N=2* SYM And
Four Manifold Invariants
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Work with JAN MANSCHOT

Modularity:
\[ S_L(\mathbb{Z}) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathbb{Z}^4 \mid ad - bc = 1 \right\} \]

Modular form
\[ f : \mathfrak{H} \to \mathbb{C} \text{ such that} \]
\[ f \left( \frac{a \tau + b}{c \tau + d} \right) = \left( c \tau + d \right)^n f(\tau) \]

\[ \vartheta(\tau) = q^{-\frac{1}{24}} \prod_{n = 1}^{\infty} (1 - q^n)^2 \]

arXiv:2104.06492
Closely related to previous work of:
Dijkgraaf, Park, Schroers (1998)
Labastida & Lozano (1998)
Labastida & Marino (1997)
Losev, Nekrasov, Shatashvili (1997)
Moore & Witten (1997)
Vafa & Witten (1994)
Witten (1988 - 1997)

Also related: Recent work of Göttsche, Kool, Nakajima, and Williams
1. Motivation

2. Preliminaries And Background

3. Physical Formulation Of Partition Function

4. Mathematical Formulation Of Partition Function

5. Coulomb Branch Integral: Measure & Evaluation

6. LEET Near Cusps & Explicit Results
Glorious History Of SYM & Four-Manifolds

Instantons (1975) →

Donaldson invariants (1982) →

TQFT (1988) →

Seiberg-Witten Invariants (1994) →

Revolution of 1995
But not all questions are answered...

Will We Ever Classify Simply-Connected Smooth 4-manifolds?

Ronald J. Stern

ABSTRACT. These notes are adapted from two talks given at the 2004 Clay Institute Summer School on Floer homology, gauge theory, and low dimensional topology at the Alfred Rényi Institute. We will quickly review what we do and do not know about the existence and uniqueness of smooth and symplectic structures on closed, simply-connected 4-manifolds. We will then list the techniques used to date and capture the key features common to all these techniques. We finish with some approachable questions that further explore the relationship between these techniques and whose answers may assist in future advances towards a classification scheme.

1. Introduction
The SW revolution was based on pure SU(2) N=2 SYM

The basic idea of topological twisting applies to any d=4, N=2 QFT

Is there more to learn about 4-manifolds from Susy QFT?
This Talk

Study 4-fold invariants for ``SU(2) N=2* theory’’

Interpolates between Donaldson & Vafa-Witten invariants

Important lessons for several future generalizations

Key to explicit evaluation:
``Coulomb branch integral’’ aka ``u-plane integral’’

Automorphic forms; indefinite theta functions; mock modular forms; Jacobi Maass forms
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Four-Manifolds We Consider

$X$: Smooth, compact, oriented, $\partial X = \emptyset$.

For simplicity: Connected, $\pi_1(X) = 0$

We assume (as in Donaldson theory) that $X$ admits an almost complex structure

$b_2^+(X)$ is odd

Important: Do not assume $X$ is spin
Almost Complex Structure

\[ T^*X \otimes \mathbb{C} \] has a basis \( e^i, (e^i)^* \) \( i = 1,2 \)

Across patches: \( e^i \rightarrow U^i_j(x) e^j \)

\[
s: U(2) \rightarrow \begin{pmatrix} \text{Re}(U) & \text{Im}(U) \\ -\text{Im}(U) & \text{Re}(U) \end{pmatrix} \in SO(4)\]
Preliminary: $Spin^c$-structure – 1/3

$Spin^c(4) := \{(u_1, u_2) | \det(u_1) = \det(u_2)\} \subset U(2) \times U(2)$

$\pi: Spin^c(4) \rightarrow SO(4)$

$x_\mu \sigma^\mu \rightarrow u_1 x_\mu \sigma^\mu u_2^{-1}$

Spin-c structure:
Give transition functions in $Spin^c(4)$
so that $\pi(u_1, u_2) = SO(4)$ transition functions of $T^*X$
Preliminary: $\text{Spin}^c$-structure – 2/3

$\text{Spin}^c(4) := \{ (u_1, u_2) \mid \det(u_1) = \det(u_2) \} \subset U(2) \times U(2)$

Has two obvious 2-dimensional reps: $2 \otimes 1$ and $1 \otimes 2$

Given a spin-c structure these define chiral spinor bundles $\mathcal{W}^\pm \to X$
\[ c(\varsigma) := c_1(\det W^\pm) \in H^2(X; \mathbb{Z}) \]

\[ \ell = \frac{c(\varsigma)^2 - 2\chi - 3\sigma}{8} \in \mathbb{Z} \]

An ACS \( \mathcal{I} \) defines a canonical spin-c structure \( \mathfrak{s}(\mathcal{I}) \)

\[ \text{Spin}^c(4) := \{ (u_1, u_2) | \det(u_1) = \det(u_2) \} \subset U(2) \times U(2) \]

\[ \phi: U(2) \to \text{Spin}^c(4): u \mapsto (u, \begin{pmatrix} 1 \\ \det u \end{pmatrix}) \]

\[ \pi \circ \phi \sim s \quad \text{For } \mathfrak{s}(\mathcal{I}) \quad \ell = 0 \]
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SU(2) N=2* SYM

\[ \mathfrak{su}(2)_+ \bigoplus \mathfrak{su}(2)_- \bigoplus \mathfrak{su}(2)_R \bigoplus \mathfrak{g}_{gauge} \]

Local Lorentz

Hypermultiplet scalars in rep:

\[ \mathcal{R} = \mathfrak{su}(2)_{gauge} \bigotimes \mathbb{H} \rightarrow SU(2)_R \]
Topological Twisting

Couple to background $SU(2)_R$ gauge field

Identify $\mathfrak{su}(2)_R$ with $\mathfrak{su}(2)_+$ in local Lorentz algebra

Hypermultiplet scalar fields become spinors under twisting

What if $X$ is not spin?

Cure the problem by introducing an "ultraviolet" spin-c structure $\mathfrak{s}_{\text{uv}}$
So with a uv spin-c structure the hypermultiplet scalars in $N = 2^*$ -theory are spinors in $W^+$.

