Three Remarks On d=4 N=2 Field Theory

Gregory Moore
Rutgers University

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1. A Little Gap In The Classification Of Line Defects
2. Comparing Computations Of Line Defect Vevs
3. Some New d=4, N=2 Superconformal Field Theories?
4. Conclusion

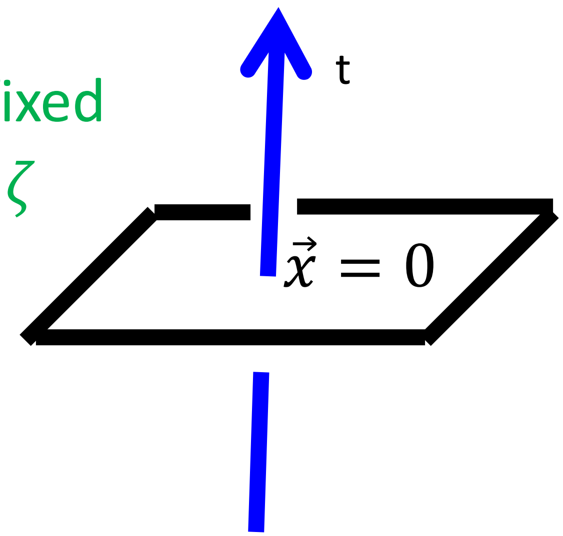
Line Defects

Supported on one-dimensional submanifold of spacetime.

Defined by UV boundary condition around small tubular neighborhood [Kapustin].

This talk: Focus on half-BPS d=4 N=2 defects on straight lines along time, sitting at points in space.

Our defects preserve $osp(4^*|2)_\zeta \subset su(2,2|2)$ fixed subalgebra under P(arity) and $U(1)_R$ rotation by ζ



$$\mathcal{R}_\alpha^A \sim Q_\alpha^A + \zeta \sigma^0_{\alpha\dot{\beta}} \bar{Q}^{\dot{\beta} A}$$

Example: 't Hooft-Wilson Lines In Lagrangian Theories

G is a compact semisimple Lie group

Denote 't Hooft-Wilson line defects $\mathbb{L}[P,Q]$ with P a representation of $G^{\vee}$ and Q a representation of G.

$$\mathbb{L}[0,Q] = \rho_Q\left(Pexp \int_{\vec{0}\times\mathbb{R}} A - Re(\zeta^{-1}\varphi)\, ds \right)$$

$$F \sim P\, vol(S^2) + \cdots$$

$\mathbb{L}[P,0]$

$$Im(\zeta^{-1}\varphi) \sim -\frac{P}{2r} + \cdots$$

Class S

$\mathfrak{g}$ = simple A,D, or E Lie algebra

$C_{g,n}$ Riemann surface with (possibly empty) set of punctures $p_1, p_2, \dots, p_n$

D = collection of ½-BPS cod=2 defects $D(p_1), \dots, D(p_n)$

Compactify d=6 (2,0) theory $S[\mathfrak{g}]$ on $M_4 \times C_{g,n}$ with partial topological twist: Independent of Kahler moduli of $C_{g,n}$.

Take limit: $A \to 0$

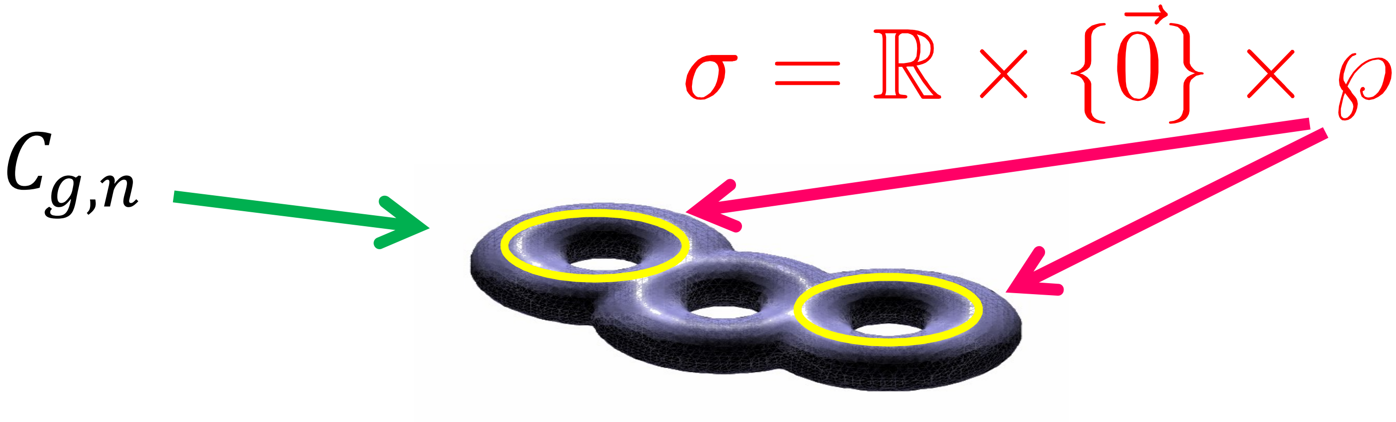
Denote these d=4 N=2 theories by $S[\mathfrak{g}, C, D]$

For suitable D the theory is superconformal.

