The RR charge of orientifolds

Oberwolfach

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and work in progress with

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Outline

1. Statement of the problem & motivation
2. What is an orientifold?
3. B-field: Differential cohomology
4. RR Fields I: Twisted KR theory
5. Generalities on self-dual theories
6. RR Fields II: Quadratic form for self-duality
7. O-plane charge
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9. Topological restrictions on the B-field
10. Precis
Statement of the Problem

What is the RR charge of an orientifold?

That’s a complicated question

a.) What is an orientifold?

b.) What is RR charge?
Motivation

- The evidence for the alleged ``landscape of string vacua'' (d=4, N=1, with moduli fixed) relies heavily on orientifold constructions.

- So we should put them on a solid mathematical foundation!

- Our question for today is a basic one, of central importance in string theory model building.

- Puzzles related to S-duality are sharpest in orientifolds

- Nontrivial application of modern geometry & topology to physics.
Question a: What is an orientifold?

Perturbative string theory is, by definition, a theory of integration over a space of maps:

\[ \varphi : \Sigma \rightarrow \mathcal{X} \]

\( \Sigma \): 2d Riemannian surface  
\( \mathcal{X} \): Spacetime endowed with geometrical structures: Riemannian,…

\[ \exp \left[ - \int_{\Sigma} \frac{1}{2} \| d\varphi \|^2 + \cdots \right] \]
What is an orbifold?

Let’s warm up with the idea of a string theory orbifold

\[ \varphi : \Sigma \to X \]

\( X \) is smooth with finite isometry group \( \Gamma \)

Gauge the \( \Gamma \)-symmetry:

\[ \hat{\Sigma} \xrightarrow{\hat{\varphi}} \hat{X} \]

\[ \Sigma \xrightarrow{\varphi} X \]

\[ \mathcal{X} = \frac{X}{\Gamma} \]

Principal \( \Gamma \) bundle

Physical worlds sheet

Spacetime groupoid
For orientifolds, $\tilde{\Sigma}$ is oriented,

In addition: $1 \to \Gamma_0 \to \Gamma \xrightarrow{\omega} \mathbb{Z}_2 \to 1$

$$\Gamma_0: \omega(\gamma) = +1 \quad \Gamma_1: \omega(\gamma) = -1$$

On $\tilde{\Sigma}$:
$\Gamma_0$: Orientation preserving
$\Gamma_1$: Orientation reversing

Orientation double cover

Unoriented
Orientifold Planes

For orientifold spacetimes $X//\Gamma$

a component of the fixed locus point of

$$g \in \Gamma_1$$

is called an "orientifold plane."
More generally, spacetime is an `orbifold,`

**In particular, \( \mathcal{X} \) is a groupoid**

(c.f. Adem, Leida, Ruan, *Orbifolds and Stringy Topology*)

There is an isomorphism

\[
\varphi^*(w) \cong w_1(\Sigma)
\]

Definition: An **orientifold** is a string theory defined by integration over such maps.
Worldsheet Measure

In string theory we integrate over "worldsheets"

For the bosonic string, space of "worldsheets" is

\[S = \{ (\Sigma, \varphi) \} = \text{Moduli}(\Sigma) \times \text{MAP}(\Sigma \rightarrow X)\]

\[
\exp[- \int_{\Sigma/S} \frac{1}{2} \| d\varphi \|^2] \cdot A_B
\]

\[A_B = \exp[2\pi i \int_{\Sigma/S} \varphi^*(B)]\]

B is locally a 2-form gauge potential…
Differential Cohomology Theory

In order to describe $B$ we need to enter the world of differential generalized cohomology theories...

If $\mathcal{E}$ is a generalized cohomology theory, then denote the differential version $\check{\mathcal{E}}$

\[
0 \rightarrow \mathcal{E}^{j-1}(M, \mathbb{R}/\mathbb{Z}) \rightarrow \check{\mathcal{E}}^j(M) \rightarrow \Omega_{\mathbb{Z}}(M; \mathcal{E}(pt) \otimes \mathbb{R})^j \rightarrow 0
\]

\[
0 \rightarrow \text{[top.triv.]} \rightarrow \check{\mathcal{E}}^j(M) \rightarrow \mathcal{E}^j(M) \rightarrow 0
\]
Variations

We will need **twisted** versions on **groupoids**

Both generalizations are nontrivial.