For $N = 2^*$ basic topological twisting needs to be supplemented with extra data.

It is not known how to twist the general $d=4$ $N=2$ theory.
Topologically Twisted Partition Function

Data needed to formulate the partition function:

\[ \tau_{uv} \sim \theta + \frac{i}{g_{uv}^2} \in \mathcal{H} \quad q_{uv} := e^{2\pi i \tau_{uv}} \]

\[ m \in \mathbb{C} \quad \Lambda: \text{UV scale} \quad t := m/\Lambda \]

(UV) Spin-c structure \( s_{uv} \),

\[ c_{uv} := c(s_{uv}) \in H^2(X, \mathbb{Z}) \]

‘t Hooft flux \( \nu \in H^2(X; \mathbb{Z}/2\mathbb{Z}) \)
and a metric $g_{\mu\nu}$

$$T_{\mu\nu} = Q(\Lambda_{\mu\nu})$$

So metric should drop out....

$$S = \int_{\mathcal{X}} \tau_{\mu\nu} \text{Tr} (F \wedge F) + Q(*)$$

so $Z$ should be holomorphic in $\tau_{\mu\nu}$....
Operators In The TQFT

$Q$ -cohomology on depends on homology

$p \in H_0(X; \mathbb{Z}) \Rightarrow p = n_1 x_1 + \cdots n_k x_k$

$S \in H_2(X; \mathbb{Z})$

$\mathcal{O}(p) = \sum_i n_i \ Tr \phi^2(x_i)$

$\mathcal{O}(S) = \int_S Tr(\phi F + \psi^2)$
Path integral defines a "function"

\[ Z_v(\tau_{uv}, c_{uv}, t): H_*(X; \mathbb{Z}) \rightarrow \mathbb{C} \]

\[ Z_v(x; \tau_{uv}, c_{uv}, t) := \langle e^{\theta(x)} \rangle_{\mathcal{N}=2^*} \]

We evaluate this function very explicitly and check some physical expectations.
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Mathematical Formulation Of The Invariants

Principal $SO(3)$ bundle $P \to X$

$A \in \mathcal{A}(P)$ \quad $M \in \Gamma(W^+ \otimes \text{ad}P \otimes \mathbb{C})$

$W^+ \to X$: Positive chirality rank two bundle associated to uv spin-c structure $z_{uv}$

$Q$ – fixed point equations

$F^+ + [M, \bar{M}] = 0 \quad \gamma \cdot D M = 0$

``adjoint SW equations``

[Labastida-Marino; Losev-Shatashvili-Nekrasov]
\( \mathcal{M}_{Q,k,\nu} \): Component of moduli ``space'' of solutions to nonabelian monopole equations

\[
\begin{align*}
  w_2(P) &= \nu \\
  k(P) &= -\frac{1}{8\pi^2} \int Tr(F \wedge F)
\end{align*}
\]
Math Definition Of Partition Function

\[ Z_{\nu}(x; \tau_{uv}, c_{uv}, t) := \langle e^{0(x)} \rangle_{\mathcal{N}=2^*} \]

\[ = \sum_{k \geq 0} q_{uv}^k \int_{\mathcal{M}_{Q,k,\nu}} e^{\mu(x)} \text{Eul}(\mathcal{E}_s; t) \]

\[ \mu: H_*(X, \mathbb{Z}) \to H^{4-\ast}(\mathcal{M}_{Q,k,\nu}; \mathbb{Q}) \]

\[ \mathcal{E}_s : \text{Obstruction bundle for elliptic complex} \]

\[ Q\text{-symmetry: Path integral} \to \int_{\mathcal{M}_{Q,k,\nu}} \cdots \]
Index Computations

\[ \nu \dim \mathcal{M}_{Q,k} = \dim G \frac{c_{uv}^2 - (2\chi + 3\sigma)}{4} = 2\ell \dim G \]

N.B. Independent of instanton number \( k \)!

\[ \dim \mathcal{M}_{\text{inst},k} = 8k - \frac{3}{2}(\chi + \sigma) \]

Index \( \gamma \cdot D = -8k + \frac{3}{8}(c_{uv}^2 - \sigma) \)

\[ \Rightarrow \text{Correlation functions on } H_*(X) \text{ infinite } q_{uv} \text{ - series, even with } x=0 \]
$U(1)_b$ Symmetry

\[ F^+ + [M, \overline{M}] = 0 \quad \gamma \cdot DM = 0 \]

\[ U(1)_b \ : \ M \rightarrow e^{i\theta} M \]

$U(1)_b$ acts on the moduli space $\mathcal{M}_{Q,k}$ of these eqs.

\[ \mathcal{O}(x) \rightarrow \mu(x) \in H^*_U(1)_b(\mathcal{M}_{Q,k}) \]

\[ t = \frac{m}{\Lambda} : U(1)_b \text{ equivariant parameter} \]

[Labastida-Marino; Losev-Shatashvili-Nekrasov]
$U(1)_b$ Localization

\[ \sum_{k \geq 0} q^k_{\mu \nu} \int_{\mathcal{M}_{Q,k,\nu}} e^{\mu(x)} Eul(\mathcal{E}_s; t) \]

\[ F^+ + [M, \bar{M}] = 0 \quad \gamma \cdot D M = 0 \]

Fixed point set for $M \rightarrow e^{i\theta} M$ has TWO branches

Branch 1: $\mathcal{M}_{\text{inst}, k, \nu}$: $M = 0 \& F^+ = 0$

Branch 2: $\mathcal{M}_{ab}$: $M \sim \begin{pmatrix} 0 & \ast \\ 0 & 0 \end{pmatrix}$
$U(1)_b$ Localization

$\int_{\mathcal{M}_{Q,k,\nu}} e^{\mu(x)}Eul(\mathcal{E}_{\tilde{s}}; t)$

$= \int_{\mathcal{M}_{\text{inst},k,\nu}} e^{\mu(x)}Eul(\tilde{\mathcal{E}}_{\tilde{s}}; t) + \int_{\mathcal{M}_{ab}} e^{\mu(x)}Eul(\tilde{\mathcal{E}}_{\tilde{s}}; t)$

