2d Categorical Wall-Crossing With Twisted Masses, And An Application To Knot Invariants

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Part of this talk is review of old things

Witten, 1982; Cecotti & Vafa, 1992

Some foundational material: D. Gaiotto, G. Moore, & E. Witten 2015 (GMW)

New things are work with AHSAN KHAN

arXiv:2010.11837
arXiv:21???.????
+ comment resulting from discussions with

With nontrivial input from Andy and Tudor
1. Supersymmetric Quantum Mechanics And Homological algebra

2. 2D N=(2,2) Landau-Ginzburg Models

3. Thimble Branes & Their Local Operators

4. Categorical Wall-Crossing

5. Generalization To Twisted Masses

6. Relation To 3d Indices
SQM & Morse-Novikov Theory (Witten: 1982)

\( M: \) Riemannian; \( \xi: \) Closed 1-form

Locally \( \xi = dh \) with \( h: M \to \mathbb{R} \) the traditional superpotential.

**But \( h \) need not be single-valued: \( \xi \) need not be exact.**

Call \( \xi \) the ``super-one-form''

**SQM:** \( \phi: \mathbb{R}_t \to M \)

\[
L = g_{Ij}(\phi) \dot{\phi}^I \dot{\phi}^J - \| \xi \|^2 + \ldots
\]
SQM & Morse-Novikov Theory (Witten: 1982)

**SQM:** \( \phi: \mathbb{R}_t \rightarrow M \)

\[
L = g_{IJ}(\phi) \dot{\phi}^I \dot{\phi}^J - \| \xi \|^2 + \ldots
\]

Classical vacua: \( \dot{\xi}(\phi_i) = 0 \) \( \text{``critical points''} \)

``Mass matrix'' \( D_I \xi_J \big|_{\phi_i} \) is invertible

Approximate quantum vacua: \( \Psi(\phi_i) \)

Fermion number: \( F(\Psi(\phi_i)) = \frac{1}{2} (d_-(\phi_i) - d_+(\phi_i)) \)
Instantons & MSW Complex

The approximate vacua are not exact because of instanton effects.

Instanton equation:

\[ \frac{d\phi^I}{d\tau} = g^{IJ}(\phi)\xi_j(\phi) \]

Use instantons to define an operator \( Q \) on approximate ground states

\[ Q\Psi(\phi_i) := \sum_{F_j=F_i+1} n_{j,i} \Psi(\phi_j) \]

Broken flows:

\[ \sum_{F_r=F_q+1=F_p+2} n_{pq}n_{qr} = 0 \]

\[ \Rightarrow \quad Q^2 = 0 \]
MSW Chain Complex

\[ \mathcal{C} := (V, F, Q) \]

SQM: \( V \subset \mathcal{H} \) : The span of the approximate ground states \( \Psi(\phi_i) \)

\( F \): Fermion number = \( Q \): Susy operator =
Homological degree Differential

\[ [F, Q] = Q \quad Q^2 = 0 \]

MSW complex: MSW(\( M, g_{IJ}, \xi_I \))

Exact ground states \( \cong H^*(V, Q) \cong H^*(M; d + \xi) \)
Homotopies Of Metric And Super-one-form

Now consider a continuous family: \( (g_{IJ}(\phi; s), \xi_I(\phi; s)) \quad s_1 \leq s \leq s_2 \)

How does the MSW complex change?

Define \( \mathcal{U}: MSW(s_1) \rightarrow MSW(s_2) \)

\[
\mathcal{U} \Psi(p; s_1) = \sum_{F_q = F_p} n_{p,q} \Psi(q; s_2) 
\]

\[
\frac{d\phi^I}{ds} = g^{IJ}(\phi; s)\xi_J(\phi; s)
\]

Claim: \( \mathcal{U} F = F \mathcal{U} \quad \mathcal{U} Q_1 = Q_2 \mathcal{U} \)
Under continuous deformation of metric and super-one-form the MSW complex changes by a chain map. Actually, it is a very special kind of chain map: A homotopy equivalence of chain complexes.
Homotopies Of Paths $\Rightarrow$ Homotopy Of Chain Maps

$$(g_{IJ}(\phi; s, u), \xi_{I}(\phi; s, u)) \quad \text{(Fixed @ s=0,1)}$$

$u = 0$ gives a chain map $\mathcal{U}_0 : (V, F, Q)_1 \rightarrow (V, F, Q)_2$

$u = 1$ gives a chain map $\mathcal{U}_1 : (V, F, Q)_1 \rightarrow (V, F, Q)_2$

$\mathcal{U}_0 - \mathcal{U}_1 = Q_2 \, E + E \, Q_1$

$$\frac{d\phi^l}{ds} = g^{IJ}(\phi; s, u)\xi_{J}(\phi; s, u)$$

$$E \, \Psi(p; s = 0, u = 0) = \sum_{F_{q}=F_{p}-1} n_{p,q} \Psi(q; s = 1, u = 1)$$
Definition: Homotopies Of Chain Maps

