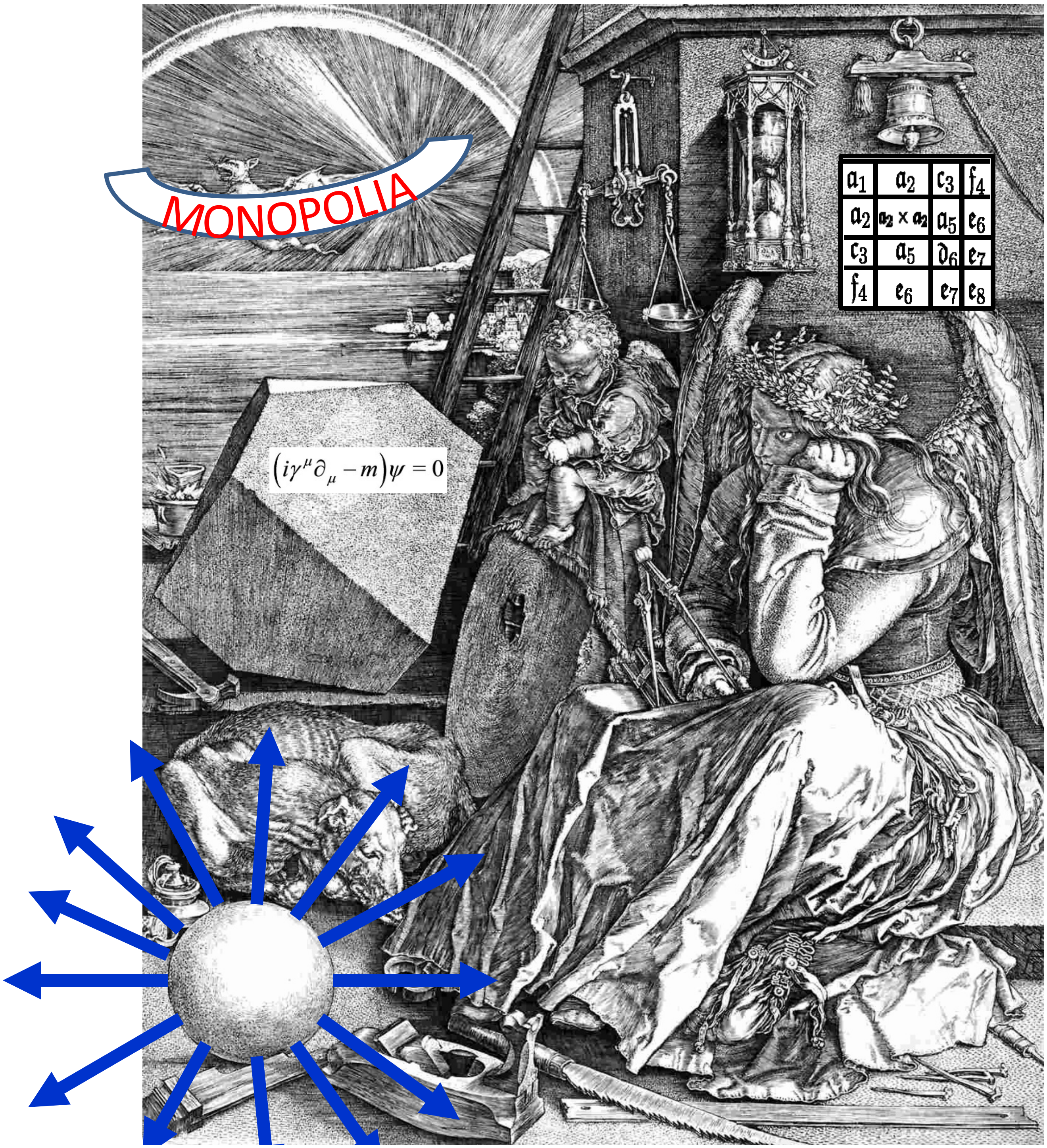
Monopolia

Gregory Moore

Nambu Memorial Symposium
University of Chicago
March 13, 2016



MONOPOLIA
$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$
a_1 a_2 c_3 f_4
a_2 $a_2 \times a_2$ a_5 e_6
c_3 a_5 d_6 e_7
f_4 e_6 e_7 e_8

