Physics 695: Advanced Topics in Mathematical Physics

Fall, 2007

September 4, 2007

1. Administrative

1. Notes for the lectures will be handed out. There is a list of useful references at the end. The notes will have many exercises but there will be no formal problem sets.

2. The grade for those taking the course for credit will be based on a short paper and possibly a presentation given at the end of the semester. I will hand out topics towards the middle of the course. The list of projects will be the same as last year's, but augmented with some new ones.

3. As a courtesy to others, please don't eat in class.

4. September 4 is an organizational meeting. No class September 7. So the first lecture is September 11.

5. I will outline the course topics. The choice of topics later in the semester depends on my interests and those of the students. Please send me an email stating who you are, what year of graduate school you are in, and a (brief!) reason why you prefer this choice. If you suggest more than one topic, order them. You can also suggest related topics you would find interesting.

2. Boundary Conditions

This course is primarily intended for graduate students intending to specialize in theory, particularly particle theory.

We try to keep prerequisites to a minimum. It would be helpful, but not absolutely necessary to know something about the following:

1 Some basic topology - definitions of homotopy groups.

2 Basic definitions of manifolds, together with a description of many important examples: projective spaces, grassmannians.

3 Quotients of manifolds by groups. Homogeneous spaces, orbifolds.

4 Definitions of differential forms, tangent bundle and cotangent bundle. Pullback and pushforward under maps.

5 Definition of DeRham cohomology and some basic properties. I will occasionally refer to aspects of cohomology over the integers. If there is a demand I can spend a few lectures outlining this more refined theory.

6 Integration of differential forms on oriented manifolds. Stokes' theorem. Periods of closed forms.

7 Definition and basic properties of metrics on manifolds.

8 Basic definitions of fiber bundle, vector bundle, principal bundle, associated bundle.

9 Last year we spent considerable time developing the theory of characteristic classes from a topological point of view. Knowing that material would be helpful, but is by no means necessary to follow my discussion this year of the Chern-Weil theory of characteristic classes.

The above material can be found in several textbooks (some listed below) as well as in my course notes for "Geometry and Modern Physics" which are available on request.

3. Tentative outline of the first part of the course

3.1. Geometry and Topology of Lie Groups

metrics. left-invariant vector fields and forms. DeRham cohomology and Lie algebra cohomology. Wess-Zumino terms. Homotopy groups.

3.2. Connections

3.2.1.Connections on fiber bundles: Definition

3.2.2.Connections on vector bundles

3.2.3. Projected connections, Adiabatic theorem, and Berry's phase

3.2.4.Connections on principal bundles

3.2.5.Holonomy

3.2.6.Curvature of a connection

3.2.7.The space of connections $\mathcal{A}/\mathcal{G}$

3.2.8.Flat connections

3.3. Chern-Weil theory of characteristic classes

3.4. Chern-Simons terms

3.5. Nonabelian Yang-Mills theory

3.6. Riemannian geometry

3.7. Kaluza Klein theories

3.8. Electromagnetic duality, Generalized Maxwell theory, and Differential Cohomology

3.9. A thorough review of Clifford algebras and spinor representations

3.10. Supersymmetry algebras and their representations

3.11. Orientation, Spin, $Pin^{\pm}$, and $Spin^c$ structures on manifolds

3.12. Dirac operator on a curved manifold

4. Possible topics for the second part of the course

Assuming time permits there are a number of interesting topics I would be interested in covering. The choice depends in part on the interest of the audience.

Possible topics include:

4.1. $N = 1$ $D = 1$ susy nonlinear sigma model: supersymmetric quantum mechanics

The general form of the supersymmetric quantum Lagrangian. Superspace

Hamiltonian and Lagrangian quantization.

Relation to the index theorems: Continue last year's discussion of topological field theory integrals (integral representation of the Thom class)

Mathai-Quillen representative of the Thom class.

Complete the SUSY QM derivation of the index theorems.

Fixed point theorems and character-valued (equivariant) index theorems.

Witten-Morse complex. Floer theory.

4.2. Symplectic geometry and geometric quantization

1. Gaussian integrals, Bogoliubov transformations, Schale's theorem
2. Kahler and hyperkahler quotients. Hamiltonian reduction in general.
3. Quantization of a Kahler manifold. Coherent states. Squeezed states.
4. Change of polarization.
5. Heisenberg groups. Clifford algebras.
6. ??Poisson manifolds. Kontsevich theorem?

4.3. K-theory and D-branes

0. Generalized cohomology theories.

1. Basic definitions in K-theory.

2. Triples and the tachyon interpretation. The ABS construction.

3. Fredholm operators, Kuiper's theorem, $K(X) = [X, \mathcal{F}]$. Fredholm operators commuting with a Clifford algera. Proof of the Bott periodicity theorem.