**Main Actors**

- **B-field**: Twisted differential cohomology
- **RR-field**: Twisted differential KR theory
Orientation & Integration

The orientation twisting of $\mathcal{E}(M)$, denoted $\tau_{\mathcal{E}}(M)$, allows us to define an "integration map" in $\mathcal{E}$-theory:

$$\int_{M}^{\mathcal{E}} : \mathcal{E}\tau_{\mathcal{E}}(M) + j(M) \to \mathcal{E}^{j}(pt)$$

Also extends to integration in differential theory.
Where does the B-field live?

For the oriented bosonic string its gauge equivalence class is in $\check{H}^3(\chi)$.

For a bosonic string orientifold its equivalence class is in $\check{H}^3 + w(\chi)$.

$A_B = \exp[2\pi i \int_{\Sigma/S} \varphi^*(\tilde{\beta})] \in \check{H}^1(S)$

Integration makes sense because $\varphi^*(w) \cong w_1(\Sigma)$.

Surprise!! For superstrings: not correct!
Superstring Orientifold B-field

Turns out that for superstring orientifolds

\[ 0 \to \tilde{H}^{3+w}(\mathcal{X}) \to \tilde{B}^{3+w}(\mathcal{X}) \to H^0(\mathcal{X}, \mathbb{Z}) \times H^1(\mathcal{X}, \mathbb{Z}_2) \to 0 \]

Necessary for worldsheet theory:
c.f. Talk at Singer85 (on my homepage) and a paper to appear soon.

That’s all for today about question (a)
Question b: What is RR Charge?

Type II strings have “RR-fields” – Abelian gauge fields whose fieldstrengths are forms of fixed degree in

\[ \Omega^*(\mathcal{X}, \mathbb{R}[u, u^{-1}]) \quad \text{deg}(u) = 2 \]

e.g. in IIB theory degree = -1:

\[ G = u^{-1}G_1 + u^{-2}G_3 + \cdots + u^{-5}G_9 \]
Naïve RR charge

In string theory there are sources of RR fields:

\[ dG = j = \text{RR current} \]

Naively: \([j] \in H^*_{\text{deRham}} = \text{RR charge}\]

This notion will need to be refined…
Sources for RR Fields

Worldsheet computations show there are two sources of RR charge:

- D-branes
- Orientifold planes

Recall that for $X/\Gamma$

a component of a fixed point locus of $g \in \Gamma_1$ is called an "orientifold plane."

Our goal is to define precisely the orientifold plane charge and compute it as far as possible.
K-theory quantization

The D-brane construction implies

\[ [j] \in K(X) \]

Minasian & Moore

So RR current naturally sits in \( \tilde{K}(X) \)

\( (X \to \mathcal{X} \text{ is a nontrivial generalization}) \)
KR and Orientifolds

Action of worldsheet parity on Chan-Paton factors

For orientifolds replace

\[ K(\mathcal{X}) \rightarrow KR(\mathcal{X}_w) \]

(Witten; Gukov; Hori; Bergman, Gimon, Horava; Bergman, Gimon, Sugimoto; Brown & Stefanski, …)

What is \( KR(\mathcal{X}_w) \)?
For $\mathcal{X} = X/\Gamma$ use Fredholm model (Atiyah, Segal, Singer)

$\mathcal{H}$: $\mathbb{Z}_2$-graded Hilbert space with stable $\Gamma$-action

$\Gamma_0$: Is linear $\quad \Gamma_1$: Is anti-linear

$\mathcal{F}$: Skew-adjoint odd Fredholms

$$KR(\mathcal{X}_w) := [X \to \mathcal{F}]^\Gamma$$

This fits well with `tachyon condensation.'
We need **twisted** KR-theory...