First focus on the instanton contribution.
$t \to 0, \infty$ Limits Of Instanton Contribution

\[
\sum_{k \geq 0} q^k_{uv} \int_{\mathcal{M}_{\text{inst}, k, v}} e^{\mu(x)} \text{Eul}(\mathcal{E}_s; t)
\]

\[
\text{Eul}(\mathcal{E}_s; t) = \prod_{i} (x_i + t) = t^{-\text{Index}(D)} \sum_n \frac{c_n(\mathcal{E}_s)}{t^n}
\]

Leading term for $m \to \infty$ : $c_0(\mathcal{E}_s) = 1$

$\Rightarrow$ Donaldson invariants

Leading term for $m \to 0$ : $c_{\text{top}}(\mathcal{E}_s)$

$s_{uv} = s(I) : \mathcal{E}_s \cong T^* \mathcal{M}_k \Rightarrow \text{``Euler character of } \mathcal{M}_{\text{inst}, k} \text{''}$
VW invariants compute the "Euler character of $\mathcal{M}_{\text{inst},k}$" and they are S-duality covariant.

Our instanton contribution also computes the Euler character (for $s_{uv} = s(J)$ and $m \to 0$,) and together with the $\mathcal{M}_{ab}$ contribution is S-duality covariant.

Natural guess: we get VW invariants.
In cases where we can compare (such as projective surfaces) \( \lim_{t \to 0} Z_\nu \) does indeed reproduce the Vafa-Witten invariants.

This is surprising since the DW and VW twists are very different.

The Q-fixed point equations are different, but can be viewed as deformation equivalent.

(long story….)
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The partition function on a compact manifold will equal a sum over all the vacua.

In particular we need to integrate over the Coulomb branch.
Coulomb Branch Integral

In principle defined for general class S theory.

\[ Z_{\nu}^{CB} = \int_{\mathcal{B}} du \, d\bar{u} \, \mathcal{H} \, \Psi \]

\( \mathcal{H} \) is **holomorphic** and **metric-independent**

\( \Psi \): **NOT holomorphic** and **metric- DEPENDENT**

``indefinite theta function``

\( \mathcal{B} \): Base of a Hitchin system

Today: \( u \in \mathbb{C} \cong \mathcal{B} \) will be identified with a modular curve
Coulomb Branch Integral: Measure & Evaluation

5a Seiberg-Witten Review

5b Formulating The Measure And Integral

5c Evaluation Using Mock This & That
\[ E_u \quad y^2 = \prod_{i=1}^{3} (x - \alpha_i) \quad \alpha_i = u e_i(\tau_{uv}) + m^2 e_i(\tau_{uv})^2 \]

e_i(\tau_{uv}) \text{ half-periods of } E_{\tau_{uv}} = \mathbb{C}/(\mathbb{Z} + \tau_{uv} \mathbb{Z})

Together with \( \lambda \in \Omega^{1,0}(E_u) \) s.t. \( \frac{d\lambda}{du} = \frac{dx}{y} \)

Discriminant \( \sim \eta^{24}(\tau_{uv}) \prod_{i=1}^{3} (u - m^2 e_i(\tau_{uv}))^2 \)
\[ u_j = m^2 e_j(\tau_{uv}) \]
Special Geometry

$H_1(E_u; \mathbb{Z})$: Fibers of a local system over $\mathcal{B}^*$

Definition: A "duality frame" is a local choice of $A, B$—cycles

Periods of $\lambda$ define homomorphism $Z_u: H_1(E_u; \mathbb{Z}) \to \mathbb{C}$

$$a(u) := \int_A \lambda \quad a_D(u) := \int_B \lambda$$

Fact: There is a locally holomorphic function $\mathcal{F}(\alpha)$

$$a_D = \frac{d\mathcal{F}}{d\alpha}$$
\[
\frac{da}{du} = \int_A \frac{dx}{y} \quad \frac{d\alpha_D}{du} = \int_B \frac{dx}{y} \quad \tau = \frac{d\alpha_D}{da} = \frac{d^2 F}{da^2}
\]

N.B.

\(\tau(u, m, \tau_{uv})\) should not be confused with \(\tau_{uv}\)

\[
\lim_{m \to 0} \tau(u, m, \tau_{uv}) = \tau_{uv} \quad \lim_{u \to \infty} \tau(u, m, \tau_{uv}) = \tau_{uv}
\]
Weak Coupling Prepotential

$u \to \infty: \exists$ Canonical duality frame (``weak coupling''):

\[ F(a, m) = \frac{1}{2} \tau_{uv} a^2 + \]
\[ + m^2 \left( \log \left( \frac{2a}{m} \right) - \frac{3}{4} + \frac{3}{2} \log \left( \frac{m}{\Lambda} \right) \right) \]

$f_n(\tau_{uv})$: polynomials:
\[ E_2, E_4, E_6 \quad \text{wt} = 2n - 2 \]

$\Lambda$ — dependence: New, and important for our story.
Derived using Nekrasov partition function in

[Minhahan, Nemeschansky, Warner; Dhoker, Phong]

[Manschot, Moore, Xinyu Zhang 2019]
Modular Parametrization

Remarkably: One can invert these equations and express periods as bimodular forms in \( \tau, \tau_{uv} \)

\[
m^2 \left( \frac{da}{du} \right)^2 = \frac{\vartheta_4^4(\tau)\vartheta_3^4(\tau_{uv}) - \vartheta_3^4(\tau)\vartheta_4^4(\tau_{uv})}{\eta^6(\tau_{uv})}
\]

\[
m^{-2} u(\tau, \tau_{uv}) = \frac{e_1^2(\tau_{uv})e_{23}(\tau) + cycl.}{e_1(\tau_{uv})e_{23}(\tau) + cycl}
\]

\[
\mathcal{B} \cong \mathcal{H}/\Gamma(2) \cong \mathcal{F}(\Gamma(2))
\]
\[ \tau = i \infty \iff u = u_1 \]

\[ \tau = 0 \iff u = u_2 \quad \tau = 1 \iff u = u_3 \]
Coulomb Branch Integral: Measure & Evaluation