Line defects in $S[\mathfrak{g}, C, D]$

Wrap surface defects of $S[\mathfrak{g}]$ on $\sigma = \mathbb{R} \times \wp$

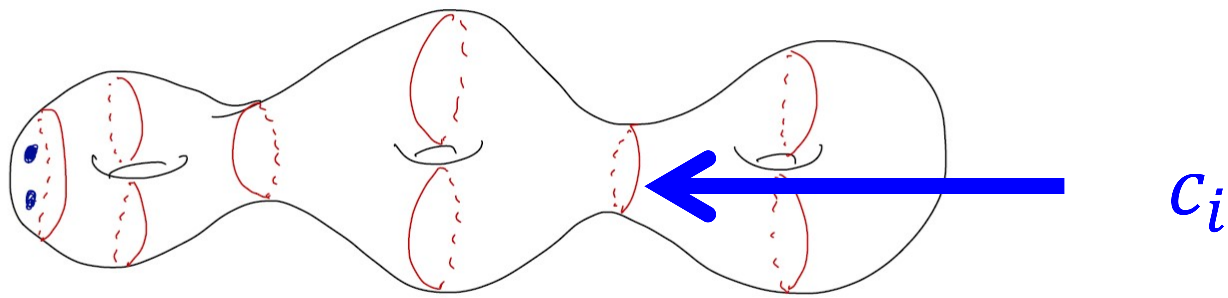
Here $\wp \subset C_{g,n}$ is a one-dimensional _**submanifold**_ of $C_{g,n}$ (not necessarily connected!)



Line defect in 4d *labeled* by $\wp$
and rep $\mathcal{R}$ of $\mathfrak{g}$ and denoted $L(\wp, \mathcal{R})$

Lagrangian Class S Theories

Weak coupling limits are defined by trinion decompositions of $C_{g,n}$



$3g - 3 + n$ cutting curves c_i

Example: $S[\mathfrak{su}(2), C_{g,n}, D]$ is a d=4 N=2 theory with gauge algebra $\mathfrak{su}(2)^r$ with lots of hypermultiplet matter.

r = 3g-3+ n

For general class S theories with a Lagrangian description:
What is the relation of $L(\wp, \mathcal{R})$ with $\mathbb{L}[P, Q]$?

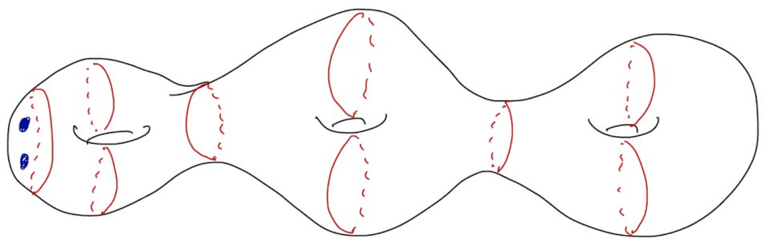
Classifying Line Defects

For $\mathfrak{g} = \mathfrak{su}(2)$ and $\mathcal{R}$ = fundamental , the Dehn-Thurston classification of isotopy classes of closed curves matches nicely with the classification of simple line operators as Wilson-'t Hooft operators: Drukker, Morrison & Okuda.

The generalization of the Drukker-Morrison-Okuda result to higher rank has not been done, and would be good to fill this gap.

But even DMO is incomplete!!

(Noted together with Anindya Dey)



For $\mathfrak{su}(2)^r$ with r 't Hooft-Wilson parameters: $\mathbb{L}(\vec{p}, \vec{q})$

$$P = \oplus_{i=1}^{r} p_i \frac{1}{2} H_{\alpha_i} \qquad \mathrm{Q} = \oplus_{i=1}^{r} q_i \frac{1}{2} \alpha_i$$

Isotopy classes of $\wp$ also classified by r-tuples $\wp(\vec{p}, \vec{q})$:
``Dehn-Thurston parameters''

