a few of the mathematical postdictions and in the next section review briefly some of the predictions.

10.1 SW=GW For Symplectic Manifolds

In some beautiful work C. Taubes shows that the SW invariants on a *symplectic manifold* can be identified with Gromov-Witten invariants counting pseudoholomorphic curves. This could have been predicted by physicists from the physics of superconductivity, since the SW equations are very similar to the equations for the Landau-Ginzburg low energy effective theory of superconductivity. The pseudoholomorphic curves in question can be thought of physically has worldsheets of Abrikosov-Gorkov flux lines. See the beautiful article by Witten [57] for an account of this. Unfortunately, for the physicists, this was a physics *postdiction*, although the history could easily have been otherwise.

10.2 The Blowup Formula

It gives a simple physical derivation of the Fintushel-Stern /Gottsche-Zagier blowup formula as a kind of "operator product expansion" of the 2-observable for the exceptional surface of a blowup. Topologically a blowup is $\hat{X} = X\#\overline{\mathbb{C}P^2}$ and one can show easily from the u-plane that

$$\exp[t\mathcal{O}(E)] = \sum_{k=1}^{\infty} t^k \mathcal{B}_k(\mathcal{O}) \tag{10.1}$$

where $\mathcal{B}_k(\mathcal{O})$ are polynomials. This is a 4d version of the familiar maneuver in 2d CFT of replacing a handle by an infinite sum over local operators. Details are in [38]. A generalization to higher rank is discussed in [10, 24].

FIGURE: COMPARISON WITH FACTORIZATION ON A HANDLE

10.3 Meng-Taubes: SW Invariants And Reidemeister-Milnor Torsion

The derivation of Z_{DW} can be extended to non-simply-connected four-manifolds, and for the pure $SU(2)$ VM theory this was done in [30, 32].

SAY MORE ABOUT WHAT IS IN THIS PAPER.

11. Applications Of The Physical Approach: Predictions

The most notable physics prediction is the one we have stressed above: Z_{DW} can be written, for manifolds of $b_2^+ > 1$ in terms of the SW invariants.

Still, it is interesting to ask if there are further physical predictions for the mathematics of four-manifolds. We briefly mention a few here.

♣ Check out the Meng-Chwan-Tan papers. What did he do? ♣

11.1 New Formulae For Class Numbers

In some good cases one can actually do the u-plane integral explicitly. Most notably, the answer for $\mathbb{C}P^2$ highlights an intriguing relation to class numbers of quadratic imaginary fields and Mock modular forms. Indeed the u-plane integral is closely related to certain kinds of "Θ-lifts" which have appeared in number theory as well as in string perturbation theory. In the latter context they have been used to give conceptual proofs of Borcherds' results on automorphic products. [Cite: HarveyMoore].

11.2 Donaldson Invariants For Other Simple Gauge Groups

As we stressed above the technique sketched above has a relatively straightforward generalization to higher rank invariants. There is an analog of the above formulae for the $SU(N)$ Donaldson invariants [31]. Dissapointingly, it is again completely expressed in terms of the classical cohomology ring and the Seiberg-Witten invariants. Kronhiemer has verified that prediction for some special X's [22].

SAY MUCH MORE. GIVEN ANALOG OF WITTEN CONJECTURE FOR GENERAL G

The relation of the topology of 4-folds to the existence of superconformal fixed points led to some nontrivial new results in topology [33, 34].

11.3 The Geography Problem

To a compact 4-fold we can associate $(\chi, \sigma, t) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2$ where t is the type, telling us whether the intersection form is even or odd.

The geography problem asks which values can occur, and for a given (χ, σ, t) how many examples (i.e. nondiffeomorphic manifolds) are there? For an excellent summary see [50, 12].

Regarding the uniqueness, it is clear we need to put some restrictions to avoid trivialities. For example,

$$n\mathbb{C}P^2 \# m\overline{\mathbb{C}P^2} \tag{11.1}$$

has $\chi = 2 + m + n$ and $\sigma = n - m$, and since $\chi + \sigma = 2(1 + b_2^+ - b_1)$ is even there are essentially no restrictions on χ, σ, except for those from $m \geq 0, n \geq 0$.

Thus we can look at:
1. Complex manifolds.
2. Symplectic manifolds.
3. Irreducible manifolds (This means $X = X_1 \# X_2$ implies X_1 or X_2 is S^4).
It is best to plot the bounds in terms of

$$c := 2\chi + 3\sigma \qquad\qquad \chi_h := \frac{\chi + \sigma}{4} = \frac{1 + b_2^+ - b_1}{2} \tag{11.2}$$

χ_h can be integer or half-integer. If it is integer X admits an almost complex structure. Then $c = c_1(X)^2$. If X is complex χ_h is the holomorphic Euler characteristic.

FIGURE1: SOME KNOWN BOUNDS. Plot minimal surfaces of general type: $2\chi_h - 6 \leq c \leq 9\chi_h$.

11.3.1 Superconformal Singularities

Now it turns out that the physics of *superconformal points* actually has some bearing on the geography problem.

The way this comes about is the following. As we have stressed, Witten's formula has a natural generalization to $SU(2)$ SYM coupled to N_f hypermultiplets with R the fundamental representation. The UV quantum field theory is only well-defined for $N_f \leq 4$, so we restrict to this case. Each hypermultiplet comes with a complex "mass parameter" m_i. (Mathematically, the m_i are parameters in equivariant cohomology, a result of Labastida and Marino [cite]).