Two chain maps \( f_0, f_1 : C_1 \to C_2 \) are **homotopic** if there is a Fermion number -1 map \( E : V_1 \to V_2 \)

\[
f_0 - f_1 = Q_2 E + E Q_1
\]

If there are chain maps...

\[
f : C_1 \to C_2 \quad \quad g : C_2 \to C_1
\]

\[
g \circ f \sim \text{Id}_{C_1} \quad \quad f \circ g \sim \text{Id}_{C_2}
\]

Then the chain complexes \( C_1 \) and \( C_2 \) are **homotopy equivalent**.
Supersymmetric Quantum Mechanics And Homological algebra

2D N=(2,2) Landau-Ginzburg Models

Thimble Branes & Their Local Operators

Categorical Wall-Crossing

Generalization To Twisted Masses

Relation To 3d Indices
Landau-Ginzburg Models $LG(X, \alpha)$

$X, g_{IJ}$: Kähler \hspace{1cm} $\alpha \in \Omega^{1,0}(X)$, $\bar{\partial} \alpha = 0$

Locally $\alpha = \partial W$ with $W: X \to \mathbb{C}$, but we will be considering multi-valued $W$

$$S = \int_{\mathbb{R}} dt \int_{D} dx \left( g_{IJ}(\phi(x, t)) \partial_{\mu} \phi^{I} \partial^{\mu} \phi^{J} - \| \alpha(\phi) \|^2 + \cdots \right)$$

Poincare invariant vacua for $D = \mathbb{R}$:

$$\mathbb{V} = \{ \phi_i \mid \alpha(\phi_i) = 0 \}$$

Branes \hspace{1cm} $D = [x_0, \infty)$
2d LG Model As 1d SQM

Consider SQM with target: $\mathcal{M} = Map(\phi: D \to X)$

$$\| \delta \phi \|^2 = \int_D g_{I\bar{J}}(\phi(x)) \delta \phi^I \delta \phi^{\bar{J}}$$

$\alpha \in \Omega^{1,0}(X)$ induces a super-one-form $\xi$ on $\mathcal{M}$

$$\xi[\phi] = \int_D [\phi^*(\omega) - \text{Re}(\zeta^{-1} \alpha_I(\phi)\delta \phi^I)] dx$$

$\text{SQM}(\mathcal{M}, \xi) = \text{LG}(X, \alpha)$
Superpotential With Twisted Masses

Usual discussion: \( \alpha = \partial W \)
with \( W: X \to \mathbb{C} \) holomorphic and Morse

If \( \alpha \) has nonzero periods there is no single-valued superpotential

``twisted masses``

\( \exists \) Minimal Abelian cover \( \pi: \hat{X} \to X \) so that \( \pi^*(\alpha) = \partial \hat{W} \)

\( \Gamma: \) Free Abelian Deck group \( \subset H_1(X; \mathbb{Z}) \)
It is often convenient to consider 
\[ LG(\hat{X}, \hat{\alpha} = \partial \hat{W}) \]
and work equivariantly wrt \( \Gamma \)

Vacua of \( LG(\hat{X}, \hat{\alpha} = \partial \hat{W}) \) :  \( \mathbb{V} = \{ \hat{\phi}_a | d\hat{W}(\hat{\phi}_a) = 0 \} \)

Abbreviate vacua \( \hat{\phi}_a, \hat{\phi}_b, ... \) simply by \( a, b, ... \)

Write free \( \Gamma \) -action on \( \mathbb{V} \) :  \( a \rightarrow a + \gamma \)

\[ \hat{W}_{a+\gamma} = \hat{W}_a + \int_{\gamma} \alpha \]
Example 1: Mirror Of The Free Chiral

\[ X = \mathbb{C}^* \quad \alpha = \left( \frac{m}{\phi} - 1 \right) d\phi \quad \mathbb{V} = \{ \phi_0 = m \neq 0 \} \]

\[ \pi: \hat{X} = \mathbb{C} \to X = \mathbb{C}^* \quad \pi: \hat{\phi} \to \phi = \exp(\hat{\phi}) \quad \Gamma \cong \mathbb{Z} \]

\[ \hat{\alpha} = d \hat{\mathbb{V}} = d \left( m \hat{\phi} - e^{\hat{\phi}} \right) \leftarrow \alpha = \left( \frac{m}{\phi} - 1 \right) d\phi \]

\[ \hat{\mathbb{V}} = \{ \hat{\phi}_a = \log m + 2\pi i a \mid a \in \mathbb{Z} \} \]

\[ \hat{\mathbb{W}}_a = m \log m + 2\pi i \ a \ m \]

\[ \hat{\mathbb{W}}_{a+n} = \hat{\mathbb{W}}_a + 2\pi i \ m \ n \]