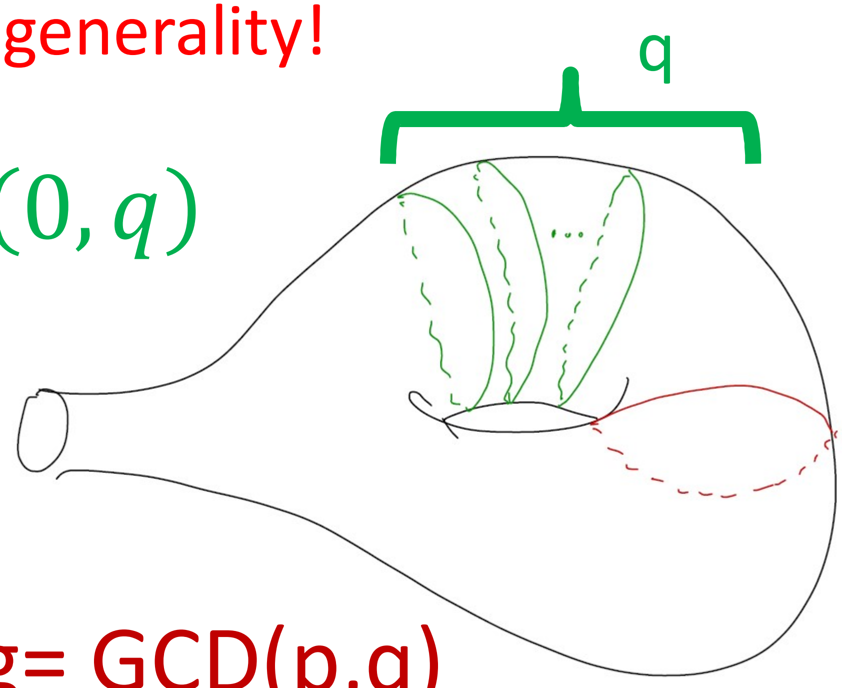
$p_i = \#(\wp \cap c_i)$ q_i ``counts twists'' around c_i

Main claim of DMO: $\mathbb{L}[\vec{p}, \vec{q}] = L(\wp(\vec{p}, \vec{q}), \mathcal{R} = \binom{1}{2})$

Main claim of DMO: $\mathbb{L}[\vec{p},\vec{q}] = L(\wp(\vec{p},\vec{q}), \mathcal{R} = \binom{1}{2})$

Actually, it cannot be true in this generality!

$\mathfrak{su}(2)\ \mathcal{N} = 2^*$ $\quad\wp(0,q)$



$$\mathbb{L}[0,q] \neq L\left(\wp(0,q), \binom{1}{2}\right)$$

For $C_{1,1}$ $\quad\wp(p,q)$ has g= GCD(p,q) connected components.

Open Problem: For ALL OTHER $C_{g,n}$ it is NOT KNOWN when $\wp(\vec{p},\vec{q})$ has a single connected connected component!

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VEV's On $\mathbb{R}^3 \times S^1$

Consider path integral with L inserted at $\{\vec{0}\} \times S^1$

$\langle L \rangle$ is a function on the SW moduli space $\mathcal{M}$
:= vacua of compactification on $\mathbb{M}^{1,2} \times S^1$

$\mathcal{M}$: Total space of an integrable system: A fibration over the Coulomb branch by torus of electric and magnetic Wilson lines.
In class S this integrable system is a Hitchin system.

$\mathcal{M}$ is a hk manifold. $\langle L \rangle$ is a holomorphic function on $\mathcal{M}$ in the complex structure selected by the phase ζ.
(The projection of the integrable system is not holomorphic.)

Part 2 of the talk focuses on exact results for these holomorphic functions.

$\langle L \rangle$ As A Trace

$$\langle L\rangle_y := Tr_{\mathcal{H}_L}(-1)^F\,(-y)^{J_3+I_3}\,e^{-2\pi R\,H+i\,\theta\cdot Q}$$

$\mathcal{H}_L$ is the Hilbert space on $\mathbb{R}^3$ in the presence of L at $\vec{x}=0$ with vacuum u at $\vec{x}=\infty$

(At y=-1 we get the vev. With $y \neq -1$ we are studying a quantization of the algebra of functions on $\mathcal{M}$.)

Math Fact: For $\zeta \neq 0,\infty$ the moduli space $\mathcal{M}$, as a complex manifold , is the space of flat $\mathfrak{g}_{\mathbb{C}}$ connections, $\mathcal{A}$, on $C_{g,n}$ with prescribed monodromy at p_i.

$$(u,\theta) \leftrightarrow \mathcal{A}$$

$$\langle L(\wp,\mathcal{R})\rangle = Tr_{\mathcal{R}} Hol(\wp) = Tr_{\mathcal{R}}\left(Pexp \oint_{\wp} \mathcal{A}\right)$$

Types Of Exact Computations

1. Localization [Pestun (2007); Gaumis-Okuda-Pestun (2011) ; Ito-Okuda-Taki (2011)]

   Applies to $\mathbb{L}[P, 0]$ in Lagrangian theories.