Once again, the quantum moduli space of vacua is the complex plane, parametrized by $u \in \mathbb{C}$, but now the curves Σ_u in the Seiberg-Witten family over the u-plane degenerate at $2 + N_f$ points u_j, $j = 1, \ldots, 2 + N_f$. At each of these points a different kind of BPS state becomes massless:

FIGURE OF U-PLANE WITH SEVERAL SINGULARITIES

For X with $b_2^+ > 1$ of SW simple type the partition function becomes:

$$Z_{DW}(p, \Sigma; m_i) = \tilde{\alpha}^{\chi} \tilde{\beta}^{\sigma} \sum_{j=1,\ldots,2+N_f} \kappa_j^{\chi_h} \left(\frac{du}{da}\right)_j^{\chi_h + \sigma} \sum_{\lambda} \mathrm{SW}(\lambda) e^{pu_j + \Sigma^2 T_j - i\left(\frac{du}{da}\right)_j \Sigma\cdot\lambda} \tag{11.3}$$

Here

1. $\tilde{\alpha}$ and $\tilde{\beta}$ are slightly different numerical constants from before. We have put $\Lambda_{N_f} = 1$.
2. κ_j is defined by $u = u_j + \kappa_j q_j + \cdots$, where $q_j = e^{2\pi i \tau_j}$ is the relevant modular parameter near the singularity u_j.

Now, $Z_{DW}(p, \Sigma; m_i)$ is a manifestly finite and well-defined expression for *generic* values of the mass parameters m_i. However, as we vary the mass parameters the points in the discriminant locus u_j will move, and they can even collide. When that collision involves massless particles which are both magnetically and electrically charged [33] there are

[33]technically, non-mutually-local

further singularities. Mathematically, this is familiar in Kodaira's classification of elliptic fibrations.

Now let us focus on $N_f = 1$. For $N_f = 1$ there is a point m_* where two singularities collide at a single point u_*. If we parametrize $m = m_* + z$ and $u = u_* + z + \delta u$ then the Seiberg-Witten curve is, to leading order:

$$y^2 = x^3 + zx + \delta u \tag{11.4}$$

up to numerical coefficients. There are then extra zeroes in κ_j and $\left(\frac{du}{da}\right)_j$. Since $\chi + \sigma$ might well be negative there are potential divergences in $Z_{DW}(p, \Sigma; m)$ as $m \to m_*$.

However, from the physical perspective there cannot be any such divergences when X is a compact manifold.

The reason is that in the IR the only singularities can come from noncompact regions in spacetime or in moduli space. But X is compact, and for $N_f = 1$ there are no such noncompact regions. (For $N_f > 1$ superconformal singularities sometimes can involve noncompact regions.)

Requiring that $Z_{DW}(p, \Sigma; m)$ as $m \to m_*$ turns out to imply nontrivial facts about the SW invariants.

11.4 Superconformal Simple Type And The Generalized Noether Inequality

A close analysis of the potential singularities of $Z_{DW}(p, \Sigma; m)$ shows that the absence of a divergence for $m \to m_*$ is guaranteed by the following mathematical criterion:

Define

$$\mathrm{SW}_X(z) := \sum_\lambda e^{2\pi i \lambda_c \cdot \lambda} \mathrm{SW}(\lambda) e^{z\lambda} \tag{11.5}$$

where we fix an integral lift $2\lambda_c$ of $w_2(X)$ and we regard powers λ^n to be in the dual space of $\mathrm{Sym}^n(\bar{H}^2(X))$. Then

If $\mathrm{SW}_X(z)$ is analytic at $z = 0$ with an of order $\geq \chi_h - c - 3$ then $Z_{DW}(p, \Sigma; m)$ is finite for $m \to m_*$.

We *define* X to be of *superconformal simple type* if $\mathrm{SW}_X(z)$ has a (nonnegative order) zero at $z = 0$ of order $\geq \chi_h - c - 3$. Reference [33] did not quite manage to prove that this is a *necessary* condition that $Z_{DW}(p, \Sigma; m)$ be finite, but it was verified that all available constructions of 4-manifolds satisfy this criterion. Recently [18] have proven that projective varieties are SST.

Pursuing this a little further leads to an interesting lower bound on the number of basic classes. Let B the the number of basic classes (where we count two nonzero classes λ and $-\lambda$ as the same. Then

$$B \geq \left[\frac{\chi_h - c}{2}\right] \tag{11.6}$$

which implies

$$c \geq \chi_h - 2B - 1 \tag{11.7}$$

A classic result of algebraic geometry is that minimal surfaces of general type satisfy

$$c \geq 2\chi_h - 6 \tag{11.8}$$

This is known as the Noether bound, so we refer to (11.7) as the "generalized Noether bound."

This leads to some new lines in the geography problem:

FIGURE: c, χ_h PLANE WITH SOME LINES WHERE THE SW SUM RULES APPLY.

12. Possible Future Directions.

1. There are interesting cases, such as $S^3 \times S^1$, where one-loop terms will contribute. However, as shown in [38], the series stops at one-loop. Recently, beautiful results on the partition function of $N = 2$ theories on $S^3 \times S^1$ have been obtained by Rastelli et. al. It would be interesting to reproduce those using the u-plane integral.