Twisted mass
Other Examples

Mirror of $\mathbb{CP}^1$: $\alpha = \left(\frac{t}{\phi^2} + \frac{m}{\phi} + t\right) d\phi \quad \phi \in X = \mathbb{C}^*$

Discussed in GMW framework in Galakhov (2021) and Khan-Moore, to appear

There are two vacua $\phi_i, \phi_j$ and rank one deck group $\Gamma \cong \mathbb{Z}$

LG models for knot homology
[Gaiotto-Witten; Galakhov-Moore; Aganagic]
Chern-Simons-Landau-Ginzburg

\( G_\mathbb{C} \): Complex Lie group

\( M_3 \): Riemannian 3-fold \( \Rightarrow \) LG model \( \text{CSLG}[G_\mathbb{C}, M_3] \)

\[ X = \{ \text{Complex } G_\mathbb{C} - \text{connections on a 3-manifold } M_3 \} \]

\[ \alpha = \int_{M_3} Tr \mathcal{F}^2 \quad \text{``} = d \text{ CS}(\mathcal{A}) \text{''} \]

Vacua on \( D = \mathbb{R} \): Flat \( G_\mathbb{C} \) - connections
Morse Theory Flows In LG Language

\( SQM(M, \xi) \) vacua:

\[
\xi = 0 \quad \iff \quad \frac{\partial \phi^I}{\partial x} = i \, \xi \, g^{IJ} \, \bar{\alpha}_j(\phi)
\]

We call this the \( \zeta \)-soliton equation

\[
\frac{\partial \phi}{\partial \tau} = \xi \quad \iff \quad \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial \tau} \right) \phi^I = i \, \xi \, g^{IJ} \, \bar{\alpha}_j(\phi)
\]

We call this the \( \zeta \)-instanton equation
Soliton Complexes For $SQM(\mathcal{M}, \xi)$ & $D = \mathbb{R}$

$\phi(x) \rightarrow \phi_i \quad \phi(x) \rightarrow \phi_j$

$R_{ij} = \left( \text{Span}\{\Psi[\phi_{ij}(x)]\}, F_{ij}, Q_{ij} \right)$

$F_{ij} \Psi_{ij} = \eta \left( D_{\phi_{ij}} \right) \Psi_{ij} \quad Q_{ij} : \text{Count } \zeta - \text{instantons}$

``Flavor Charge'': \quad [\phi_{ij}(\mathbb{R})] = \gamma_{ij} \in \Gamma_{ij}$

$\Gamma_{ij} = \text{paths in } X \text{ from } \phi_i \text{ to } \phi_j \quad \text{--- up to homology.}$
Adding Charges

Composition of curves defines $\gamma_{ij} + \gamma_{jk} \in \Gamma_{ik}$

Abelian group structure on $\Gamma_{ii} \cong \Gamma$.

$\Gamma_{ij}$ is a $\Gamma$–torsor
\[ R_{ij} \text{ is graded by } \Gamma - \text{torsor } \Gamma_{ij} \quad R_{ij} = \bigoplus_{\gamma_{ij} \in \Gamma_{ij}} R_{\gamma_{ij}} \]

``BPS index’’

\[ \mu_{\gamma_{ij}} := Tr_{R_{\gamma_{ij}}} e^{i\pi F_{ij}} \]

Central charge: \[ Z_{\gamma_{ij}} = \oint_{\gamma_{ij}} \alpha \]

Generalizes the standard (\(\alpha\) exact) \[ Z_{ij} = W_i - W_j \]

``Twisted mass property’’

\[ Z_{\gamma_{ij} + \gamma} = Z_{\gamma_{ij}} + Z_\gamma \]
Periodic Solitons

Qualitatively new feature with twisted masses:

\[ \phi(x) \rightarrow \phi_i \]

When \( \alpha \) is not exact there can be nontrivial solutions!

\[ Z_{\gamma_{ii}} = \oint_{\gamma} \alpha = \oint_{\phi_{ii}(\mathbb{R})} \alpha \]

\( R_{ii} \) can be nontrivial!
Main new ingredient in categorified wall-crossing with twisted masses involves Fock spaces constructed from $R_{ii}$
Wall-Crossing When $\alpha = \partial \mathcal{W}$

$$
\left( g_{I\bar{J}}(\phi; s), \mathcal{W}(\phi; s) \right) \quad 0 \leq s \leq 1
$$

$\mu_{ij}(s)$ is only piecewise constant: CFIV, CV 1991, 1992

Happens when two central charges are parallel.