Following Witten and Bouwknegt & Mathai, we will interpret the B-field as a (differential) twisting of (differential) KR theory.

It is nontrivial that this is compatible with what we found from the worldsheet viewpoint.

As a bonus: This point of view nicely organizes the zoo of K-theories associated with various kinds of orientifolds found in the physics literature.
Twistings

• We will consider a special class of twistings with geometrical significance.

• We will consider the degree to be a twisting, and we will twist by a ``graded gerbe."

• We now describe a simple geometric model
Double-Covering Groupoid

Spacetime $\mathcal{X}$ is a groupoid:

$\mathcal{X}: X_0 \leftrightarrow X_1 \leftrightarrow X_2$

Homomorphism: $\epsilon_w : X_1 \rightarrow \mathbb{Z}_2$

$\epsilon_w(gf) = \epsilon_w(g) + \epsilon_w(f)$

Double cover: $X_{w,1} := \ker \epsilon_w$

Defines $\mathcal{X}_w$
Twisting KR Theory

Def: A twisting of $KR(X_w)$ is a quadruple $\tau = (d, L, \epsilon_a, \theta)$

Degree $d : X_0 \to \mathbb{Z}$

$L$ is a line bundle on $X_1$ $\mathbb{Z}_2$-grading: $\epsilon_a : X_1 \to \mathbb{Z}_2$

Cocycle: $\theta_{g,f} : \epsilon_w(f) L_g \otimes L_f \to L_{gf}$

$\epsilon V := \begin{cases} V & \epsilon = 0 \\ \bar{V} & \epsilon = 1 \end{cases}$
Twistings of KR

Topological classes of twistings of $KR(\mathcal{X}_w)$

$$H^0(\mathcal{X}; \mathbb{Z}) \times H^1(\mathcal{X}; \mathbb{Z}_2) \times H^{3+w}(\mathcal{X}; \mathbb{Z})$$

Abelian group structure:

$$(d_1, a_1, h_1) + (d_2, a_2, h_2)$$

$$= (d_1 + d_2, a_1 + a_2, h_1 + h_2 + \tilde{\beta}(a_1 a_2))$$
The Orientifold B-field

So, the B-field is a geometric object whose topological class is

\[
[\beta] = (d, a, h) \in H_\mathbb{Z}^0 \times H_{\mathbb{Z}_2}^1 \times H_{\mathbb{Z}}^{w+3}
\]

\(d=0,1 \mod 2: \text{ IIB vs. IIA.}\)

\(a: \text{Related to } (-1)^F \text{ & Scherk-Schwarz}\)

\(h: \text{is standard}\)
Bott Periodicity

For $\mathcal{X} = \varnothing := pt//\mathbb{Z}_2$

$$H^0(\varnothing; \mathbb{Z}) \times H^1(\varnothing; \mathbb{Z}_2) \times H^{3+w}(\varnothing; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_4$$

We refer to these as ``universal twists”

$$B = d + \beta_\ell$$

Bott element $u \in KR^{2+\beta_1}(pt)$

Note $(\mathbb{Z} \oplus \mathbb{Z}_4)/\langle\langle(2, 1)\rangle\rangle \cong \mathbb{Z}_8$

So, there are 8 distinct universal $B$-fields
Twisted KR Class

1. $\mathbb{Z}_2$ graded $E \to X_0$ with odd skew Fredholm with graded $C\ell(d)$ action

2. On $X_1$ we have gluing maps:

\[ f : x_0 \to x_1 \quad \psi_f : \varepsilon_w(f) \left( L_f \otimes E_{x_0} \right) \to E_{x_1} \]

3. On $X_2$ we have a cocycle condition:
Where RR charge lives

The RR current is

\[ [\tilde{j}] \in \mathcal{K}R^\beta (\mathcal{X}_w) \]

The charge is an element of

\[ KR^\beta (\mathcal{X}_w) \]

Next: How do we define which element it is?
The RR field is self-dual

The key to defining and computing the background charge is the fact that the RR field is a **self-dual theory**.