♣WRONG TOPOLOGICAL TWIST? ♣

♣Check this. There might be no specialization of their index which corresponds to the Donaldson-Witten twist. ♣

2. Families of 4-manifolds and $H^*(BDiff)$. Recent progress [CITE:SEIBERG et. al.] on coupling rigid SUSY theories to background supergravity should help.

3. Give expression for the "u-plane integral" for theories of class S. What is the UV equation whose intersection theory we are computing? Can we use the vast new array of superconformal theories to learn new things, perhaps along the lines of the superconformal simple type story?

4. When $\chi_h = \frac{1-b_1+b_2^+}{2}$ is half-integral all the SW invariants vanish. Can physics really be blind to half the world of four-manifolds?

These appendices contain extra material on the mathematics of four-manifolds.

A. Orientation, Spin, Spin^c, $Pin^{\pm}$, And Pin^c Structures On Manifolds

A.1 Reduction Of Structure Group: General Discussion

Given two compact Lie groups G_1, G_2 and a homomorphism $\phi : G_1 \to G_2$ we can define a functor F_ϕ from principal G_1 bundles on M to principal G_2 bundles on M by taking principal G_1 bundle $G_1 \to P \to M$ to $(P \times_{G_1} G_2) \to M$. Recall that $(P \times_{G_1} G_2)$ is the set of pairs $(p, g) \in P \times G_2$ with equivalence relation $(ph, g) = (p, \phi(h)g)$ for $h \in G_1$, and this clearly admits a free right G_2 action.

Definition If $G_2 \to P_2 \to M$ is a principal G_2 bundle, a *reduction to* G_1 *under* $\phi : G_1 \to G_2$ is a principal G_1 bundle $G_1 \to P_1 \to M$ together with an isomorphism ψ such that we have the commutative diagram:

$$\begin{tikzcd} (P_1 \times_{G_1} G_2) \arrow[r, "\psi"] \arrow[dr] & P_2 \arrow[d] \\ & M \end{tikzcd} \tag{A.1}$$

Working through the definitions one can give a description in terms of transitions functions on patch overlaps $\mathcal{U}_{\alpha\beta}$. If $h_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \to G_1$ are the transition functions of P_1 then there is a bundle isomorphism of P_2, as a principal G_2 bundle to a bundle with transition functions $\phi(h_{\alpha\beta}) : \mathcal{U}_{\alpha\beta} \to G_2$.

Examples

1. If $\phi : H \to G$ is the inclusion of a subgroup then given a G-bundle $P \to M$, H acts freely, so we can consider $G/H \to P/H \to M$, a bundle of homogeneous spaces. In this case a section of the bundle P/H gives a reduction of P to an H bundle, which is in fact a subbundle. As a special case, take $H = \{1\}$. This is the familiar fact that a global section of a principal G bundle trivializes the bundle.

2. Take $H = O(n)$, $G = GL(n, \mathbb{R})$. A metric gives a reduction of the frame bundle to the orthonormal frame bundle $\mathcal{B}_O(M)$. Clearly the bundle of orthonormal frames is a subbundle of the frame bundle.

3. Suppose M is a Riemannian manifold, $H = SO(n), G = O(n)$, and ϕ is the inclusion. Then $\mathcal{B}_O(M)/H$ is the orientation double cover. If M is *orientable* then the orientation bundle has a section. Indeed, if M is connected $\mathcal{B}_O(M)$ has two components and there are two sections. A choice of section gives a reduction of the bundle to an $SO(n)$ bundle of oriented frames. The choice of section is the choice of orientation of M.

4. Now suppose ϕ is a covering map, i.e. $\phi : \tilde{G} \to G$ is *surjective* with kernel K. Then a "reduction" of a principal G bundle P to a principal $\tilde{G}$ bundle $\tilde{P}$ is an isomorphism

$$\begin{tikzcd} \tilde{P}/K \arrow[r, "\psi"] \arrow[dr] & P \arrow[d] \\ & M \end{tikzcd} \tag{A.2}$$

Note the word "reduction" in the general definition is misleading since $\tilde{P}$ is really a covering of P. Put differently, a "reduction of structure group" of a principal G bundle P to $\tilde{G}$ is the same thing as a principal $\tilde{G}$ bundle $\tilde{P}$ that covers P:

$$\tilde{P} \to P \to M \tag{A.3}$$

so that the fiber above every point $m \in M$ "looks like" the covering

$$1 \to K \to \tilde{G} \to G \to 1 \tag{A.4}$$

5. A choice of *spin structure* on an oriented manifold is a special case of the previous remark, (A.2) for the case $\phi : Spin(n) \to SO(n)$. In order to classify spin structures we begin by classifying principal $\mathbb{Z}_2$ bundles over $\mathcal{B}_{SO}(M)$ by $z \in H^1(\mathcal{B}_{SO}(M); \mathbb{Z}_2)$. The spin structures are those which restrict to the fibers to the double cover $Spin(n) \to SO(n)$. The double cover $Spin(n) \to SO(n)$ corresponds to $z_s \in H^1(SO(n); \mathbb{Z}_2) \cong \mathbb{Z}_2$. Accordingly, the spin structures are the double covers of $\mathcal{B}_{SO}(M)$ which restrict to the fibers of $\mathcal{B}_{SO}(M) \to M$ to give the class z_s. Note that the difference of two spin structures $z_1 - z_2$ is therefore trivial on the fibers, and hence pulls back from a class in $H^1(M; \mathbb{Z}_2)$. Thus, the spin structures on M form a torsor for $H^1(M; \mathbb{Z}_2)$.