CVWCF: Tells how $\mu_{ij}(s)$ jump.

Categorified CVWCF: Describe how the homotopy equivalence class of $R_{ij}$ jumps.
Remarks

1. We claim that the homotopy equivalence class is physically meaningful so this is a well-posed question.

2. Moreover, the homotopy equivalence class of $R_{ij}$ is a nontrivial refinement of $\mu_{ij}$.
Remarks

It is often said that the only thing we can hope to compute exactly in interacting, non-integrable QFTs are Witten indices.

For general $\text{LG}(X, W)$ are interacting & non-integrable

So what we are doing here has some tension with this standard folklore.
Wall-Crossing Formula With Twisted Masses: 1/3

````
``Vacuum Groupoid Algebra’’:
For each $\gamma_{ij} \in \Gamma_{ij}$ introduce a variable $x_{\gamma_{ij}}$

$$x_{\gamma_{ij}} x_{\gamma_{kl}} = \delta_{jk} \ x_{\gamma_{ij}+\gamma_{jk}}$$
````
Wall-Crossing Formula With Twisted Masses: 2/3

\[ \forall \ i \neq j \ & \ \forall \ \gamma_{ij} \in \Gamma_{ij} \ \ S_{\gamma_{ij}} := 1 + \mu_{\gamma_{ij}} x_{\gamma_{ij}} \]

\[ \forall \ \gamma \in \Gamma \ \ K_{\gamma} := \sum_{i} \prod_{k}(1 - x_{\gamma})^{-\mu_{k\gamma_{ii}}} x_{u_{i}} \]

Additive identity

For any half-plane \( \mathbb{H} \subset \mathbb{C} \)

\[ S(\mathbb{H}) =: \prod_{Z_{\gamma_{ij}}, Z_{\gamma} \in \mathbb{H}} S_{\gamma_{ij}} K_{\gamma} : \text{ Phase-ordered product} \]
Wall-Crossing Formula With Twisted Masses: 3/3

Wall-crossing statement:
\[ S(\mathbb{H}) \text{ is invariant} \]
provided no BPS rays enter/leave the half-plane \( \mathbb{H} \)

Very similar to the mathematics of the 2d-4d WCF.

[Kontsevich, Soibelman 2008; Gaiotto, Moore, Neitzke 2010]
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Branes

Use the \((A_\infty \text{ algebra/category of) Branes})\.

The homotopy class of the category of branes is invariant.
Lefshetz Thimbles

Choose a half plane $\mathbb{H}$ and phase $\zeta$. For each vacuum $\phi_i$ there is a canonical brane $\mathcal{Z}_i$

Consider all values $\phi(x_0) \in X$ so there is a solution

$$\mathcal{L}_i^{\text{right}}(\zeta) \subset X$$

$$\frac{\partial \phi^I}{\partial x} = i \zeta g^{IJ} \bar{\alpha}_J$$

$$\phi(x_0) \to \phi_i$$

$$x \to +\infty$$

$\mathcal{L}_i^{\text{right}}(\zeta) \subset X$ are Lagrangian subspaces and provide nice half-susy bc’s. [Hori-Iqbal-Vafa]
Example 1 of Lefshetz Thimbles

\[ W = \frac{1}{2} \phi^2 \]

\[ \phi_i = 0 \]

\[ \frac{d\phi}{dx} = i \zeta \bar{\phi} \]

\[ \phi(x) = c_0 \sqrt{\pm i \zeta} e^{ \pm x } \]

\[ c_0 \in \mathbb{R} \]
Boundary Condition-Changing Local Operators

Choose a half-plane $\mathbb{H} \subset \mathbb{C}$, and a phase $\zeta$

$$\hat{R}_{ij} := \text{Vector space of local bc changing operators between } \mathcal{Z}_j(\zeta) \text{ and } \mathcal{Z}_i(\zeta)$$

$\hat{R}_{ij}$ is a chain complex
⊕_{ij} \hat{R}_{ij} \text{ Is An Algebra

ρ_{ijk}^{(2)}: \hat{R}_{ij} \otimes \hat{R}_{jk} \rightarrow \hat{R}_{ik}
⊕_{ij} \hat{R}_{ij} \text{ has higher } \text{``OPE Products''}

Differential/Susy operator is \( \rho^{(n)}(\delta_1, \ldots, \delta_n) \)

Differential/Susy operator is \( \rho^{(1)} \)
$\bigoplus_{ij} \hat{R}_{ij}$ Is An $A_\infty$ -Algebra

Sum with signs must vanish:

Quadratic relations on the $\rho^{(n-k+1)}$ : $A_\infty$ – relations
These multiplications can be constructed explicitly using the \textit{``web formalism''} of GMW.
Sources For The Web Formalism

Algebra of the Infrared: String Field Theoretic Structures in Massive $\mathcal{N} = (2,2)$ Field Theory In Two Dimensions

Algebra of the infrared and secondary polytopes

Davide Gaiotto, Gregory W. Moore, Edward Witten

August 13, 2014

Only 51 pages !!!
There is a notion of homotopy equivalence of $A_\infty$-algebras. It extends the notion of homotopy equivalence of chain complexes, and says how the OPE’s are related to each other.