How to formulate self-duality?
Generalized Maxwell Theory

(A naïve model for the RR fields )

\[
\dim \mathcal{X} = n \quad [\tilde{A}] \in \tilde{H}^{d-1}(\mathcal{X})
\]

\[
dF = J_m \in \Omega^d(\mathcal{X})
\]

\[
d \ast F = J_e \in \Omega^{n+2-d}(\mathcal{X})
\]

Self-dual setting: \( F = \ast F \) & \( J_m = J_e \)
Consideration of three examples:

1. Self-dual scalar: \( n=2 \) and \( d=2 \)

2. M-theory 5-brane: \( n=6 \) and \( d=4 \)

3. Type II RR fields: \( n=10 \) & \( \text{GCT} = K \)

has led to a general definition (Freed, Moore, Segal)

We need 5 pieces of data:
General Self-Dual Theory: Data

1. Poincare-Pontryagin self-dual mult. GCT

\[ \mathcal{E}^\tau(M, \mathbb{R}/\mathbb{Z}) \times \mathcal{E}^{\tau\varepsilon}(M) - \tau - s(M) \to \mathbb{R}/\mathbb{Z} \]

degree \( \to \tau \quad \iota : \mathcal{E}^{-s}(pt; \mathbb{R}/\mathbb{Z}) \to I^0(pt; \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z} \)

For a spacetime \( \mathcal{X} \) of dimension \( n \)

\[ [\tilde{j}] \in \tilde{\mathcal{E}}^\tilde{\tau}(\mathcal{X}) \]
2. Families of Spacetimes

\[ \dim \mathcal{X}/\mathcal{P} = n \]

\[ \dim \mathcal{Y}/\mathcal{P} = n + 1 \]

\[ \dim \mathcal{Z}/\mathcal{P} = n + 2 \]
3. Isomorphism of Electric & Magnetic Currents

\[ \theta_0 : \tilde{\mathcal{E}}^\tau (\mathcal{X}) \rightarrow \tilde{\mathcal{E}}^\tau (\mathcal{X}) - \tau + 2 - s (\mathcal{X}) \]

\[ \theta_1 : \tilde{\mathcal{E}}^\tau (\mathcal{Y}) \rightarrow \tilde{\mathcal{E}}^\tau (\mathcal{Y}) - \tau + 1 - s (\mathcal{Y}) \]

\[ \theta_2 : \tilde{\mathcal{E}}^\tau (\mathcal{Z}) \rightarrow \tilde{\mathcal{E}}^\tau (\mathcal{Z}) - \tau - s (\mathcal{Z}) \]

4. Symmetric pairing of currents:

\[ b_0 (\tilde{j}_1, \tilde{j}_2) = i \int_{\mathcal{X}} \mathcal{E} \theta_0 (\tilde{j}_1) \tilde{j}_2 \in \tilde{I}^2 (\mathcal{P}) \]

\[ b_1 (\tilde{j}_1, \tilde{j}_2) = i \int_{\mathcal{Y}} \mathcal{E} \theta_1 (\tilde{j}_1) \tilde{j}_2 \in \tilde{I}^1 (\mathcal{P}) \]

\[ b_2 (\tilde{j}_1, \tilde{j}_2) = i \int_{\mathcal{Z}} \mathcal{E} \theta_2 (\tilde{j}_1) \tilde{j}_2 \in \tilde{I}^0 (\mathcal{P}) \]
5. Quadratic Refinement

\[ q_i(\tilde{j}_1 + \tilde{j}_2) - q_i(\tilde{j}_1) - q_i(\tilde{j}_2) + q_i(0) = b_i(\tilde{j}_1, \tilde{j}_2) \]

\[ q_0(\tilde{j}) \in \tilde{I}^2(\mathcal{P}) = \mathbb{Z}_2\text{-graded line bundles over } \mathcal{P} \text{ with connection} \]

\[ q_1(\tilde{j}) \in \tilde{I}^1(\mathcal{P}) = Map(\mathcal{P}, \mathbb{R}/\mathbb{Z}) \]

\[ q_2(\tilde{j}) \in \tilde{I}^0(\mathcal{P}) = Map(\mathcal{P}, \mathbb{Z}) \]
Formulating the theory

Using these data one can formulate a self dual theory.