Seiberg-Witten Review

Formulating The Measure And Integral

Evaluation Using Mock This & That
Coulomb Branch Measure

\[ Z^\text{CB}_V = \int_{\mathcal{F}(\Gamma(2))} \Omega \]

\[ \Omega = d\tau \wedge d\bar{\tau} \quad \mathcal{H} \quad \Psi \]

Begin with Maxwell partition function \( \Psi \)

\[ \Psi \sim \sum_{\text{fluxes}} e^{-S_{\text{class}}} \]

Frame dependent.
Not holomorphic.
Metric dependent.
The ```Period Point'' \( J \)

\[
\begin{align*}
\; b_2^+ &> 1 \Rightarrow Z_v^{CB} = 0 \\
\; b_2^+ & = 1 \quad Z_v^{CB} \neq 0
\end{align*}
\]

\[ H^2(X; \mathbb{R}) \]

\(* J = J \]

\[ J^2 = 1 \]

\( J \in \text{Forward Light Cone} \)
Maxwell Partition Function

\[ \psi^J \sim \sum_{\text{fluxes}} e^{-\int \bar{\tau}(u)f_+^2 + \tau(u) f_-^2} \]

Sum over the first Chern class
\[ \lambda \in 2L + \bar{\nu} , \quad L = H^2(X; \mathbb{Z}) \]

\[ \psi^J_\nu = \sum_{\lambda \in 2L + \bar{\nu}} \partial_{\bar{\tau}} E^J_\lambda \quad q^{\frac{1}{4}\lambda^2} \quad e^{\pi i \lambda \cdot \bar{z}} \]

\[ z = c_{uv} \nu(\tau, \tau_{uv}) + S \frac{du}{da} \quad \nu := \frac{d^2 F}{d\alpha d\text{d}m} \]
Maxwell Partition Function

\[ \Psi^J_\nu = \sum_{\lambda \in 2L+\nu} \partial_\tau E^J_\lambda \ q^{\frac{1}{4} \lambda^2} \ e^{\pi i \lambda \cdot z} \]

\[ z = c_{uv} \nu(\tau, \tau_{uv}) + S \frac{du}{da} \quad \nu := \frac{d^2 F}{dadm} \]

\[ E^J_\lambda = \text{Erf}(x_\lambda) \quad \text{Erf}(x) := \int_0^x e^{-\pi t^2} dt \]

\[ x_\lambda = \sqrt{\text{Im} \tau}(\lambda + \frac{\text{Im} z}{\text{Im} \tau}) \cdot J \]
Maxwell Coupling To $\varsigma_{uv}$

$$\sim \exp\left(\int_X \nu F^+_b f^+ + \bar{\nu} F^-_b f^-ight)$$

$$\nu := \frac{d^2 \mathcal{F}}{d\alpha d\mu m}$$

Nonholomorphic!

6d derivation will involve $\mathbb{C}^*$-valued quadratic refinement
Remarkable Equation For $\nu(\tau, \tau_{uv})$

$$\nu := \frac{d^2F}{dadm} = (a_D - a\tau)/m$$

$$\frac{\vartheta_2(\nu, 2\tau)}{\vartheta_3(\nu, 2\tau)} = \frac{\vartheta_2(0, 2\tau_{uv})}{\vartheta_3(0, 2\tau_{uv})}$$

Determines bimodular $\nu(\tau, \tau_{uv})$
Holomorphic Part Of Measure

\[ \mathcal{H}_{bare} = A_1^\sigma A_2^\chi A_3^{c_{uv}} \]

Include observables:

\[ \mathcal{H} = \mathcal{H}_{bare} A_4^p A_5^{c_{uv} \cdot S} A_6^S \]

Depend on duality frame – but the local system has nontrivial monodromy.
Local Topological Interactions

\[ A_1^8 = \prod_{i} (u - u_i) = \]

\[ (2m)^6 \eta(\tau_{uv})^{24} \eta(\tau)^{12} \]
\[ \frac{(\vartheta_4(\tau)^4 \vartheta_3(\tau_{uv})^4 - \vartheta_3(\tau)^4 \vartheta_4(\tau_{uv})^4)^3}{m^2 \eta^6(\tau_{uv})} \]

\[ A_2^{-4} = \frac{\vartheta_4^4(\tau) \vartheta_3^4(\tau_{uv}) - \vartheta_3^4(\tau) \vartheta_4^4(\tau_{uv})}{m^2 \eta^6(\tau_{uv})} \]

\[ A_3 := \exp \left( -2\pi i \frac{d^2 F}{dm^2} \right) = \left( \frac{\Lambda}{m} \right)^{\frac{3}{2}} \frac{\vartheta_1(2\tau, 2\nu)}{\vartheta_2^2(\tau_{uv}) \vartheta_4(2\tau)} \]
With all these ingredients we can now check that the CB measure is indeed monodromy invariant and hence well-defined. (Nontrivial!)
The measure on the Coulomb branch is physical and must be single-valued.

Even though several couplings in the LEET are multi-valued.

Even though there is no global duality frame.

*Interesting constraint on low-energy couplings*
What about defining the integral of the measure?
\[ u \rightarrow u_j \]

\[ \mathcal{H} \rightarrow q_j^{-2} F(\tau_{uv}) \left( 1 + \mathcal{O}(q_j) \right) \]

\[ u \rightarrow \infty \quad \text{i.e.} \quad \tau \rightarrow \tau_{uv} \]

\[ (\tau - \tau_{uv})^{\ell-\frac{3}{2}} \sum_{\lambda} \ldots e^{-\frac{m}{\Lambda} (\tau - \tau_{uv})^{-\frac{1}{2}}} S \cdot \lambda \]

Do the phase integral first.
(as in string theory)
Coulomb Branch Integral: Measure & Evaluation

Seiberg-Witten Review

Formulating The Measure And Integral

Evaluation Using Mock This & That
Relation To Mock Modular Forms -1.1

$Z^C_B \nu$ : A sum of integrals of the form

$$I_f = \int_{\mathcal{F}_\infty} d\tau d\bar{\tau} \ (Im \ \tau)^{-s} \ f(\tau, \bar{\tau})$$