Categorical wall-crossing will involve the homotopy equivalence of these $A_\infty$-algebras.
An $\mathcal{A}_\infty$ - Category $\hat{\mathcal{R}}$ (When $\alpha = \partial \mathcal{W}$)

Choose $\zeta, \mathbb{H} \subset \mathbb{C}$

Objects = thimbles $\mathcal{I}_i$

$\text{Hop}(\mathcal{I}_i, \mathcal{I}_j) = \begin{cases} \hat{\mathcal{R}}_{ij} & Z_{ij} \in \mathbb{H} \\ \mathbb{Z} & i = j \\ 0 & Z_{ij} \notin \mathbb{H} \end{cases}$
The Product Formula ($\alpha = \partial \mathcal{W}$)

\[ \hat{R}_{ij} \text{ can be written in terms of } R_{ij} \quad [\text{GMW}] \]

\[ \hat{R} := \bigoplus_{ij} \hat{R}_{ij} e_{ij} = \bigotimes_{z_{ij} \in \mathbb{H}} (\mathbb{Z} \ 1 + R_{ij} e_{ij}) \]

phase ordered!

\[ \hat{R}_{ik} = R_{ik} \bigoplus R_{ij} \bigotimes R_{jk} \bigoplus \cdots \]
\[ \hat{R}_{ik} = R_{ik} \oplus R_{ij} \otimes R_{jk} \oplus \cdots \]

Summands correspond to sequences of central charges \( Z_{i,i_1}, Z_{i_1,i_2}, \ldots, Z_{i_n,k} \) whose phases are clockwise ordered in the half plane \( \mathbb{H} \).
Naïve Differential On $\hat{R}_{ik}$

$$\hat{R}_{ik} = R_{ik} \oplus R_{ij} \otimes R_{jk} \oplus \cdots$$

$\hat{Q}^{naive} = Q_{ik} \oplus ( Q_{ij} \otimes 1 + 1 \otimes Q_{jk} ) \oplus \cdots$

PHYSICALLY WRONG!

We missed important instanton effects
Domain Wall Junctions: $\zeta$ — instanton equation

\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial \tau} \right) \phi^I = i \zeta \ g^{IJ} \ t^j(\phi)
\]

$\phi \approx \phi_i$

$\phi \approx \phi_j$

$\phi \approx \phi_k$

$\parallel Z_{ij} \parallel\phi_{ij}(x)$

$\parallel Z_{ki} \parallel\phi_{ki}(x)$

$\parallel Z_{jk} \parallel\phi_{jk}(x)$

Note: $Z_{ij}, Z_{jk}, Z_{ki}$ clockwise ordered....
Explicit examples studied in


Interior Amplitude

\[ \phi_{ij} \otimes \phi_{jk} \otimes \phi_{ki} \in R_{ij} \otimes R_{jk} \otimes R_{ki} \]

Counting solutions defines an "interior amplitude"

\[ \beta_{ijk} \in R_{ij} \otimes R_{jk} \otimes R_{ki} \]

Summing over all such cyclic fans defines an \( L_\infty \) \(-\)algebra

\[ R_c = \bigoplus_{\text{cyclic fans}} R_{i_1 i_2} \otimes \cdots \otimes R_{i_n i_1} \]

\( \beta \) is a Maurer-Cartan element in an \( L_\infty \) algebra.

(generalizes the "broken flows identity." )
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Categorical Wall-Crossing ($\alpha = \partial \mathcal{W}$)

IF: \[ (X, g_{IJ}, \alpha, \zeta)_1 \sim (X, g_{IJ}, \alpha, \zeta)_2 \]

THEN: \[ \hat{R}(X, g_{IJ}, \alpha, \zeta)_1 \sim \hat{R}(X, g_{IJ}, \alpha, \zeta)_2 \]

⇒ how the $R_{ij}$ complexes change (up to h.e.)
Definition: Cones In Homological Algebra

If \( f : C_1 \to C_2 \) is a chain map

\[
C_1 := (V_1, F_1, Q_1) \quad C_2 := (V_2, F_2, Q_2)
\]