The topological data of $q_2$ and $\theta_2$ in $(n + 2)$ dimensions determines $q_1, q_0$.
Physical Interpretation: Holography

Generalizes the well-known example of the holographic duality between 3d abelian Chern-Simons theory and 2d RCFT.

See my Jan. 2009 AMS talk on my homepage for this point of view, which grows out of the work of Witten and Hopkins & Singer, and is based on my work with Belov and Freed & Segal.
Holographic Formulation

\[ \hat{A} \in \mathcal{E}^\tau(\mathcal{Y}) : \text{Chern-Simons gauge field.} \]

\[ q_1(\hat{A}) \in \text{Map}(\mathcal{P}, \mathbb{R}/\mathbb{Z}) : \text{Chern-Simons action.} \]

\[ \hat{A}|_x = \hat{j} \]

Edge modes = self-dual gauge field

Chern-Simons wavefunction = Self-dual partition function

\[ \Psi(\hat{A}|_x) = Z(\hat{j}) \]
Defining the Background Charge - I

Identify automorphisms of $\tilde{j}$ with $\alpha \in \mathcal{E}^{r-2}(\mathcal{X}, \mathbb{R}/\mathbb{Z})$

Identify these with global gauge transformations

Automorphisms act on CS wavefunction

$$(\alpha \cdot \Psi)(\tilde{j}) = e^{2\pi i q_1 (\tilde{j} + \tilde{\alpha} \tilde{t})} \Psi(\tilde{j})$$

Global gauge transformations:

$$(\alpha \cdot \Psi)(\tilde{j}) = e^{2\pi i \alpha \cdot \mathcal{Q}} \Psi(\tilde{j})$$
Defining the Background Charge - II

\[ e^{2\pi i \alpha \cdot \mathcal{Q}} \Psi(\tilde{j}) = e^{2\pi i q_1 (\tilde{j} + \tilde{\alpha} \tilde{t})} \Psi \]

\[ q_1 (\tilde{j} + \tilde{\alpha} \tilde{t}) = \int \theta(j) \alpha + q_1 (\tilde{\alpha} \tilde{t}) \]

\[ \alpha \rightarrow q_1 (\tilde{\alpha} \tilde{t}) \text{ is linear} \]

Poincare duality: \[ q_1 (\tilde{\alpha} \tilde{t}) := \iota \int_{\mathcal{X}} \theta(\mu) \alpha \]

\[ \mu \in \mathcal{E}^\tau (\mathcal{X}) \quad \text{``background charge''} \]
Computing Background Charge

A simple argument shows that twice the charge is computed by

$$q_1(y) - q_1(-y) = \int_{\mathcal{E}} \theta_0(-2\mu) \alpha$$

$$y = \alpha t$$

Heuristically:

$$q_1(y) = \frac{1}{2} (y - \mu)^2 + \text{const.}$$
Self-Duality for Type II RR Field

Now $[\tilde{j}] \in K(\mathbb{Z})$ and $\dim \mathbb{Z}/\mathcal{P} = 12$

It turns out that

$$q_2(j) = \int_{\mathbb{Z}}^{K_{\mathbb{O}}} \tilde{j}j \in \mathbb{Z}$$

correctly reproduces many known facts in string theory and M-theory

Witten 99, Moore & Witten 99, Diaconescu, Moore & Witten, 2000, Freed & Hopkins, 2000, Freed 2001
Self-duality for Orientifold RR field

Now \( j \in KR^\beta(\mathcal{Z}) \) and \( \dim \mathcal{Z}/\mathcal{P} = 12 \)

We want to make sense of a formula like

\[
q_2(j) = \int_{\mathcal{Z}}^{KO} \bar{jj} \quad \bar{jj} \in \mathbb{Z}
\]

But \( \bar{jj} \in KR^{\bar{\beta} + \beta}(\mathcal{Z}) \), not in \( KO \)