3. RR charges and K-theory: anomaly cancellation. physical interpretation of the Atiyah-Hirzebruch spectral sequence.

4. RR charges and K-theory: Boundary CFT approach.

5. Twisted K-theory, Dixmier-Douady class, and B-fields.

6. Twisted real KR theory and orientifolds.

7. Integration in K-theory, Thom isomorphism in K-theory, and the index theorems.

8. Self-dual cohomology theories. Poincare-Pontryagin duality.

9. Differential K-theory and RR fields.

10. K theory of algebras. Noncommutative geometry and K-theory. D-branes as noncommutative solitons. Noncommutative tachyons.

11. Quantum theory of self-dual fields.

4.4. Quantization of Chern-Simons theories

1. Quantization of 3d Chern-Simons theory:

Chern-Simons-Witten; Relation to Rational Conformal Field theory. knot polynomials. Relation to the quantum Hall effect.

2. Quantization of higher-dimensional Chern-Simons theories.

5. Some sources

There is no formal textbook. The following is a list of sources I have used. A star means it is an especially useful pedagogical reference and you are encouraged to read it.

*1. M. Nakahara, *Geometry, Topology and Physics* Institute of Physics publishing.

*2. Eguchi-Gilkey-Hanson, "Gravitation, gauge theories and differential geometry," Phys. Rep. **66**(1980)213.

*3. R. Bott and L.W. Tu, *Differential forms in algebraic topology.* Springer

Those marked with * are the primary sources.

For an interesting historical essay on this general subject see:

C. Nash, "Historical essay on geometry and physics," hep-th/9709135

Other books on geometry and topology aimed at physicists:

*3. A.S. Schwarz, *Topology for Physicists*, Springer (Top Sch 952t)

13. R. Bott and J. Mather, lectures at Battelle Rencontres

8. Trautman, Differential geometry for physicists

9. Isham, Modern Differential Geometry for Physicists

12. Monastyrsky, Topology of gauge fields...

13. Raoul Bott, Collected Works, vol. 4: Mathematics Related to Physics.

14. Michael Atiyah, Collected Works, vol. 5: Mathematical Physics

Much useful material can also be found in

Green, Schwarz, and Witten, *Superstring theory*, vol. 2

Other math books useful for this material:

1. G. Bredon, *Topology and Geometry*, Springer GTM 139

2. Dubrovin, Fomenko, and Novikov, *Modern Geometry*, vols. 1,2,3

3. H.B. Lawson and M.L. Michelsohn *Spin Geometry*

14. Madsen and Tornehave, *From Calculus to Cohomology*

10. Kobayashi + Nomizu, *Differential Geometry*

For material specifically on complex manifolds we recommend:

1. R.O. Wells, *Differential Analysis on Complex Manifolds*

2. Griffiths and Harris, *Principles of Algebraic Geometry*

3. Chern, Complex manifolds without potential theory

4. Morozov and Perelomov, "String theory and complex geometry," Phys. Rep. 1999

For supersymmetry:

10. J. Bagger and J. Wess, *Supersymmetry and Supergravity.* Princeton

11. P. Freund, *Supersymmetry.* Cambridge

12. J. Lykken - TASI lectures hepth/9612114

13. P. West, *Introduction to supersymmetry and supergravity.*

14. M. Sohnius, "Introducing supersymmetry," Phys. Rep. **128**(1985) 39.

15. J. Strathdee, "Extended superpoincare supersymmetry" Int. J. Mod. Phys. **A2**(1987) 273

16. A. Van Proeyen, "Tools for supersymmetry," hep-th/9910030

17. A. Bilal, "Introduction to supersymmetry," hep-th/0101055

18. S. Weinberg, *Quantum Theory of Fields, vol. 3*

Some reviews covering material on susy and geometry or topics closely related are in:

0. O. Alvarez, "Lectures on quantum mechanics and the index theorem," Park City Lectures

1. Cordes, Moore, and Ramgoolam, Les Houches Lectures hep-th/...

2. Dijkgraaf, Les Houches lectures hep-th/....

3. Labastida and Lozano, hep-th/9709192; Topological field theory, Donaldson-Witten theory.

4. Peeters and Waldron, hep-th/9901016; Susy QM derivation of APS index theorem.

5. P. Woit, Review of Borel-Weil-Bott, hep-th/02...

6. K. Iga, "What do Topologists want from Seiberg–Witten theory? (A review of four-dimensional topology for physicists)," hep-th/0207271

7. A. Mostafazadeh, hep-th/9405048 Title: Supersymmetry, Path Integration, and the Atiyah-Singer Index Theorem

8. T. Hollowood and T. Kingaby, χ_y genus, hep-th/0303018

Some recent texts with useful mathematical material for string theorists:

1. Hori et. al. *Mirror Symmetry*