Then \( \text{Cone}(f) \) is the new chain complex with

\[
V := V_2 \oplus V_1[-1] \quad Q_{\text{Cone}(f)} = \begin{pmatrix} Q_2 & f \\ 0 & -Q_1 \end{pmatrix}
\]
Interior Amplitude Induces A Chain Map

\[ \beta_{ijk} \in R_{ij} \otimes R_{jk} \otimes R_{ki} \]

\( \text{CPT} \Rightarrow \exists \ (\text{deg}=-1) \) contraction: \( K: R_{ij} \otimes R_{ji} \rightarrow \mathbb{Z} \)

Chain map:

\[ M(\beta_{ijk}): R_{ik} \rightarrow R_{ij} \otimes R_{jk} \]

Corrects \( \hat{Q}^{\text{naive}} \): off-diagonal component of differential on \( \hat{R}_{ik} \)

\[ M(\beta_{ijk}) = \hat{Q}_{ik}: R_{ik} \rightarrow R_{ij} \otimes R_{jk} \]
Defines a chain map $M^L(\beta_{ijk}): R^L_{ik}[1] \to R^L_{ij} \otimes R^L_{jk}$

Defines a chain map $M^R(\beta_{ikj}): R^R_{ij} \otimes R^R_{jk} \to R^R_{ik}$
Solving Cat. Wall Crossing For $\alpha = \partial W$

An elegant way of solving the wall-crossing constraint

$$R^L_{ik} \sim \text{Cone}(M^L(\beta_{ijk}): R^L_{ik}[1] \to R^L_{ij} \otimes R^L_{jk})$$

$$R^R_{ik} \sim \text{Cone}(M^R(\beta_{ikj}): R^R_{ij} \otimes R^R_{jk} \to R^R_{ik})$$

Conversely, if $\hat{R}^L \sim \hat{R}^R$ then, up to homotopy, $R^L_{ik}$ and $R^R_{ik}$ are related by cone constructions as above
\[ R_{ik}^R \sim \text{Cone}(M^L(\beta_{ijk}): R_{ik}[1] \to R_{ij}^L \otimes R_{jk}^L) \]

\[ R_{ik}^L \sim \text{Cone}(M^R(\beta_{ikj}): R_{ij}^R \otimes R_{jk}^R \to R_{ik}^R) \]

Taking Euler characters gives the Cecotti-Vafa wall-crossing formula:

\[ \mu_{ik}^R = \mu_{ik}^L + \mu_{ij}^L \mu_{jk}^L \]

\[ \mu_{ik}^L = \mu_{ik}^R - \mu_{ij}^R \mu_{jk}^R \]
Cecotti-Vafa Cones

How do they generalize to the case with twisted masses?
Thimble Branes & Their Local Operators

Categorical Wall-Crossing

Supersymmetric Quantum Mechanics And Homological algebra

2D N=(2,2) Landau-Ginzburg Models

Generalization To Twisted Masses

Relation To 3d Indices
Generalization To Twisted Masses

Work in progress with Ahsan Khan

Definition: A **cyclic fan of charges** is a cyclically-ordered set

\[
\{ \gamma_{i_0 i_1}, \gamma_{i_1 i_2}, \ldots, \gamma_{i_n i_0} \}
\]

So that the phases of \( Z_{\gamma_k,\gamma_{k+1}} \) are **monotonically decreasing** (clockwise)

New ingredient: Successive \( Z_{\gamma_k,\gamma_{k+1}} \) can be parallel.
Representations Of (Irreducible) Webs

Irreducible fans: \( \{ \gamma_{i_0 i_1}, \gamma_{i_1 i_2}, \ldots, \gamma_{i_n i_0} \} \quad i_k \neq i_{k+1} \)

GMW web formalism applies to give \( L_\infty \) algebra structure on the sum over all these pictures

\[
R_{\gamma_{i_0 i_1}} \bigotimes R_{\gamma_{i_1 i_2}} \bigotimes \cdots \bigotimes R_{\gamma_{i_n i_0}}
\]
Vertices For Generalized Webs

\[ \{ \gamma_{i_0i_1}, \gamma_{i_1i_1}, \gamma_{i_1i_1}, \ldots, 2\gamma_{i_1i_1}, \ldots, \gamma_{i_1i_2}, \ldots, \gamma_{i_ni_0} \} \]
Representations Of Generalized Webs

\[ \{ \gamma_{i_0i_1}, \gamma_{i_1i_1}, \gamma_{i_1i_1}, \ldots, 2\gamma_{i_1i_1}, \ldots, \gamma_{i_1i_2}, \ldots, \gamma_{i_ni_0} \} \]

\[ \mathcal{F}_{\gamma_{jj}} := \text{Graded Fock space on } R_{\gamma_{jj}} \]

\[ i \neq j \neq k \neq i \quad \mathcal{F}_{\gamma_{ij}, \gamma_{jk}} := \bigotimes_{\gamma_{ij} < \gamma_{jj} < \gamma_{jk}} \mathcal{F}_{\gamma_{jj}} \]
Conjecture: Has a natural $L_\infty$ structure associated with generalized webs.