6. We can proceed in this way with other structures. For a manifold M we can speak of $\mathrm{Pin}^c, \mathrm{Spin}^c, \mathrm{Pin}^\pm, \mathrm{Spin}$ structures based on the above concept applied to the homomorphisms

$$\phi : \mathrm{Pin}^c \to O(n) \tag{A.5}$$
$$\phi : \mathrm{Spin}^c \to O(n) \tag{A.6}$$
$$\phi : \mathrm{Pin}^\pm \to O(n) \tag{A.7}$$
$$\phi : \mathrm{Spin} \to O(n) \tag{A.8}$$
$$\tag{A.9}$$

Note that these homomorphisms are in general neither injective nor surjective. For a BLOTZ structure on an *oriented* manifold we apply the analogous homomorphisms to $SO(n)$ for the bundle of oriented frames.

A.2 Obstructions To Spin And Pin Structures

It is worthwhile translating the above somewhat abstract description into the language of transition functions for the tangent bundle of a manifold X. Let $\{\mathcal{U}_{\alpha\beta}\}$ be a coordinate atlas for X. Using the metric we can form orthonormal frames and these will have transition functions

$$g_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \to O(n) \tag{A.10}$$

if $\dim_{\mathbb{R}} X = n$. If we can modify these with cocycles

$$\tilde{g}_{\alpha\beta} = h_\alpha g_{\alpha\beta} h_\beta^{-1} \tag{A.11}$$

with $h_\alpha : \mathcal{U}_\alpha \to O(n)$ so that $\tilde{g}_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \to SO(n)$ then the manifold is orientable. The only obstruction to orientability is provided by a Cech 2-cochain $\det g_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \to \mathbb{Z}_2$ which defines a cohomology class $w_1(X) \in H^1(X;\mathbb{Z}_2)$. Note that if X is simply connected then $H^1(X;\mathbb{Z}_2) = 0$ and hence it must be orientable.

When X is orientable we can take $g_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \to SO(n)$. If we choose a *good cover*, meaning that all the intersections $\mathcal{U}_{\alpha\beta\cdots}$ are contractible then, on each overlap $\mathcal{U}_{\alpha\beta}$ we can choose lifts $\tilde{g}_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \to Spin(n)$. The only problem is that the cocycle condition for these lifts might fail. Because we have chosen lifts and the kernel of the covering $Spin(n) \to SO(n)$ is just the group $\{\pm 1\}$ we know for sure that on $\mathcal{U}_{\alpha\beta\gamma}$

$$\tilde{g}_{\alpha\beta}(x)\tilde{g}_{\beta\gamma}(x)\tilde{g}_{\gamma\alpha}(x) := \xi_{\alpha\beta\gamma} \in \{\pm 1\} \subset Spin(n) \qquad \forall x \in \mathcal{U}_{\alpha\beta\gamma} \tag{A.12}$$

The signs $\xi_{\alpha\beta\gamma}$ define a Cech 3-cocycle and this defines a cohomology class in $H^2(X;\mathbb{Z}_2)$. It is one (very concrete) definition of w_2. Note that any modification of the lifts $g_{\alpha\beta}$ by a cocycle, or different choice of lift $\tilde{g}_{\alpha\beta}$ only changes $\{\xi_{\alpha\beta\gamma}\}$ by a coboundary. Almost by definition, this is the only obstruction to the existence of a spin structure.

1. Example: Two spin structures on the circle and relation to the two double covers of the circle. EXPLAIN.

2. An example: $BSO(2) = BU(1) = \mathbb{C}P^\infty$. w_2 is the reduction mod two of c_1, which is the cochain dual to the 2-cell. So explain why c_1 mod two is an obstruction to the spin structure. On $\mathbb{C}P^2$ $c_1 = 3x$. The complexified tangent bundle admits a reduction of the $SO(4) = SU(2) \times SU(2)/\mathbb{Z}_2$ structure group to $U(2) = SU(2) \times U(1)/\mathbb{Z}_2$. Restricting to a $\mathbb{C}P^1 \subset \mathbb{C}P^2$ the tangent bundle splits as $\mathcal{O}(2) \oplus \mathcal{O}(1)$ where $\mathcal{O}(2)$ is the tangent bundle of $\mathbb{C}P^1$ and $\mathcal{O}(1)$ is the normal bundle. The structure group is further reduced to $U(1) \times U(1)$. Clearly, the principal $U(1)$ bundle associated to the normal bundle $\mathcal{O}(1)$ does not admit a two-fold covering restricting to a double covering of $U(1)$ over $U(1)$.