2. AGT-type [Alday,Gaiotto,Gukov,Tachikawa,Verlinde (2009); Drukker,Gaumis,Okuda,Teschner (2009)]

   Should apply to $L(\wp, \mathcal{R})$ in general class S.

3. Darboux expansion

Darboux Expansion

$$\langle L \rangle = \sum_{\gamma \in \Gamma_L} \overline{\underline{\Omega}}(L,\gamma)\, \mathcal{Y}_\gamma$$

$\overline{\underline{\Omega}}(L,\gamma)$ Framed BPS state degeneracies.

$\mathcal{Y}_\gamma$ Locally defined holomorphic functions on $\mathcal{M}$

At weak coupling, or at large R we can write them explicitly in terms of (u,θ) and parameters in the Lagrangian:

$$\log \mathcal{Y}_\gamma = \frac{R}{\zeta} Z_\gamma + i\, \gamma \cdot \theta + R\, \zeta \overline{Z}_\gamma + \mathcal{O}(e^{-\left(\frac{R}{g^2}\right)})$$

A Set Of ``Darboux Coordinates''

$$\mathcal{Y}_{\gamma_1}\mathcal{Y}_{\gamma_2} = \pm\,\mathcal{Y}_{\gamma_1+\gamma_2}$$

Choose basis γ_i for Γ gives a set of coordinates

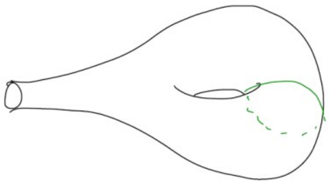
Conjecture: Same as:

Shear/Thurston/Penner/Fock-Goncharov coordinates

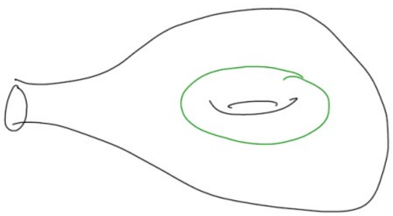
Checked in many cases.

$\langle L \rangle$ is a Laurent polynomial in these coordinates

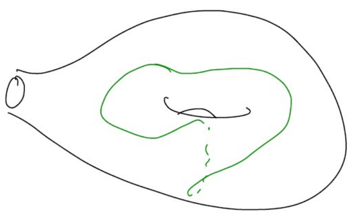
Example: SU(2) $\mathcal{N} = 2^*$



$$\langle L_{0,1} \rangle = Tr\ A = \alpha$$



$$\langle L_{1,0} \rangle = Tr\ B = \beta$$



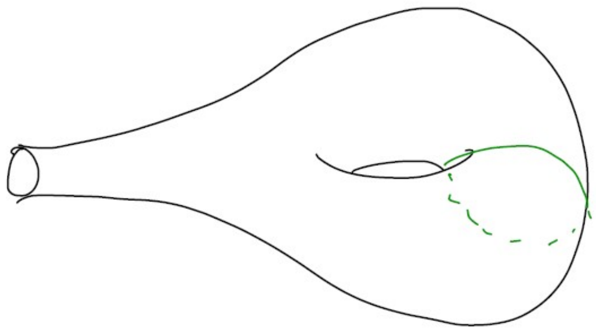
$$\langle L_{1,1} \rangle = Tr\ AB = \gamma$$

Can reduce $Tr(W)$ W = any word in $A^{\pm 1}, B^{\pm 1}$ to polynomial in α, β, γ

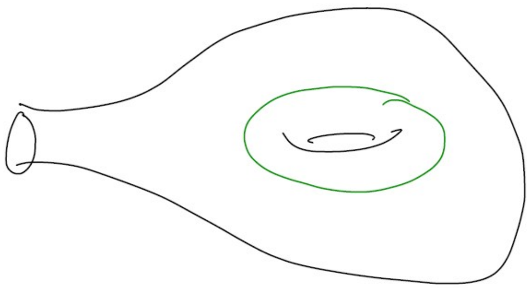
$$x \in SL(2, \mathbb{C}) \Rightarrow \quad x + x^{-1} = 1 \cdot Tr(x)$$

$$e^{2\pi i\, m} + e^{-2\pi i\, m} = Tr(ABA^{-1}B^{-1}) = \alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 2$$