And, we need a \( KO \) density!
The real lift

Lemma: There exists maps

\[ \mathcal{R} : \text{Twist}_{KR}(\mathcal{M}_w) \rightarrow \text{Twist}_{KO}(\mathcal{M}) \]

\[ \rho : KR^\beta(\mathcal{M}_w) \rightarrow KO^{\mathcal{R}(\beta)}(\mathcal{M}) \]

So that under complexification:

\[ \rho(j) \rightarrow u^{-d\bar{j}j} \]

\[ \mathcal{R}(\beta) \rightarrow \beta + \bar{\beta} - d\tau(u) \]
Twisted Spin Structure - I

In order to integrate in KO,

\[ \rho(j) \in KO^{\mathbb{R}(\beta)}(\mathcal{M}) \]

Must be an appropriately twisted density

For simplicity now take \( \mathcal{M} = M/\mathbb{Z}_2 \)

\[ \int_{M}^{KO_{\mathbb{Z}_2}} : KO^{\tau KO_{\mathbb{Z}_2}}(M) + j \rightarrow KO^{j}_{\mathbb{Z}_2}(pt) \]
Twisted Spin Structure- II

Definition: A twisted spin structure on $\mathcal{M}$ is

$$\kappa : \mathcal{R}(\beta) \cong \tau_{KO}(TM - \dim M)$$

Note: A spin structure on $M$ allows us to integrate in $KO$. It is an isomorphism

$$0 \cong \tau_{KO}(TM - \dim M)$$

Existence of tss  Topological conditions on B
Orientifold Quadratic Refinement

\[ \int_{\mathbb{Z}}^{KO} \kappa \rho(j) \in KO_{\mathbb{Z}_2}^{-12}(pt) \]

\[ KO_{\mathbb{Z}_2}^{-12}(pt) \cong KO^{-4}(pt) \otimes (\mathbb{Z} \oplus \mathbb{Z}_\varepsilon) \]

\[ \iota : KO^{-4}(pt) \rightarrow I^0(pt) \cong \mathbb{Z} \]

Definition: \( q_2(j) := [\iota \int_{\mathbb{Z}}^{KO} \kappa \rho(j)]_\varepsilon \)
At this point we have defined the "background RR charge" of an orientifold spacetime.

How about computing it?
Localization of the charge on $X/\mathbb{Z}_2$

$$q_1(y) - q_1(-y) = \int_X \theta_0(-2\mu) \alpha$$

$$\left\{ \iota \int_{Y}^{K\mathbb{O}\mathbb{Z}_2} [\rho(y)) - \rho(-y)] \right\}_\varepsilon = \iota \int_{Y}^{K\mathbb{R}} \theta(-2\mu) y$$

Localize wrt $S = \{(1 - \varepsilon)^n\} \subset R(\mathbb{Z}_2)$

Atiyah-Segal localization theorem

Background charge with 2 inverted localizes on the O-planes.
K-theoretic O-plane charge

\[ \mu = i_*(\Lambda) \in KR^\beta[\frac{1}{2}](X) \]

\(i : F \hookrightarrow X\) \(\nu = \text{Normal bundle}\)

\[ \Psi(\Lambda) = 2^d \frac{C(F)}{\text{Euler}(\nu)} \]

``Adams Operator'' \(\Psi : KR^\beta[\frac{1}{2}](Y) \to S^{-1}KO_{\mathbb{Z}_2}^{\text{Re}(\beta)}(Y)\)

\(C(F)\) KR-theoretic Wu class generalizing Bott's cannibilistic class
Special case: Type I String

Freed & Hopkins (2001)