3. The obstruction to a $Spin^c$ structure is W_3, the image of w_2 under the Bockstein map. Therefore, it vanishes when $w_2(X) \in H^2(X;\mathbb{Z}_2)$ has an integral lift. Using the fact that, for all $\sigma \in H_2(X;\mathbb{Z})$

$$\int_\sigma w_2(X) = \sigma \cdot \sigma \bmod 2 \tag{A.13}$$

where $\sigma \cdot \sigma$ is the oriented <u>integral</u> intersection number one can show that indeed such an integral lift exists. See [52] for the details. The analogous statement fails in

♣Also look up proof in Gompf, Stipsicz, 4-Manifolds and Kirby Calculus ♣

the unorientable case: $\mathbb{RP}^2 \times \mathbb{RP}^2$ does not admit a Pin^c structure. Moreover, there are orientable five-dimensional manifolds which are not $Spin^c$. A simple example is the space of symmetric $SU(3)$ matrices, which is diffeomorphic to $SU(3)/SO(3)$. See [28] for an explanation. ♣Give proof. ♣

A.3 $Spin^c$ Structures On Four-Manifolds

The group $\mathrm{Spin}^c(4)$ is defined to be

$$\mathrm{Spin}^c(4) := (\mathrm{Spin}(4) \times U(1))\,/\mathbb{Z}_2 = (SU(2) \times SU(2) \times U(1))\,/\mathbb{Z}_2 \tag{A.14}$$

where we divide by the group $\mathbb{Z}_2$ embedded as $(-1,-1,-1)$. The bundle of oriented ON frames of X, $OrFr(X)$ is a principal $SO(4)$ bundle and a spin-c structure is - by definition - a reduction of structure group to a principal $\mathrm{Spin}^c(4)$ bundle defined by the obvious homomorphism $\mathrm{Spin}^c(4) \to SO(4)$. Working out the definition this means that a spin-c structure is defined by a principal $\mathrm{Spin}^c(4)$ bundle $\mathfrak{P}$ with a projection $\mathfrak{P} \to OrFr(X)$ which along the fibers looks like the exact sequence

$$1 \to U(1) \to \mathrm{Spin}^c(4) \to SO(4) \to 1 \tag{A.15}$$

A spin-c structure exists when $w_2(X)$ has an integral lift. As mentioned above, this is indeed true for every compact orientable four-manifold.

The group homomorphism $\mathrm{Spin}^c(4) \to U(2) \times U(2)$ given by $[(v_L, v_R, \zeta)] \mapsto (\zeta v_L, \zeta v_R)$ defines an isomorphism

$$\mathrm{Spin}^c(4) \cong \{(u_L, u_R) |\det u_L = \det u_R\} \subset U(2) \times U(2) \tag{A.16}$$

and the latter presentation makes it obvious that there are two inequivalent rank 2 representations $W^{\pm}$ of $\mathrm{Spin}^c(4)$ simply defined by the fundamental representations of each of the two $U(2)$ factors. Given a spin-c structure there are therefore two associated complex rank two bundles $W^{\pm} \to X$ and we can identify

$$W^{\pm} = S^{\pm} \otimes L \tag{A.17}$$

in the discussion of section 4.7.2 above. Conversely such a pair of bundles $W^{\pm}$ defines a spin-c structure. Consequently the space of spin-c structures is a torsor for the group of line bundles, since give a line bundle $\mathcal{L}$ we can always take

$$W^{\pm} \to W^{\pm} \otimes \mathcal{L} \tag{A.18}$$

so that

$$c_1(\det W^{\pm}) \to c_1(\det W^{\pm}) + 2c_1(\mathcal{L}) \tag{A.19}$$

Note that given an almost complex structure on X there is a canonical spin-c structure

$$W^+ = \Omega^{0,0}(X) \oplus \Omega^{0,2}(X) \qquad\qquad W^- = \Omega^{0,1}(X) \tag{A.20}$$

A.4 't Hooft Flux

At several points in the notes we used the second Stiefel-Whitney class $w_2(P)$ where P is a principal $SO(3)$ bundle. In general, an $SO(n)$ bundle P over a manifold M has a characteristic class $w_2(P) \in H^2(M;\mathbb{Z}_2)$. One way to define it follows the discussion of spin structures above: Choose a good cover $\{\mathcal{U}_\alpha\}$ of M and a trivialization of P on this cover. On patch overlaps $\mathcal{U}_{\alpha\beta}$ choose lifts $\tilde{g}_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \to Spin(n)$ and measure the failure of the cocycle condition on $\mathcal{U}_{\alpha\beta\gamma}$ to define a class in $H^2(M;\mathbb{Z}_2)$. If $w_2(P)$ is nonzero then there is no reduction of structure group of P from $SO(n)$ to $Spin(n)$.

A simple example of an $SO(n)$ bundle that does not lift to a $Spin(n)$ bundle is obtained by considering M to be a two-dimensional compact surface. Choose a point $p \in M$ and a small disk around p. Define an $SO(n)$ bundle by taking the transition function around the boundary of the disk to define a nontrivial closed loop in $SO(n)$.

B. The 11/8 Conjecture

Even unimodular forms must be of the type $Q = mE_8 \oplus nH$, where m is an integer of either sign. Note that $E_8 \oplus -E_8 \cong 8H$.

Rokhlin's theorem: If X is smooth and $w_2(X) = 0$ then $\sigma(X) = 0 \mathrm{mod} 16$. Equivalently, if X is smooth and Q_X is even then $\sigma(X) = 0 \mathrm{mod} 16$.