Support of $c$ is bounded below

$$f(\tau, \bar{\tau}) = \sum_{m-n \in \mathbb{Z}} c(m, n) q^m \bar{q}^n$$

Strategy: Find $\hat{h}(\tau, \bar{\tau})$ such that

$$\partial_{\bar{\tau}} \hat{h} = (Im \ \tau)^{-s} \ f(\tau, \bar{\tau})$$

$\hat{h}(\tau, \bar{\tau})$ is modular of weight (2,0)
We choose an explicit solution
\[ \partial_{\bar{\tau}} R = (\text{Im}\tau)^{-s} f(\tau, \bar{\tau}) \]
vanishing exponentially fast at \( \text{Im}\tau \to \infty \)

\( R \) is not modular, but it’s failure to be modular must be holomorphic.

\[ \hat{h}(\tau, \bar{\tau}) = h(\tau) + R \]

\( h(\tau) : \) mock modular form

\[ h(\tau) = \sum_{m \in \mathbb{Z}} d(m)q^m \quad q = e^{2\pi i\tau} \]
Doing The Integral

\[ I_f = \int_{\mathcal{F}_\infty} d\tau d\bar{\tau} \; y^{-s} \; f(\tau, \bar{\tau}) \]

\[ \partial_{\bar{\tau}} \hat{h} = y^{-s} \; f(\tau, \bar{\tau}) \]

\[ I_f = d(0) \]

\[ h(\tau) = \sum_{m \in \mathbb{Z}} d(m)q^m \]

Note: \( d(0) \) undetermined by diffeq but fixed by the modular properties: Subtle!
Evaluation Of CB Integral?

\[ Z_{v}^{CB} = \int_{\mathcal{F}(\Gamma(2))} \Omega \quad \Omega = d\tau \wedge d\bar{\tau} \quad \mathcal{H} \quad \Psi_{v}^{J} \]

\[ \Psi_{v}^{J} = \sum_{\lambda \in 2L+\nu} \partial_{\bar{\tau}} E_{\lambda}^{J} \quad q^{-\frac{1}{4}\lambda^2} \quad e^{-2\pi i \lambda \cdot z} \]

\[ Z = c_{uv} \quad \nu(\tau, \tau_{uv}) + S \quad \frac{du}{da} \]

\[ \Omega = d \Lambda \quad \Lambda = d\tau \quad \mathcal{H} \quad \hat{G} \quad \Psi_{v}^{J} = \partial_{\bar{\tau}} \hat{G} \]
Evaluation Of CB Integral?

\[ \Psi^J_\nu = \sum_{\lambda \in 2L + \nu} \partial_{\bar{\tau}} E^J_\lambda \ q^{-\frac{1}{4}\lambda^2} \ e^{-2\pi i \lambda \cdot z} \]

\[ \Psi^J_\nu = \partial_{\bar{\tau}} \hat{G} \]

\[ \hat{G} = \sum_{\lambda \in 2L + \nu} E^J_\lambda \ q^{-\frac{1}{4}\lambda^2} \ e^{-2\pi i \lambda \cdot z} \]

??? NO!!! \[ \lim_{|\lambda_+| \to +\infty} E^J_\lambda = \pm 1 \]
Evaluating Difference Of CB Integrals

$$\Psi^{J_1} - \Psi^{J_2} = \partial_{\tilde{\tau}} \hat{G}^{J_1,J_2}$$

$$\hat{G}^{J_1,J_2} = \sum_{\lambda \in 2L + \nu} E^J_{1,J_2} q^{-\frac{1}{4}\lambda^2} e^{-2\pi i \lambda \cdot z}$$

$$E^J_{\lambda} = Erf(x^J_{\lambda}) - Erf(x^J_{\lambda})$$

Converges nicely!

\[ \Rightarrow \text{Can use this to evaluate the difference} \]

$$Z^{CB,J_1}_v - Z^{CB,J_2}_v \text{ by a sum of residues.}$$
Metric & Holomorphic Anomaly

Wall crossing involves modular functions

For the boundary at $u \to \infty$ the modular parameter $\tau \to \tau_{uv}$. This leads to **continuous** metric dependence.

Closely related: Nonholomorphic in $\tau_{uv}$

$$\frac{\partial}{\partial \bar{\tau}_{uv}} Z_{\nu}^{CB} = y^{-\frac{3}{2}} \eta^{-2\chi} \sum_{\lambda} K[\lambda_+, \lambda_-] \bar{q}_{uv}^{\lambda_+} \bar{q}_{uv}^{-\lambda_-}$$
The Special Period Point

For any manifold with $b_2^+ = 1$

$\exists$ special $J_0$ such that $\Psi_{\nu}^{J_0}$ factorizes:

$$\Psi_{\nu}^{J_0} = f_{\nu} \Theta_{L-} (\tau, z)$$

$$f_{\nu} = \sum_{\lambda \in 2\mathbb{Z} - \nu} \partial_{\tau} E_{\lambda}^{J} q^{-\frac{1}{4} \lambda^2} e^{-2\pi i \lambda \cdot z}$$
Measure As A Total Derivative

\[ \Omega = d \Lambda \quad \Lambda = d\tau \mathcal{H} \hat{\mathcal{G}} \]

Where we can write \( \hat{\mathcal{G}} \) explicitly so that \( \Lambda \) is:

1. Well-defined
2. Nonsingular away from \( \tau \in \{ 0, 1, i \infty, \tau_{uv} \} \)
3. Good \( q_i \) expansion near cusps
Harmonic Jacobi-Maass Forms

These conditions determine \( \hat{G} \) uniquely.