Robbert Dijkgraaf’s Thesis Frontispiece

``Making the World a Stabler Place''

The BPS Times

Late Edition

Today, BPS degeneracies, wall-crossing formulae. **Tonight**, Sleep. **Tomorrow**, K3 metrics, BPS algebras, p.B6

Est. 1975 www.bpstimes.com SEOUL, FRIDAY, JUNE 28, 2013 ₩ 2743.75

INVESTIGATORS SEE NO EXOTICS IN PURE SU(N) GAUGE THEORY

Use of Motives Cited

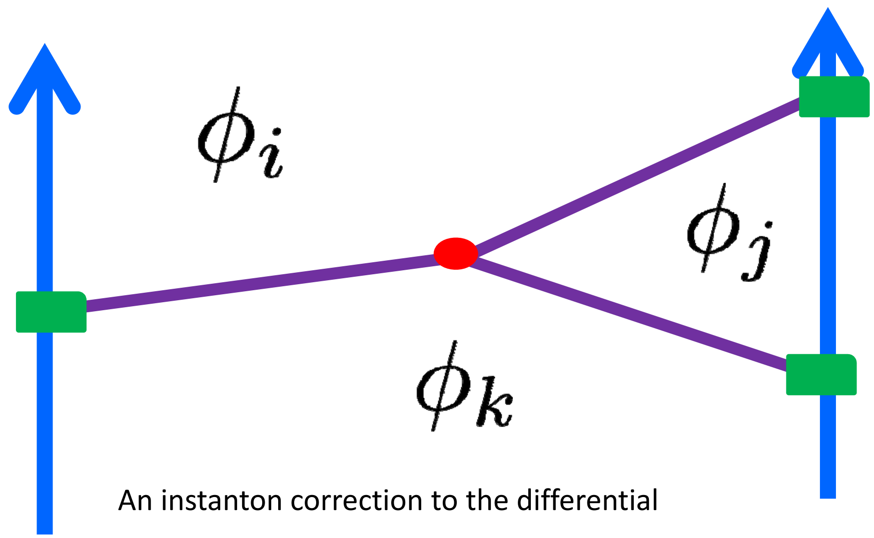
By E. Diaconescu, et. al.

RUTGERS – An application of results on the motivic structure of quiver moduli spaces has led to a proof of a conjecture of GMN. p.A12

Semiclassical, but Framed, BPS States

By G. Moore, A. Royston, and D. Van den Bleeken

RUTGERS – Semiclassical framed BPS states have been constructed as



An instanton correction to the differential

Operadic Structures Found in Infrared Limit of 2D LG Models

NOVEL CONSTRUCTION OF d ON INTERVAL

Hope Expressed for Categorical WCF

By D. Gaiotto, G. Moore, and E. Witten

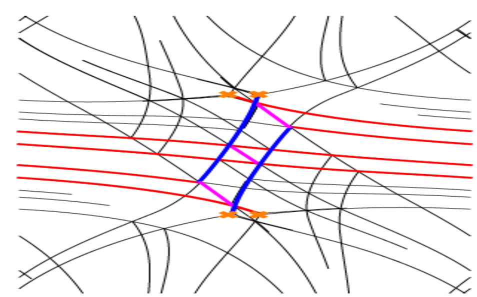
PRINCETON - A Morse-theoretic formulation of LG models has revealed ∞-structures familiar from String Field Theory. LG models are nearly trivial in

WILD WALL CROSSING IN SU(3)