Therefore, for a smooth manifold with even indefinite form $Q = 2mE_8 \oplus nH$. From Donaldson we know that $n > 0$.

Connected sums with $S^2 \times S^2$ increases H, so the interesting question is: What is the *minimal* number of H's needed for Q to be Q_X for a *smooth* 4-fold X?

Note that

$$X = K3^{\#\ell} \# (S^2 \times S^2)^{\# t} \qquad \Rightarrow \qquad Q_X = -2\ell E_8 \oplus (3\ell + t) H \tag{B.1}$$

so in this case $m = -\ell$ and $n = 3\ell + t$ with $\ell, t \geq 0$. Therefore there exist smooth X with

$$n \geq 3|m| \tag{B.2}$$

and the bound is saturated by taking $t = 0$.

The so-called "11/8 conjecture" says that $K3^{\#\ell}$ is the optimal four manifold for this inequality:

Conjecture: Every smooth X with even intersection form must have $n \geq 3|m|$, or, equivalently

$$b_2(X) \geq \frac{11}{8} |\sigma(X)| \tag{B.3}$$

Note that for $Q = mE_8 + nH$, where WLOG we can take $n \geq 0$, we have

$$\begin{aligned} b_2(Q) &= 16|m| + 2n \\ |\sigma(Q)| &= 16|m| \end{aligned} \tag{B.4}$$

so $n \geq 3|m|$ iff

$$b_2(Q) \geq 22|m| = \frac{11}{8} |\sigma(Q)| \tag{B.5}$$

A stronger conjecture is the 3/2 conjecture, which applies to *irreducible* X. This is a manifold that does not split as a connected sum of two 4-folds one of which is not S^4.

Conjecture: Let X be a smooth irreducible 4-fold with even intersection form. Then

$$n \geq 4|m| - 1 \tag{B.6}$$

(Again note that $K3$ with $m = -1$ saturates the inequality.) Equivalently

$$b_2(X) \geq 16|m| + 8|m| - 2 = 24|m| - 2 = \frac{3}{2}|\sigma(X)| - 2 > \frac{3}{2}|\sigma(X)| \tag{B.7}$$

Some known results in this direction:

1. Theorem[Donaldson] The forms H and $H \oplus H$ are the only even forms that can appear for smooth X with $b_2^+ = 1$ or $b_2^+ = 2$. (So, if there is an E_8 then $n \geq 3$.)

2. Theorem[Furuta] For X smooth with even intersection form $n \geq 2|m| + 1$, that is

$$b_2(X) \geq \frac{10}{8}|\sigma(X)| + 2 \tag{B.8}$$

Furuta et. al. also show that $\pm 4E_8 \oplus 5H$ cannot be the intersection form of any smooth X. So the first unkown case allowed by Furuta's theorem, but violating the 11/8 conjecture is $|m| = 3$ and $n = 7, 8$.

The method of Furuta is to exploit a Pin(2) invariance of the SW equations....

C. Theorems On SW Moduli Space

CITE REFERENCES! AT LEAST write the key Weitzenbock identity.

1. $b_2^+ \geq 1$, generic metric: $\mathcal{M}(\lambda)$ is either empty or a smooth manifold of the expected dimension.

2. $b_2^+ \geq 2$: For generic paths g_t we get a smooth cobordism between smooth manifolds.

3. $\mathcal{M}(\lambda)$ is only nonempty for a finite number of spin-c structures.

4. $\mathcal{M}(\lambda)$ when nonempty are compact.

5. $\mathcal{M}(\lambda)$ are orientable, depending on an orientation on $H^1 \oplus H^{2,+}$ (as in Donaldson theory).

6. Simple type conjecture: For any simply connected X with $b_2^+ \geq 2$, if $\mathcal{M}(\lambda)$ is nonempty then it is zero-dimensional.

7. Simple type conjecture is true for symplectic manifolds.

8. There are examples of $b_2^+ = 1$ and non-simply connected X showing that $\mathcal{M}(\lambda)$ can have arbitrarily high dimension.

D. Constraints On π_1 Of Three-Manifolds

Any orientable three-manifold admits a Heegard decomposition. This means we can find an embedded closed oriented surface $\Sigma \subset Y$ so that Y is the gluing of two handlebodies (bordisms of Σ to the emptyset) using a diffeomorphism of Σ. Such a Heegard decomposition strongly constraints the fundamental group, thanks to the Seifert-van-Kampen theorem: A set of generators is given by a set of generators of the two handlebodies. The fundamental group of the handlebody for Σ of genus g is just a free group on g generators. (The "A-cycles" contract to a point, leaving the "B-cycles" with no relation.) After gluing to the other handlebody there will be g relations expressing the contractibility of the new "A-cycles" of the second handlebody.

Thus, for any orientable three-manifold Y, $\pi_1(Y, y_0)$ admits a presentation where the number of generators is equal to the number of relations. The existence of such a presentation is certainly not possible for a general finitely generated group.

If the universal cover $\tilde{Y}$ of Y is a rational homology sphere we can same more. [34] (From the Poincaré conjecture we know it is S^3, but we don't actually need that for the following argument.) So, π_1 must act continuously on S^3.