$R_c = \bigoplus \{ \gamma_{i_0 i_1}, \ldots, \gamma_{i_n i_0} \}$ irred

$\mathcal{F}_{\gamma_{i_n i_0}, \gamma_{i_0 i_1}} \otimes R_{\gamma_{i_0 i_1}} \otimes \cdots \otimes \mathcal{F}_{\gamma_{i_{n-1} i_n}, \gamma_{i_n i_0}} \otimes R_{\gamma_{i_n i_0}}$

Checked in several special cases.
\(A_\infty\) — Category Of Branes

\[\hat{R}_{ij} := \text{Sum over all half-plane fans with Fock spaces of periodic jj solitons inserted between factors } R_{\gamma ij} \text{ and } R_{\gamma jk}\]

Conjecture: \((\hat{R}, R_c)\) has the structure of an \(LA_\infty\)-category (= open closed homotopy category)
Example: Mirror Of $\mathbb{CP}^1$

$$\phi \in X = \mathbb{C}^* \quad \alpha = \left( \frac{t}{\phi^2} + \frac{m}{\phi} + t \right) d\phi$$

Two vacua: $\phi_i$ & $\phi_j$ & $\Gamma = \mathbb{Z}\gamma$ is rank one.

$\mathbb{H} = \text{UHP}$

Peacock pattern

\[ Z_{\gamma_{ij}+n\gamma} \]
\[ Z_{\gamma_{ij}+\gamma} \]
\[ Z_{\gamma_{ji}+m\gamma} \]
\[ Z_{\gamma_{ji}+2\gamma} \]
\[ Z_{\gamma_{ji}+\gamma} \]
\[ \mathbb{H} = \text{UHP} \]

\[ \hat{R} = \bigotimes_{(n=0)}^{\infty} \left( \begin{array}{cc} \mathbb{Z} & R_{\gamma_{ij}+n\gamma} \\ 0 & \mathbb{Z} \end{array} \right) \bigotimes_{k \geq 1} \left( \begin{array}{cc} \mathcal{F}[R_{k\gamma_{ii}}] & 0 \\ 0 & \mathcal{F}[R_{k\gamma_{jj}}] \end{array} \right) \bigotimes_{n=\infty}^{1} \left( \begin{array}{cc} \mathbb{Z} & 0 \\ R_{\gamma_{ji}+n\gamma} & \mathbb{Z} \end{array} \right) \]
1. Supersymmetric Quantum Mechanics And Homological algebra
2. 2D N=(2,2) Landau-Ginzburg Models
3. Thimble Branes & Their Local Operators
4. Categorical Wall-Crossing
5. Generalization To Twisted Masses
6. Relation To Three-Dimensional SQFT
RELATION TO 3D INDEX

With nontrivial input from Andy and Tudor
Motivation

Recent striking conjecture by Garoufalidis, Gu, and Marino, `Peacock Patterns And Resurgence In Complex Chern-Simons Theory’’

They observed a relation between the 3d index

\[ I_T(q, x) = Tr_{\mathcal{H}_T}(-1)^F q^{\frac{1}{2}R + j_3} x^e \]

\(T\): 3d SUSY class R theory associated with a hyperbolic knot complement \(T(M_K)\)

and Stokes matrices related to thimbles in complex Chern-Simons theory on \(M_K\)
We give it a natural context and state a conjecture about PDE’s (Kapustin-Witten equations) which implies the GGM conjecture.

In fact, the conjecture has been stated before by Victor Mikhaylov (2017) for different reasons.
$T(M_3)$

$T(M_3)$ is a theory obtained by reduction of 6d (2,0) on $M_3$ with topological twist (class R) [Dimofte, Gaiotto, Gukov; Terashima-Yamazaki]

Consider $T(M_3)$ on $\mathcal{C} \times \mathbb{R}_t$

With `holomorphic-topological twist’’ [Witten; Oh-Yagi] we can identify the index $I_T(q, x)$ with the trace over the Q-cohomology of local operators at the tip of the cigar.
KK Reduction of Cigar $\Rightarrow$ LG model With A Brane

Get a LG theory on half space with boundary condition $\mathcal{B}_{Cigar}$

$Q$ — closed local operators on $C \times \mathbb{R}_t$

Boundary operators for $\mathcal{B}_{cigar}$

$$I_{T(M_3)}(q, x) = Tr_{Hop}(\mathcal{B}_{cigar}, \mathcal{B}_{cigar}) (-1)^F q^{J_1} x^{J_2}$$
LG Model = CSLG$[M_3]$  