Modular completion of an Appel-Lerche sum

\[
F(\tau, z) \sim \frac{e^{-2\pi i z}}{\vartheta_4(2\tau)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{n^2 - \frac{1}{4}}}{1 + e^{4\pi i z} q^{2n-1}}
\]

\[
z = c_{uv} \nu(\tau, \tau_{uv}) + S \frac{du}{d\alpha}(\tau, \tau_{uv})
\]
Technical Comment On Poles

\[ \Lambda \] must be nonsingular away from \( \tau \in \{ 0, 1, i \infty, \tau_{uv} \} \)

\[ F(\tau, z) \text{ has poles for} \]

\[ z = \frac{1}{4}(a + b \tau) \quad a = 1 \mod 2 \quad b = 2 \mod 4 \]

But for \( z = c_{uv} \nu(\tau, \tau_{uv}) \) one can add a meromorphic form to \( F(\tau, z) \) to cancel unwanted poles
The Integral Is a Mock Modular Form

For $s_{uv} = s(J)$ we find $Z_{v}^{CB,J_0} =$

$$
\hat{g}_v(\tau_{uv}, \bar{\tau}_{uv}) \ \Theta_{L-}(\tau_{uv})/\eta^{2\chi}(\tau_{uv})
$$

$$
g_v = 3 \sum_{n \geq 0} H(4n - 2\mu) q_{uv}^{n-\frac{\nu}{2}}
$$
\[
\text{... but other } \nabla \text{ generalize ...}
\]

For \( \mathbb{CP}^2 \) & \( c_{uv} = 1 \) (acs \( \Rightarrow \) \( c_{uv} = 3 \))

\[
\frac{\partial}{\partial \bar{\tau}_{uv}} Z_v = y_{uv}^{\frac{3}{2}} \eta^{-2} \overset{\text{\(\bar{E}_2\)}}{\Theta_v} (\bar{\tau}_{uv})
\]
Including Observables

\begin{tabular}{|c|c|}
\hline
\textbf{n} & \textbf{Hol. part of } \eta(\tau_{uv})^6 \Phi^{P^2}_{1/2}[\phi^n_D] \\hline
0 & \frac{3}{4} q_{uv} + 3 q_{uv}^{7/4} + 3 q_{uv}^{11/4} + 6 q_{uv}^{15/4} + \ldots \\hline
1 & -m^2 \left( \frac{3}{5} q_{uv}^{7/4} + 12 q_{uv}^{11/4} + 35 q_{uv}^{15/4} + \ldots \right) \\hline
2 & m^4 \left( \frac{19}{16} q_{uv}^{7/4} + \frac{31}{2} q_{uv}^{11/4} + 89 q_{uv}^{15/4} + \ldots \right) \\hline
3 & -m^6 \left( \frac{15}{4} q_{uv}^{11/4} + \frac{971}{16} q_{uv}^{15/4} + \ldots \right) \\hline
4 & m^8 \left( \frac{85}{32} q_{uv}^{11/4} + \frac{15151}{256} q_{uv}^{15/4} + \ldots \right) \\hline
\end{tabular}
Contributions Of The Cusps $u_j$

Near each cusp $u_j$, $j = 1,2,3$

the description of the vacuum changes:

We have a U(1) VM coupled to a charge 1 HM.

(In the appropriate duality frame) [Seiberg-Witten 94]

There is a separate contribution to the path integral coming from the path integral of these three LEET.

We add the contributions, because we sum over vacua:

$$Z_v = Z_v^{CB} + \sum_{j=1}^{3} Z_{v,j}^{SW}$$
When $b_2^+ > 1 \ Z_v^{\text{CB}}$ vanishes – we get true topological invariants:

\[ Z_v = \sum_{j=1}^{3} Z_{v,j}^{\text{SW}} \]

So it is quite interesting to determine the three effective actions
\[ u_j = m^2 e_j(\tau_0) \]
MW97: The behavior of the CB integral at $u_j$ uniquely fixes $Z_{\nu,j}^{SW}$
Determination Of Effective Action

\[ Z_{v,j}^{SW} = \sum_{c_{ir}} SW(c_{ir}) \prod_{n=1}^{12} F_{n,j}(\tau_{uv}; t)^{\Delta n} \]

\[ \chi_h = \frac{1}{4} (\chi + \sigma) \quad \lambda = 2\chi + 3\sigma \quad \ell = \frac{c_{uv}^2 - \lambda}{8} \]

\[ x^2 := \left( \frac{c_{uv} - c_{ir}}{2} \right)^2 \]

\[ S \cdot c_{ir} \quad S^2 \quad S \cdot c_{uv} \]
\[ Z_{SW,1, \mu}(\tau_{uv}) = (-2 \eta (2\tau_{uv})^{12})^{-\lambda_h} \left( 4 t^3 \eta (\tau_{uv})^4 \vartheta_3(2\tau_{uv})^4 \right)^{-\epsilon} \left( \frac{\eta(\tau_{uv})^2}{\vartheta_3(2\tau_{uv})} \right)^{\lambda} \times \sum_{x=2\mu \mod 2L} \text{SW}(c_{ir}) \left( \frac{\vartheta_3(2\tau_{uv})}{\vartheta_2(2\tau_{uv})} \right)^{x^2} \]

+ 

\[ Z_{SW,2, \mu}(\tau_{uv}) = 2 \left( 2 \eta (\tau_{uv}/2)^{12} \right)^{-\lambda_h} \left( -t^3 \eta (\tau_{uv})^4 \vartheta_3(\tau_{uv}/2)^4 \right)^{-\epsilon} \left( \frac{2\eta(\tau_{uv})^2}{\vartheta_3(\tau_{uv}/2)} \right)^{\lambda} \times \sum_{x \in L} \text{SW}(c_{ir}) (-1)^{2B(x, \mu)} \left( \frac{\vartheta_3(\tau_{uv}/2)}{\vartheta_4(\tau_{uv}/2)} \right)^{x^2} \]

+ 

\[ Z_{SW,3, \mu}(\tau_{uv}) = 2 e^{2\pi i \mu^2} \left( -t^3 \eta (\tau_{uv})^4 \vartheta_3((\tau_{uv} + 1)/2)^4 \right)^{-\epsilon} \times (2 \eta((\tau_{uv} + 1)/2)^{12})^{-\lambda_h} \left( \frac{2\eta(\tau_{uv})^2}{\vartheta_3((\tau_{uv} + 1)/2)} \right)^{\lambda} \times \sum_{x \in L} \text{SW}(c_{ir}) (-1)^{2B(x, \mu)} \left( \frac{\vartheta_3((\tau_{uv} + 1)/2)}{\vartheta_4((\tau_{uv} + 1)/2)} \right)^{x^2}. \]
Including Observables