EXPONENTIAL GROWTH OF Ω

By D. Galakhov, P. Longhi, T. Mainiero, G. Moore, and A. Neitzke

AUSTIN – Some strong coupling regions exhibit wild wall crossing. ``I didn't think this could happen,'' declared Prof. Nathan Seiberg of the Institute for Advanced Study in Princeton. *Continued on p.A4*



Semiclassical framed BPS states — arXiv:1512.08924

L^2-Kernels Of Dirac-Type Operators On Monopole Moduli Spaces — arXiv:1512.08923

Brane bending and monopole moduli — arXiv:1404.7158

Parameter counting for singular monopoles on R^3 — arXiv:1404.5616

Goal Of Our Project

Recently there has been some nice progress in understanding BPS states in d=4, N=2 supersymmetric field theory:

No Exotics Theorem, Wall-Crossing Formulae, Exact Results For Line Defect Vev's

What can we learn about the differential geometry of monopole moduli spaces from these results?

1. Introduction
2. Monopoles & Monopole Moduli Space
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4. Singular Monopole Moduli: Dimension & Existence
5. Semiclassical N=2 d=4 SYM: Collective Coordinates
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8. Application 2: Wall-crossing & Fredholm Property
9. Future Directions

Lie Algebra Review: 1/4

Let G be a compact simple Lie group with Lie algebra $\mathfrak{g}$.

$X \in \mathfrak{g}$ is ***regular*** if Z(X) has minimal dimension.

Then Z(X) = $\mathfrak{t}$ is a Cartan subalgebra.

$T = \exp[2\pi \mathfrak{t}]$ is a Cartan subgroup.

$\Lambda_G^\vee := \mathrm{Hom}(T, U(1))$ character lattice

$\Lambda_G := \mathrm{Hom}(U(1), T)$ $\exp(2\pi X) = 1$

$$\Lambda_{rt} \subset \Lambda_G^\vee \subset \Lambda_{wt} \subset \mathfrak{t}^\vee$$