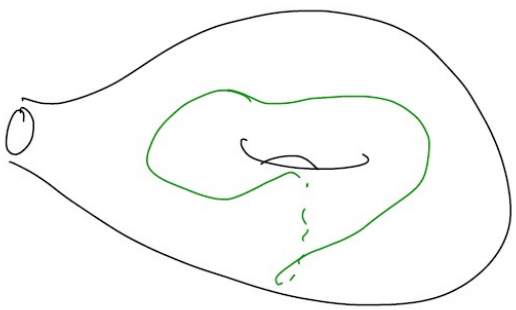
Shear Coordinates On $\mathcal{M}$



$$\langle L \rangle = yz + \frac{1}{yz} + \frac{\mathrm{z}}{\mathrm{y}}$$



$$\langle L \rangle = xz + \frac{1}{xz} + \frac{\mathrm{x}}{\mathrm{z}}$$



$$\langle L \rangle = xy + \frac{1}{xy} + \frac{\mathrm{y}}{\mathrm{x}}$$

$x, y, z \sim \mathcal{Y}_{\gamma_i}$ $\qquad$ $xyz = i\, e^{-i\,\pi\, m}$

Relation Of Shear Coordinates To Physical Quantities

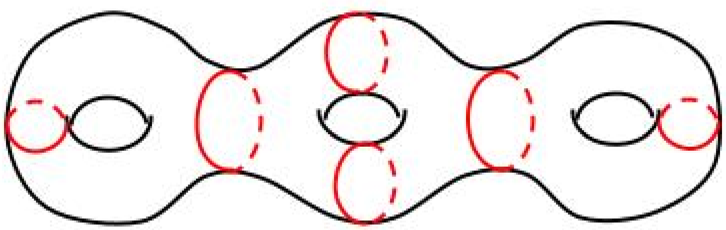
$$\log x = \frac{R}{2\zeta}(m - a) - \frac{i}{2}\theta_e + \frac{R\zeta}{2}(\bar{m} - \bar{a}) + NP$$

$$\log y = -\frac{R}{2\zeta}a_D - \frac{i}{2}\theta_m - \frac{R\zeta}{2}\overline{a_D} + NP$$

$$\log z = \frac{R}{2\zeta}(a_D + a) + \frac{i}{2}(\theta_e + \theta_m) + \frac{R\zeta}{2}(\overline{a_D} + \bar{a}) + NP$$

Complexified Fenchel-Nielsen Coordinates

Localization and AGT formulae are expressed in terms of CFN coords:



[Nekrasov, Rosly, Shatashvili; Dimofte & Gukov]

Half the coordinates: $Pexp \oint_{c_i} \mathcal{A}$ $\qquad e^{2\pi i\, \mathfrak{a}_i} \in \mathfrak{t}_{\mathbb{C}}$

$\mathcal{M}$ is holomorphic symplectic: $\varpi := \int_C Tr(\delta\mathcal{A} \wedge \delta\mathcal{A})$

Darboux-conjugate coordinates: $\varpi = \sum_i \langle d\mathfrak{a}_i \wedge d\mathfrak{b}^i \rangle$

$$\mathfrak{b} \to \mathfrak{b} + f(\mathfrak{a})$$

General Form Of Localization Answers

$$\langle \mathbb{L}[P,0] \rangle_y = \sum_{v \in \Lambda_{cochar}(G)} e^{2\pi i\, v \cdot \mathfrak{b}} Z_{P,v}(\mathfrak{a}, y)$$

GOP [For S^4] IOT [For $\mathbb{R}^3 \times S^1$]

$$Z_{P,v}(\mathfrak{a}, y) = Z_{P,v}^{1-loop}(\mathfrak{a}, y) Z_{P,v}^{monopole}(\mathfrak{a}, y)$$

$Z_{P,v}^{monopole}(\mathfrak{a}, y)$ Sums over tuples of Young diagrams

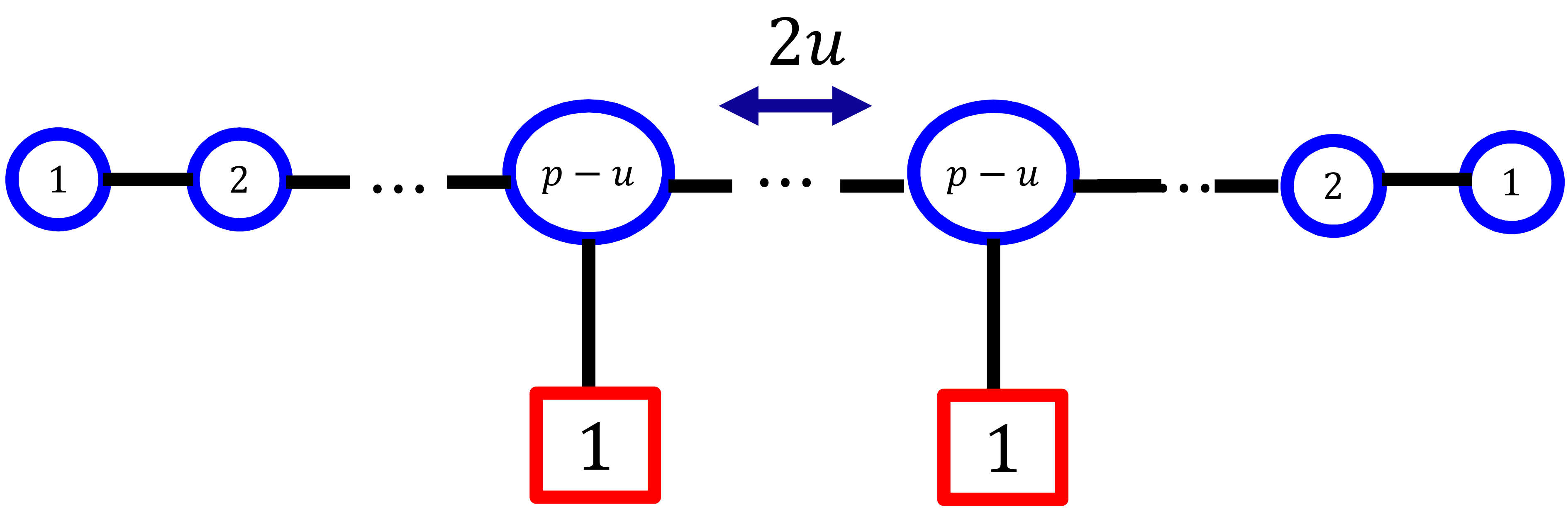
Localization of path integral to some subset $\mathcal{M}(P,v)$ of a monopole bubbling locus in the sense of Kapustin & Witten.