Now consider the Gysin sequence of $EG \times_{\pi_1} S^3$. On the one hand, this is homotopy equivalent to S^3/π_1 which cannot have cohomology above degree three. On the other hand, it is a sphere bundle over BG. Now consider the Gysin sequence for the pair (D, S) where S is the above sphere bundle over BG and D is the corresponding disk bundle. (Extend the action by π_1 into the disk.)

Of course, the disk bundle contracts to BG and $H^*(D, S) \cong H^{*+4}(BG)$ by Thom isomorphism, so in the Gysin sequence we get $H^k(BG) \to H^{k+4}(BG)$ which is sandwhiched between zeroes for $k > 3$ since $H^k(S) \cong H^k(S^3/\pi_1)$. In fact, $H^k(BG) \to H^{k+4}(BG)$ is multiplication by the Euler class. This shows that above some low degree the cohomology of BG must be periodic. But already for sufficiently many products of $\mathbb{Z}_2$'s we see that the cohomology is not 4-fold periodic. Therefore, such groups cannot arise as π_1 of any 3-fold.

References

[1] M. Atiyah, N. Hitchin, and I. Singer, "Self-Duality in Four-Dimensional Riemannian Geometry," Proc. Royal Soc. (London) A 362 (1978) 425-461

[2] M. Atiyah, "New Invariants Of 3- And 4- Dimensional Manifolds," in *The Mathematical Heritage of Hermann Weyl*, Proc. Symp. Pure Math. **48** AMS 1988

[3] M. F. Atiyah and L. Jeffrey, "Topological Lagrangians and cohomology," J. Geom. Phys. **7**, 119 (1990).

[4] S. Cordes, G. W. Moore and S. Ramgoolam, "Lectures on 2-d Yang-Mills theory, equivariant cohomology and topological field theories," Nucl. Phys. Proc. Suppl. **41**, 184 (1995) [hep-th/9411210].

[5] A. Daemi and Y. Xie, "Sutured Manifolds and Polynomial Invariants from Higher Rank Bundles," arXiv:1701.00571 [math.GT]

[6] P. Deligne and J.W. Morgan, *Notes on Supersymmetry (following Joseph Bernstein* in *Quantum Fields and Strings: A Course for Mathematicians*, Vol. 1, pp.41-96 AMS

[7] S.K. Donaldson, "Yang-Mills Invariants of Four-Manifolds," in *Geometry of Low-Dimensional Manifolds: Gauge theory and algebraic surfaces*, S.K. Donaldson and C.B. Thomas eds., Lond Math. Soc. Lecture Note Series 150.

[34] This was explained to me by Graeme Segal.

[8] S. Donaldson and P. Kronheimer, *The Geometry of Four-Manifolds*, Oxford 1990.

[9] S. Donaldson, Bull. Amer. Math. Soc. Vol. 33 (1996) p. 45

[10] J. D. Edelstein, M. Gomez-Reino and M. Marino, "Blowup formulae in Donaldson-Witten theory and integrable hierarchies," Adv. Theor. Math. Phys. **4**, 503 (2000) [hep-th/0006113].

[11] Paul M. N. Feehan, Peter B. Kronheimer, Thomas G. Leness, Tomasz S. Mrowka, "PU(2) monopoles and a conjecture of Marino, Moore, and Peradze," arXiv:math/9812125 [math.DG]

[12] Six lectures on four manifolds

[13] M.H. Freedman, A. Kitaev, C. Nayak, J.K. Slingerland, K. Walker, Z. Wang, "Universal manifold pairings and positivity," arXiv:math/0503054 [math.GT]

[14] P. Feehan and T. Leness, SERIES OF PAPERS.

[15] D.S. Freed and K.K. Uhlenbeck, *Instantons and Four-Manfiolds* MSRI Publications 1984

[16] R. Friedman and J. Morgan, *Smooth Four-Manifolds and Complex Surfaces*, Springer 1994

[17] Gompf and Stipsicz

[18] L. Göttsche, H. Nakajima, and K. Yoshioka, "Donaldson=Seiberg-Witten From Mochizuki's Formula and Instanton Counting," arXiv:1001.5024.

[19] Allyn Jackson, "A Revolution in Mathematics," Notices AMS 1995 ??

[20] D. Kotschick, "Gauge Theory is Dead! - Long Live Gauge Theory!" Notices AMS ???

[21] P. B. Kronheimer and T. S. Mrowka," Embedded surfaces and the structure of Donaldsons polynomial invariants," J. Differential Geom. 41 (1995), no. 3, 573734.

[22] P. Kronheimer, "Four-manifold invariants from higher-rank bundles," arXiv.math/0407518

[23] P.B. Kronheimer, "Some non-trivial families of symplectic structures," On his homepage.

[24] J. Labastida and M. Marino, "Topological quantum field theory and four manifolds," (Mathematical physics studies. 25)

[25] D. Leites, "Introduction to supermanifolds," 1980 Russ. Math. Surv. 35 1.

[26] A. Losev, N. Nekrasov and S. L. Shatashvili, "Issues in topological gauge theory," Nucl. Phys. B **534**, 549 (1998) [hep-th/9711108].

[27] T.-J. Li and Ai-Ko Liu, "Family Seiberg-Witten invariants and wall crossing formulas," arXiv:math/0107211

[28] J. M. Maldacena, G. W. Moore and N. Seiberg, "D-brane instantons and K theory charges," JHEP **0111**, 062 (2001) doi:10.1088/1126-6708/2001/11/062 [hep-th/0108100].