<table>
<thead>
<tr>
<th>( j )</th>
<th>( B(S, c_{\text{ir}}): \log(S_j) )</th>
<th>( S^2: \log(T_j) )</th>
<th>( B(S, c_{\text{uv}}): \log(U_j) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-\frac{it}{4} \vartheta_3^2 \vartheta_4^2)</td>
<td>(\frac{t^2}{144} \left[ -3 \vartheta_3^4 \vartheta_4^4 + E_2 (\vartheta_3^4 + \vartheta_4^4) + E_2^2 \right])</td>
<td>(\frac{t}{12i} [\vartheta_3^4 + \vartheta_4^4 + 2E_2])</td>
</tr>
<tr>
<td>2</td>
<td>(-\frac{it}{4} \vartheta_2^2 \vartheta_3^2)</td>
<td>(\frac{t^2}{144} \left[ -3 \vartheta_2^4 \vartheta_3^4 - E_2 (\vartheta_2^4 + \vartheta_3^4) + E_2^2 \right])</td>
<td>(\frac{t}{12i} [-\vartheta_2^4 - \vartheta_3^4 + 2E_2])</td>
</tr>
<tr>
<td>3</td>
<td>(-\frac{t}{4} \vartheta_2^2 \vartheta_4^2)</td>
<td>(\frac{t^2}{144} \left[ 3 \vartheta_2^4 \vartheta_4^4 + E_2 (\vartheta_2^4 - \vartheta_4^4) + E_2^2 \right])</td>
<td>(\frac{t}{12i} [\vartheta_2^4 - \vartheta_4^4 + 2E_2])</td>
</tr>
</tbody>
</table>
Full Modular Transformation Law

\[
Z_v \left( \tilde{p}, \tilde{S}, \frac{a \tau_{uv} + b}{c \tau_{uv} + d} \right) = (c \tau_{uv} + d)^w \sum_{\mu} B_{\mu, \nu}(\gamma)Z_{\mu}(p, S, \tau_{uv})
\]

\[
w = -\frac{\chi}{2} - 4\ell
\]

\[
\tilde{S} = \frac{S}{(c \tau + d)^2}
\]

\[
\tilde{p} = \frac{1}{(c \tau + d)^2} \left( p - 2\pi i \frac{c}{c \tau + d}S^2 \right)
\]
CONCLUDING REMARKS
Relation To Previous Results

For $\xi(I)$ and $m \to 0$ we recover and generalize formulae of [VW;DPS] for VW invariants.

For $c_{uv} = 0$ we recover formulae of Labastida-Lozano

For $m \to \infty, q_{uv} \to 0$ after suitable renormalization we recover the ``Witten conjecture” for the Donaldson invariants in terms of the Seiberg-Witten invariants.

Recover and generalize explicit evaluation of u-plane integral for $\mathbb{CP}^2, S^2 \times S^2$ of Moore-Witten, Malmendier-Ono

A generalization and unification of the 1990’s formulae:
Recent Discussions Of Holomorphic Anomaly

Duality and Mock Modularity

Atish Dabholkar, Pavel Putrov, Edward Witten

Gauge theories on compact toric manifolds

Giulio Bonelli, Francesco Fucito, Jose Francisco Morales, Massimiliano Ronzani, Ekaterina Sysoeva, Alessandro Tanzini
VIRTUAL REFINEMENTS OF THE VAFA-WITTEN FORMULA

LOTHAR GÖTTSCHE AND MARTIJN KOOL

with an appendix by Lothar Göttsche and Hiraku Nakajima

VERLINDE FORMULAE ON COMPLEX SURFACES I:
$K$-THEORETIC INVARIANTS

L. GÖTTSCHE, M. KOOL, AND R. A. WILLIAMS

REFINED SU(3) VAFA-WITTEN INVARIANTS AND MODULARITY

LOTHAR GÖTTSCHE AND MARTIJN KOOL

VIRTUAL SEGRE AND VERLINDE NUMBERS OF PROJECTIVE SURFACES

L. GÖTTSCHE AND M. KOOL

SHEAVES ON SURFACES AND VIRTUAL INVARIANTS

L. GÖTTSCHE AND M. KOOL
\[ Z_{SW,1,\mu}(\tau_{uv}) = \left( -2 \eta (2\tau_{uv})^{12} \right)^{-\chi_h} \left( 4 t^3 \eta (\tau_{uv})^4 \vartheta_3 (2\tau_{uv})^4 \right)^{-\ell} \left( \frac{\eta (\tau_{uv})^2}{\vartheta_3 (2\tau_{uv})} \right) \lambda \times \sum_{x=2\mu \mod 2L} \text{SW}(c_{ir}) \left( \frac{\vartheta_3 (2\tau_{uv})}{\vartheta_2 (2\tau_{uv})} \right)^2 \]

\[ + \]

\[ Z_{SW,2,\mu}(\tau_{uv}) = 2 \left( 2 \eta (\tau_{uv}/2)^{12} \right)^{-\chi_h} \left( -t^3 \eta (\tau_{uv})^4 \vartheta_3 (\tau_{uv}/2)^4 \right)^{-\ell} \left( \frac{2\eta (\tau_{uv})^2}{\vartheta_3 (\tau_{uv}/2)} \right)^\lambda \times \sum_{x \in L} \text{SW}(c_{ir}) (-1)^{2B(x,\mu)} \left( \frac{\vartheta_3 (\tau_{uv}/2)}{\vartheta_4 (\tau_{uv}/2)} \right)^2 \]

\[ + \]

\[ Z_{SW,3,\mu}(\tau_{uv}) = 2 e^{2\pi i \mu^2} \left( -t^3 \eta (\tau_{uv})^4 \vartheta_3 ((\tau_{uv} + 1)/2)^4 \right)^{-\ell} \times \left( 2 \eta ((\tau_{uv} + 1)/2)^{12} \right)^{-\chi_h} \left( \frac{2\eta (\tau_{uv})^2}{\vartheta_3 ((\tau_{uv} + 1)/2)} \right)^\lambda \times \sum_{x \in L} \text{SW}(c_{ir}) (-1)^{2B(x,\mu)} \left( \frac{\vartheta_3 ((\tau_{uv} + 1)/2)}{\vartheta_4 ((\tau_{uv} + 1)/2)} \right)^2 \]
Comparing With GKNW

\( \mathcal{X} \): General four-manifold admitting acs.