Some New Results

Work in progress with Anindya Dey & Daniel Brennan

$\mathcal{M}(P, v)$ is just a quiver variety

Example: $G = SU(2)$ $\quad P = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix}$ $\quad v = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix}$



General Prescription

Kronheimer correspondence: Identify singular monopoles with U(1)-invariant instantons on TN

Bubbling locus: U(1) invariant instantons at NUT point

Identify with U(1)-invariant instantons on $\mathbb{C}^2$

Make ADHM complex U(1) equivariant: As U(1) modules:

$$W(P) = W(v) + (\rho - 2 + \bar{\rho})V$$

Kapustin & Witten

$$\iota : \mathbb{Z}_n \hookrightarrow U(1)$$

$$\iota^*\left(W(v) \otimes \rho_q\right) = \oplus_{i=0}^{n-1} W_i \otimes R_i$$

$$\iota^*\left(V \otimes \rho_q\right) = \oplus_{i=0}^{n-1} V_i \otimes R_i$$

Stabilizes for $n > N_0(v, q)$.

Expressions For $Z_{P,v}^{monopole}$

Moreover, we observe that for SU(N) $\mathcal{N} = 2^*$,
the answer found by IOT also agrees with the Witten
index of the SQM for this quiver:

$$Z_{P,v}^{monopole} = Z_{quiver\ SQM} = \int_{\mathcal{M}(P,v)} e^{\omega+\mu\cdot\mathfrak{a}} \chi_y(\mathfrak{a})$$

$$= \oint_{\mathrm{t}} [d\phi] Z^{vm} Z^{hm}$$

[Moore, Nekrasov, Shatashvili 1997]

Remark: The same functions are claimed by
Bullimore-Dimofte-Gaiotto to appear in an ``abelianization
map'' for monopole operators in d=3 N=4 gauge theories.

Relation Between Coordinates?

Both shear and CFN coordinates are holomorphic Darboux coordinates

$\langle L \rangle$ has a finite Laurent expansion in both.

But the relation between them is very complicated !

Comparison with Darboux expansion in shear coordinates in a weak-coupling regime shows:

$$2\pi i\, \mathfrak{a} = \frac{R}{\zeta} a + i\, \theta_e + R\zeta \bar{a} + NonPerturbative$$

$$2\pi i\, \mathfrak{b} = \frac{R}{\zeta} a_D + i\, \theta_m + R\zeta \overline{a_D} + NonPerturbative$$

N.B. Literature misses the nonperturbative corrections.

Localization Results For SU(2) $\mathcal{N} = 2^*$

$$\langle L_{0,1} \rangle = \lambda + \lambda^{-1} \qquad \lambda = e^{2\pi i\, \mathfrak{a}}$$

$$\langle L_{1,0} \rangle = (\beta + \beta^{-1})F \qquad \langle L_{1,1} \rangle = (\beta\lambda + \beta^{-1}\lambda^{-1})F$$

$$\beta = e^{2\pi i\, \mathfrak{b}}$$

$$\ell = e^{i\,\pi m} \qquad F = \frac{(\lambda^2 + \lambda^{-2} - \ell^2 - \ell^{-2})^{\frac{1}{2}}}{\lambda - \lambda^{-1}}$$

$$\langle L_{2,q} \rangle = (\beta^2\, \lambda^q + \beta^{-2}\lambda^{-q})F^2 + (\lambda + \lambda^{-1})(F^2 - 1)$$

Valid for q odd.

Heroic computation by Anindya Dey using AGT approach.

Can also be done in shear coordinates
but with more complicated answer.