$6d \ (2,0)/ \ M_3 \times C \times \mathbb{R}_t$  

$T[M_3] / C \times \mathbb{R}_t \overset{M_3}{\longrightarrow} \mathcal{B}_{cigar}$  

$U(1)_C \overset{L G / \mathbb{R}_+ \times \mathbb{R}_t}{\longrightarrow} \mathcal{B}_{cigar}$  

$\cong$  

$5D \ SYM / M_3 \times \mathbb{R}_+ \times \mathbb{R}_t \overset{M_3}{\longrightarrow} \mathcal{B}_{Nahm}$
Relation to KW Equations With Nahm bc’s.

\[ 6d \ (2,0) / M_3 \times \mathbb{C} \times \mathbb{R}_t \quad \xrightarrow{U(1)_C} \quad 5d \ \text{SYM} / M_3 \times \mathbb{R}_+ \times \mathbb{R}_t \]

\[ ds^2 = g_{IJ}(x)dx^I dx^J + dy^2 + dt^2 \]

Witten: BPS equations = KW equations for \( \mathcal{A} \) on \( M_3 \times \mathbb{R}_+ \)

\[ y \to 0 \quad \text{Im}(\mathcal{A}^a) = \frac{e^a}{y} + \mathcal{O}(1) \quad \text{Re}(\mathcal{A}^a) = \omega^a + \mathcal{O}(y) \]

\( e^a, \omega^a \) : Dreibein and spin connection for Riemannian metric on \( M_3 \) & \( G_c = SL(2, \mathbb{C}) \)
Nahm = $\sum$ Chan-Paton $\times$ Lefshetz

Vacua of CSLG[$M_3$] : Flat connections $\sigma_i$

$$\mathcal{B}_{Nahm} \cong \sum_i \mathcal{E}_{\sigma_i} \mathcal{I}_{\sigma_i}$$

KW equations with $\mathcal{A}_y = 0$ are the $\zeta$ --soliton equations for CSLG[$M_3$]

Chan-Paton complex $\mathcal{E}_{\sigma_i}(\mathcal{B}_{Nahm})$ is MSW for KW & Nahm bc’s $@ y \rightarrow 0$ & $\mathcal{A} \rightarrow \sigma_i$ $@ y \rightarrow \infty$
\[ O = (-1)^F q^{J_1} x^{J_2} \]

\[
\begin{align*}
T r_{H o p}(\mathcal{B}_{c i g a r}, \mathcal{B}_{c i g a r}) O \\
= \sum_i T r_{\varepsilon_{\sigma_i}} O \times T r_{H o p}(\varepsilon_{\sigma_i}, \varepsilon_{\sigma_j}) O \times T r_{\varepsilon_{\sigma_j}} O
\end{align*}
\]

\[
T r_{H o p}(\varepsilon_{\sigma_i}, \varepsilon_{\sigma_j}) O = S_{\sigma_i, \sigma_j}(q, x)
\]

Stokes matrices for Chern-Simons thimbles
Specialize to Hyperbolic Knot Complement

\[ M_3 = M_K \] knot complement in \( S^3 \) of a hyperbolic knot.

Among flat \( SL(2, \mathbb{C}) \) connections on \( M_K \) there is a distinguished one: \( \sigma_1 \)

\( \sigma_1 \): corresponds to the complete hyperbolic metric. \( \mathcal{A} = \omega + i \epsilon \)

Conjecture: \( \mathcal{E}_{\sigma_i}(\mathcal{B}_{Nahm}) \cong \begin{cases} \mathbb{Z} & \sigma_i = \sigma_1 \\ 0 & \text{Else} \end{cases} \)
Interesting Generalization

$L$: Colored (by reps of $SL(2)$) link in $M_3$

Witten: Modify Nahm boundary condition in 5d SYM with 't Hooft line $L$

Corresponds to a brane $\mathcal{B}(L)$ in CSLG$[M_3]$

Up to homotopy equivalence, it only depends on isotopy class of $L \subset M_3$
Potentially New Knot Invariants

**Conjecture:**

a.) The h.e. class of the $A_\infty$—category of $\mathcal{Br}(CSLG(M_3))$ is a 3-manifold invariant

b.) The h.e. class of $A_\infty$ - algebras $Hop(\mathcal{B}(L), \mathcal{B}(L))$ are (new?) colored link invariants.
Conclusion

Using the framework of GMW we derived a categorified version of the Cecotti-Vafa WCF. The framework can be extended to the case with twisted masses. Here there are qualitatively new features involving Fock spaces of periodic solitons.

This circle of ideas applied to CSLG[$M_3$] gives a natural framework for interpreting a recent striking conjecture of GGM, and moreover suggests potentially new colored link invariants.