Arbitrary spin-c structure \( s_{uv} \)

???!?!?!?!?!

\( \mathcal{X} \): Projective algebraic surface

Canonical spin-c structure determined by the complex structure

Further refinement of invariants computing \( \chi_y \) -genus
VERLINDE FORMULAЕ ON COMPLEX SURFACES I: K-THEORETIC INVARIANTS

L. GöttscHe, M. Kool, AND R. A. Williams

Conjecture 1.2. Let $S$ be a smooth projective surface with $p_g(S) > 0$, $b_1(S) = 0$, and $L \in \text{Pic}(S)$. Let $H, c_1, c_2$ be chosen such that there are no rank 2 strictly Gieseker $H$-semistable sheaves on $S$ with Chern classes $c_1, c_2$. Let $v_d$ be defined by (1). Then $y^{-\frac{v_d}{2}} \chi_{-y}(M^H_S(2, c_1, c_2), \mu(L))$ equals the coefficient of $x^{v_d}$ of

\[
4 \left( \frac{1}{2} \prod_{n=1}^{\infty} \frac{1}{(1 - x^{2n})^{10}(1 - x^{2n}y)(1 - x^{2n}y^{-1})} \right) \chi(O_S) \left( \frac{2\eta(x^4)^2}{\theta_3(x, y^{\frac{1}{2}})} \right)^{K_S^2} \\
\cdot \left( \prod_{n=1}^{\infty} \left( \frac{(1 - x^{2n})^2}{(1 - x^{2n}y)(1 - x^{2n}y^{-1})} \right)^{\frac{n^2}{2}} \left( \frac{L^2}{\theta_3(x, y^{\frac{1}{2}})} \right) \right) \left( \prod_{n=1}^{\infty} \left( \frac{1 - x^{2n}y^{-1}}{1 - x^{2n}y} \right)^n \right)^{L K_S} \\
\cdot \sum_{a \in H^2(S, \mathbb{Z})} (-1)^{c_1 a} \text{SW}(a) \left( \frac{\theta_3(x, y^{\frac{1}{2}})}{\theta_3(-x, y^{\frac{1}{2}})} \right)^{a K_S} \\
\cdot \left( \prod_{n=1}^{\infty} \left( \frac{(1 - x^{2n-1}y^{\frac{1}{2}})(1 + x^{2n-1}y^{-\frac{1}{2}})}{(1 - x^{2n-1}y^{-\frac{1}{2}})(1 + x^{2n-1}y^{\frac{1}{2}})} \right)^{2n-1} \right)^{\frac{L(K_S - 2a)}{2}}.
\]
Comparison ($t$ large) shows:

\[ \int_{\mathcal{M}_{ab}} \ldots \sim Z_{\nu,1}^{SW} \quad u_1 \sim \frac{m^2}{6} \]

\[ \int_{\mathcal{M}_{\text{inst}}} \ldots \sim Z_{\nu,2}^{SW} + Z_{\nu,3}^{SW} \quad u_2 \sim -\frac{m^2}{12} + \Lambda_0^2 \]

\[ u_3 \sim -\frac{m^2}{12} - \Lambda_0^2 \]

Showing that the instanton contribution alone cannot be S-duality covariant.
Future Directions

$X$ complex: Compute Refined Versions From Physics

With X. Zhang: Interesting generalization to 5d SYM

Derivation from 6d (2,0) theory?

Generalization of these techniques to class $S$
1. Motivation

2. Preliminaries And Background

3. Summary Of Main Results

4. Mathematical Formulation Of Partition Function

5. Coulomb Branch Integral: Measure & Evaluation

6. LEET Near Cusps & Explicit Results
FAQ 1

SW94 showed the SW curve for N=2* is invariant under S-duality. What about the partition functions?

Partition functions are suitably S-duality covariant with some interesting nontrivial details.

See previous talks for details.
FAQ 2:

SW94: Set $q_{uv} m^4 = \Lambda_0^4$ fixed and take $m \to \infty$. This gives the $N_f = 0$ SW curve.

Does limit of $Z_\gamma$ exist and give the DW partition function?

Yes, sort of.

The limit does not exist.

But $Z_\gamma$ is naturally a sum of three terms. Throwing one away, and renormalizing the others, there is a well-defined limit. It reproduces the DW function. (With an interesting orientation issue.)
FAQ 3:

\[ S = \int_X \tau_{uv} \text{Tr} (F \wedge F) + Q(*) \]

Is the partition function metric independent and holomorphic in \( \tau_{uv} \) ?

Yes, when \( b_2^+ > 1 \).

Absolutely not when \( b_2^+ = 1 \).

In fact, most correlators vary continuously with metric.
\[ \langle \mathcal{O}(S)^r \mathcal{O}(\rho)^n \rangle \]

Varies continuously with metric when

\[
\begin{cases} 
\ell \leq n + r/2, & \text{if } r \text{ even,} \\
\ell \leq n + (r + 1)/2, & \text{if } r \text{ odd.}
\end{cases}
\]

\[
\ell = \frac{c(s)^2 - 2\chi - 3\sigma}{8} \in \mathbb{Z}
\]

(i.e. all but finitely many correlators)

We derive very explicit formulae for the holomorphic and metric anomalies.
FAQ 4:

Again using $Q$ –symmetry the coupling to the background spin-c connection is expected to be holomorphic in $u = \langle Tr(\phi^2) \rangle$

$$S_{\text{LEET}} = \int_X \kappa_1(u) f \wedge F_\flat + \kappa_2(u) F_\flat^2$$

Shapere & Tachikawa

$F_\flat$ : Fieldstrength of background spin-c connection

$f$ : dynamical U(1) fieldstrength of the Coulomb branch LEET

Yes, for $\kappa_2(u)$. No for $\kappa_1(u)$. 
This has important implications for the class S generalization.

We also give explicit formulae for these couplings.