[29] M. Marino and G. W. Moore, "Integrating over the Coulomb branch in N=2 gauge theory," Nucl. Phys. Proc. Suppl. **68**, 336 (1998) [hep-th/9712062].

[30] M. Marino and G. W. Moore, "Donaldson invariants for nonsimply connected manifolds," Commun. Math. Phys. **203**, 249 (1999) [hep-th/9804104].

[31] M. Marino and G. W. Moore, "The Donaldson-Witten function for gauge groups of rank larger than one," Commun. Math. Phys. **199**, 25 (1998) [hep-th/9802185].

[32] M. Marino and G. W. Moore, "Three manifold topology and the Donaldson-Witten partition function," Nucl. Phys. B **547**, 569 (1999) [hep-th/9811214].

[33] M. Marino, G. W. Moore and G. Peradze, "Superconformal invariance and the geography of four manifolds," Commun. Math. Phys. **205**, 691 (1999) [hep-th/9812055].

[34] M. Marino, G. W. Moore and G. Peradze, "Four manifold geography and superconformal symmetry," math/9812042.

[35] J. Milnor, "On simply connected 4-manifolds." 1958 Symposium internacional de topologa algebraica International symposium on algebraic topology pp. 122128 Universidad Nacional Autnoma de Mxico and UNESCO, Mexico City

[36] T. Mochizuki, "A theory of the invariants obtained from the moduli stacks of stable objects on a smooth polarized surface," arXiv:math/0210211 [math.AG]

[37] G. W. Moore, "2-D Yang-Mills theory and topological field theory," hep-th/9409044.

[38] G. W. Moore and E. Witten, "Integration over the u plane in Donaldson theory," Adv. Theor. Math. Phys. **1**, 298 (1998) [hep-th/9709193].

[39] G. W. Moore, N. Nekrasov and S. Shatashvili, "Integrating over Higgs branches," Commun. Math. Phys. **209**, 97 (2000) doi:10.1007/PL00005525 [hep-th/9712241].

[40] G. W. Moore, N. Nekrasov and S. Shatashvili, "D particle bound states and generalized instantons," Commun. Math. Phys. **209**, 77 (2000) doi:10.1007/s002200050016 [hep-th/9803265].

[41] G. W. Moore, A. B. Royston and D. Van den Bleeken, "L^2-Kernels Of Dirac-Type Operators On Monopole Moduli Spaces," arXiv:1512.08923 [hep-th].

[42] J.W. Morgan, *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds*, Princeton University Press Mathematical Notes, 1996.

[43] N. A. Nekrasov, "Seiberg-Witten prepotential from instanton counting," Adv. Theor. Math. Phys. **7**, no. 5, 831 (2003) doi:10.4310/ATMP.2003.v7.n5.a4 [hep-th/0206161].

[44] N. Nekrasov and S. Shadchin, "ABCD of instantons," Commun. Math. Phys. **252**, 359 (2004) doi:10.1007/s00220-004-1189-1 [hep-th/0404225].

[45] N. Nekrasov and V. Pestun, "Seiberg-Witten geometry of four dimensional N=2 quiver gauge theories," arXiv:1211.2240 [hep-th].

[46] N. Seiberg and E. Witten, "Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory," Nucl. Phys. B **426**, 19 (1994) [Erratum-ibid. B **430**, 485 (1994)] [hep-th/9407087].

[47] N. Seiberg and E. Witten, "Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD," Nucl. Phys. B **431**, 484 (1994) [hep-th/9408099].

[48] A. Scorpan, *The Wild World of Four-Manifolds*, AMS,...

[49] A. D. Shapere and Y. Tachikawa, "Central charges of N=2 superconformal field theories in four dimensions," JHEP **0809**, 109 (2008) doi:10.1088/1126-6708/2008/09/109 [arXiv:0804.1957 [hep-th]].

[50] Will we every classify 4-folds?

[51] Y. Tachikawa, "N=2 supersymmetric dynamics for pedestrians," Lect. Notes Phys. **890** (2014) doi:10.1007/978-3-319-08822-8 [arXiv:1312.2684 [hep-th]].

[52] P. Teichner and E. Vogt, "ALL 4-MANIFOLDS HAVE SPINc STRUCTURES," http://people.mpim-bonn.mpg.de/teichner/Papers/spin.pdf

[53] E. Witten, "Topological Quantum Field Theory," Commun. Math. Phys. **117**, 353 (1988).

[54] E. Witten, "Supersymmetric Yang-Mills theory on a four manifold," J. Math. Phys. **35**, 5101 (1994) [hep-th/9403195].

[55] E. Witten, "Monopoles and four manifolds," Math. Res. Lett. **1**, 769 (1994) [hep-th/9411102].

[56] E. Witten, "On S duality in Abelian gauge theory," Selecta Math. **1**, 383 (1995) [hep-th/9505186].

[57] E. Witten, "From Superconductors and Four-Manifolds to Weak Interactions," Bull. Amer. Math. Soc. Vol. 44 (2007) 361.

[58] E. Witten, "Notes on Supermanifolds and Integration," arXiv:1209.2199.