Comparison Of Coordinates In SU(2) $\mathcal{N} = 2^*$

$$x = \frac{\frac{i}{\ell}\left(\tilde{\beta} - \tilde{\beta}^{-1}\right)}{\tilde{\beta}\lambda - \left(\tilde{\beta}\lambda\right)^{-1}} \qquad y = i\,\frac{\tilde{\beta}\lambda - \tilde{\beta}^{-1}\lambda^{-1}}{\lambda - \lambda^{-1}} \qquad z = -i\,\frac{\lambda - \lambda^{-1}}{\tilde{\beta} - \tilde{\beta}^{-1}}$$

$$\tilde{\beta} = \beta\left(\frac{\lambda\ell - \lambda^{-1}\ell^{-1}}{\lambda\ell^{-1} - \lambda^{-1}\ell}\right)^{\frac{1}{2}}$$

Dimofte & Gukov, 2011

Inverting these equations and using the weak coupling expansion of x,y,z gives weak coupling expansion of complexified FN coordinates.

It's the only way I know to express CFN coordinates in a weak-coupling expansion.

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New Superconformal Theories From Old

Given a superconformal theory T and a $\beta = 0$ subgroup $H \subset Glob(T)$ we can gauge it to form a new superconformal theory T/H.

In particular, given two theories with a common subgroup $H \subset Glob(T_1)$ and $H \subset Glob(T_2)$ and a $\beta = 0$ embedding:

$$H \hookrightarrow Glob(T_1) \times Glob(T_2)$$

Gauge the embedded H with gauge-coupling q to produce $T_1 \times_{H,q} T_2$

Argyres-Seiberg, 2007

Class S

$\mathfrak{g}$ = simple A,D, or E Lie algebra

$C_{g,n}$ Riemann surface with (possibly empty) set of punctures $p_1, p_2, \dots, p_n$

D = collection of ½-BPS cod=2 defects $D(p_1), \dots, D(p_n)$

For suitable D the theory $S[\mathfrak{g}, C, D]$ is superconformal

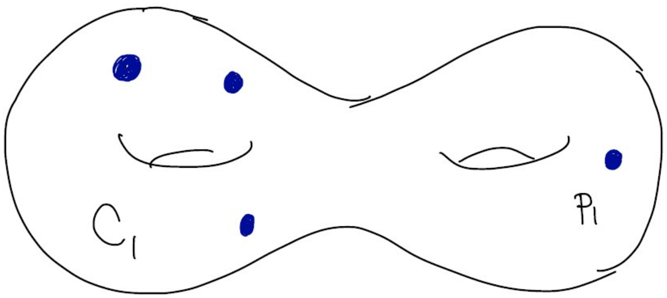
Lie algebra of global symmetry contains:

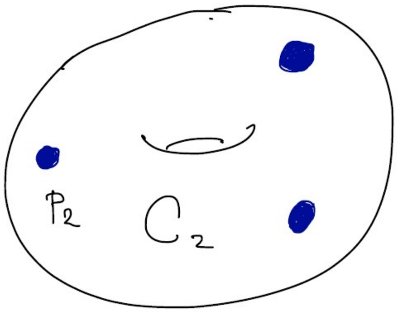
$$\oplus_{p_i} \mathfrak{f}(D(p_i))$$

``Full (maximal) puncture'' : $\mathfrak{f}(D) = \mathfrak{g}$

Gaiotto Gluing – 1/2

Given $S[\mathfrak{g}, C_1, D_1]$ & $S[\mathfrak{g}, C_2, D_2]$





Suppose we have full punctures $D(p_1)$ & $D(p_2)$ with $p_1 \in C_1$ & $p_2 \in C_2$

The diagonal $\mathfrak{g}$ – symmetry $\mathfrak{g}_{diag} \subset \mathfrak{g} \oplus \mathfrak{g}$ has $\beta = 0$

Gauge it to produce a new superconformal theory:

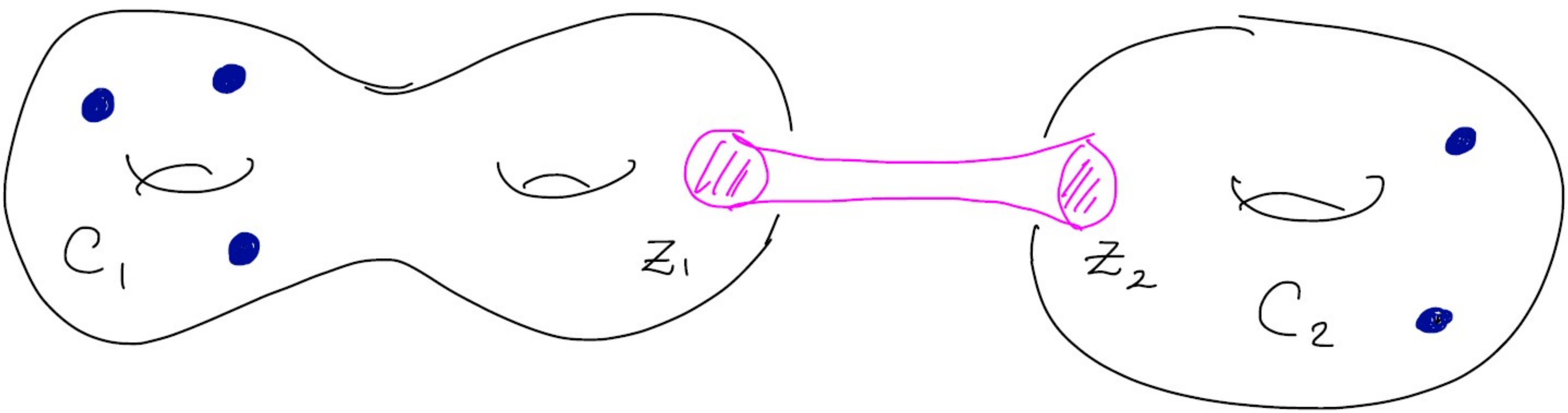
$$S[\mathfrak{g}, C_1, D_1] \times_{\mathfrak{g},q} S[\mathfrak{g}, C_2, D_2] \qquad q = e^{2\pi i \tau}$$

Gaiotto Gluing -2/2

$$S[\mathfrak{g}, C_1, D_1] \times_{\mathfrak{g},q} S[\mathfrak{g}, C_2, D_2]$$

$$=$$

$$S[\mathfrak{g}, C_1 \times_q C_2, D_1 \cup D_2 - \{D(p_1), D(p_2)\}]$$



$$z_1 z_2 = q$$

Theories Of Class H

Ongoing work with J. Distler, A. Neitzke, W. Peelaers & D. Shih.

$$S[\mathfrak{g}_1, C_1, D_1] \quad \& \quad S[\mathfrak{g}_2, C_2, D_2]$$

$$\mathfrak{g}_1 \neq \mathfrak{g}_2$$

$$\mathfrak{h} \subset \mathfrak{f}(D(p_1)) \quad \& \quad \mathfrak{h} \subset \mathfrak{f}(D(p_2))$$

$$\mathfrak{h}_{diag} \subset \mathfrak{f}(D(p_1)) \oplus \mathfrak{f}(D(p_2)) \qquad \beta(\mathfrak{h}_{diag}) = 0$$

$$S[\mathfrak{g}_1, C_1, D_1] \times_{\mathfrak{h}_{diag}, q} S[\mathfrak{g}_2, C_2, D_2]$$

Partial No-Go Theorem

Important class of punctures: ``Regular Punctures''

$$D(\mathfrak{g}, \omega, \rho) \qquad \rho: \mathfrak{su}(2) \to (\mathfrak{g}^{\omega})^{\vee}$$

Theorem: Gluing two regular punctures is only superconformal for the case of full punctures. In particular: $\mathfrak{g}_1 = \mathfrak{g}_2$

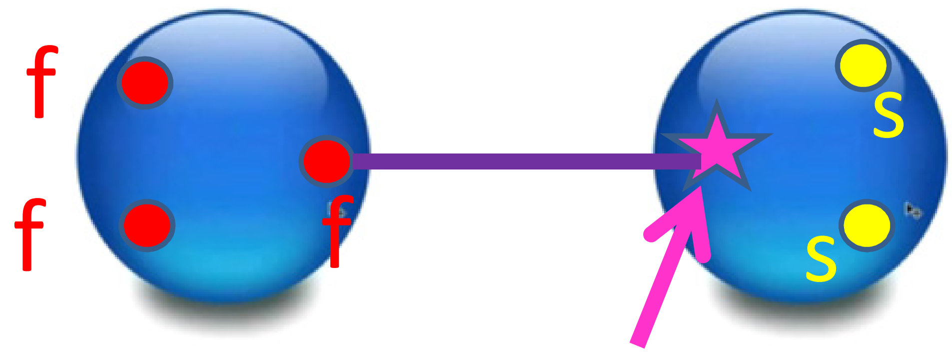
Proof: Condition for $\beta\left(\mathfrak{h}_{diag}\right) = 0$:

$$-4h^{\vee}(\mathfrak{h}) + \kappa_1 + \kappa_2 = 0$$

Use nontrivial formulae for κ from Chacaltana, Distler, and Tachikawa.

Other Punctures

But! There are other types of punctures!



"Superconformal irregular puncture" (SIP)

If you can now insert SIP's just like other punctures then there appear to be Hippogriff theories.

Geometrical interpretation?

Seiberg-Witten curve?

AdS duals?

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