Type I theory: $\mathcal{X} = X//\mathbb{Z}_2$

With $\mathbb{Z}_2$ acting trivially and $\beta = 0$

$$2\mu = -\Xi(X)$$

$\Xi(F)$: KO-theoretic Wu class

$$\int_F^{KO} \psi_2(x) = \int_F^{KO} \Xi(F)x$$

$$-\mu = TX + 22 + \text{Filt}(\geq 8)$$
The physicists’ formula

Taking Chern characters we get the physicist’s (Morales-Scrucca-Serone) formula for the charge in de Rham cohomology:

\[ -\sqrt{A(TX)ch(\mu)} = \pm 2^k \iota_* \sqrt{\frac{\tilde{L}(TF)}{\tilde{L}(\nu)}} \]

\[ \tilde{L}(V) := \prod_i \frac{x_i/4}{\tanh(x_i/4)} \]

\[ k = \dim F - 5 \]
Topological Restrictions on the B-field

One corollary of the existence of a twisted spin structure is a constraint relating the topological class of the B-field to the topology of $X$

\begin{align*}
  w_1(X) &= dw \\
  w_2(X) &= \frac{d(d+1)}{2} w^2 + aw
\end{align*}

\[
  [\beta] = (d, a, h)
\]

This general result unifies scattered older observations in special cases.
Examples

**Zero B-field**
If $[\beta]=0$ then we must have IIB theory on $X$ which is orientable and spin.

**Op-planes**

$X = \mathbb{R}^{p+1} \times \mathbb{R}^r \mod \mathbb{Z}_2 \quad p + r = 9$

Compute:

$w_1(X) = rw \quad w_2(X) = \frac{r(r-1)}{2}w^2$

$d = r \mod 2 \quad a = \begin{cases} 0 & r = 0, 3 \mod 4 \\ w & r = 1, 2 \mod 4 \end{cases}$
Pinvolutions

\[ \mathcal{X} = X/\!/\mathbb{Z}_2 \]

Deck transformation \( \sigma \) on \( X \) lifts to \( \text{Pin}^- \) bundle.

<table>
<thead>
<tr>
<th>( r )</th>
<th>( KR^0(\mathcal{X}_w) )</th>
<th>( KR^{\beta_2}(\mathcal{X}_w) )</th>
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<tr>
<td>( r = 0 )</td>
<td>( KR^{1+\beta_1}(\mathcal{X}_w) )</td>
<td>( KR^{1+\beta_3}(\mathcal{X}_w) )</td>
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\( r = \text{cod. mod 4 of orientifold planes} \)
# Older Classification

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<th>K-group</th>
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<td>$KR_\pm(S^{9,0}) = \mathbb{Z} \oplus \mathbb{Z}_2$</td>
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<td>$O_1^-$</td>
<td>$KR^{-1}(S^{8,0}) = \mathbb{Z} \oplus \mathbb{Z}_2$</td>
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<td>$KR(S^{7,0}) = \mathbb{Z} \oplus \mathbb{Z}$</td>
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<td>$KH_\pm^{-1}(S^{2,0}) = \mathbb{Z}$</td>
</tr>
<tr>
<td>$O_8^+$</td>
<td>$KH_\pm(S^{1,0}) = \mathbb{Z}$</td>
</tr>
</tbody>
</table>

**Table 2:** Orientifield K-theory groups for RR fields.

(Bergman, Gimon, Sugimoto, 2001)
Orientifold Précis : NSNS Spacetime

1. $\mathcal{X}$: 10-dimensional Riemannian orbifold with dilaton.

2. Orientifold double cover $\mathcal{X}_w$, $w \in H^1(\mathcal{X}, \mathbb{Z}_2)$.

3. $B$: Differential twisting of $\tilde{KR}(\mathcal{X}_w)$

4. Twisted spin structure:
   \[ \kappa : \mathcal{R}(\beta) \cong \tau_{KO}(T\mathcal{X} - \dim \mathcal{X}) \]
Orientifold Précis: Consequences

1. Well-defined worldsheet measure.

2. K-theoretic definition of the RR charge of an orientifold spacetime.

3. RR charge localizes on O-planes after inverting two, and
   \[ \mu = i_*(\Lambda) \quad \Psi(\Lambda) = \frac{2^d C(F)}{\text{Euler}(\nu)} \]

4. Well-defined spacetime fermions and couplings to RR fields.

5. Possibly, new NSNS solitons.
Conclusion

The main future direction is in applications

• Destructive String Theory?
• Tadpole constraints (Gauss law)
• Spacetime anomaly cancellation
• S-Duality Puzzles