$$\Lambda_{cr} \subset \Lambda_G \subset \Lambda_{mw} \subset \mathfrak{t}$$

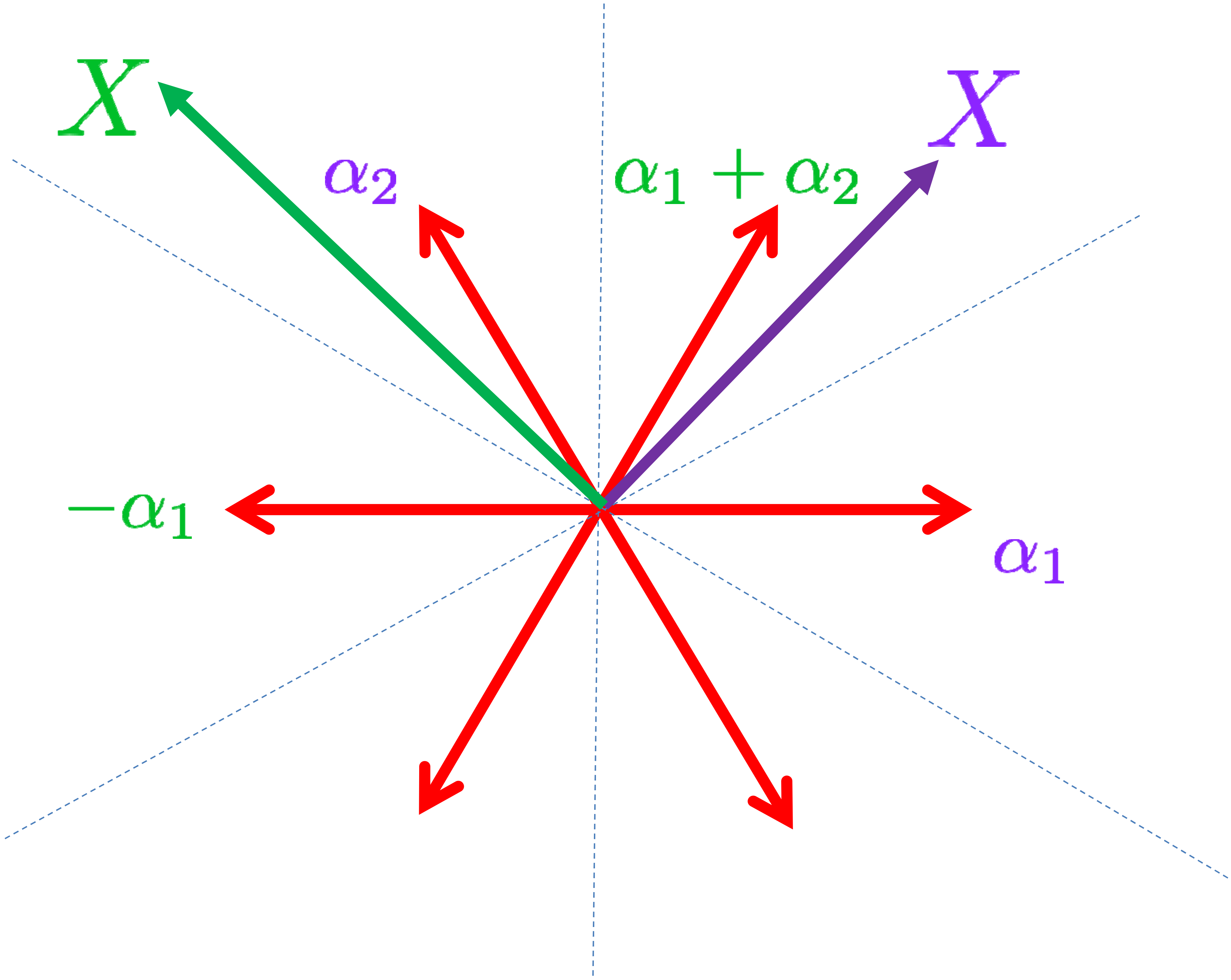
Lie Algebra Review: 2/4

Moreover, a regular element X determines a set of simple roots $\alpha_I \in \mathfrak{t}^\vee$

and simple coroots $H_{\alpha_I} := H_I \in \mathfrak{t}$

$$\Lambda_{rt} = \oplus_I \mathbb{Z}\alpha_I \subset \mathfrak{t}^\vee$$

$$\Lambda_{cr} = \oplus_I \mathbb{Z} H_I \subset \mathfrak{t}$$



X
X
α2
α1 + α2
−α1
α1

Nonabelian Monopoles

Yang-Mills-Higgs system for compact simple G

$$(A, X) \qquad \int_{\mathbb{R}^4} \mathrm{Tr}(F * F + DX * DX)$$

$$F = *DX \qquad \text{on } \mathbb{R}^3$$

$$F = \gamma_m \mathrm{vol}(S^2) + \cdots \qquad X \to X_\infty - \frac{\gamma_m}{2r} + \cdots$$

$X_\infty \in \mathfrak{g}$ regular ➡ $\mathfrak{t} \quad \alpha_I \quad H_I$

$$\gamma_m \in \Lambda_{cr} \subset \mathfrak{t} \subset \mathfrak{g}$$

$$\gamma_m = \sum_{I=1}^{r} n_m^I H_I \qquad n_m^I \in \mathbb{Z}$$

Monopole Moduli Space

$\mathcal{M}(\gamma_m; X_\infty)$ SOLUTIONS/GAUGE TRANSFORMATIONS

If $\mathcal{M}$ is nonempty then [Callias; E. Weinberg]:

$$\dim \mathcal{M}(\gamma_m; X_\infty) = 4 \sum_I n_m^I$$

Known: $\mathcal{M}$ is nonempty iff all magnetic charges nonnegative and *at least one* is positive (so $4 \leq \dim \mathcal{M}$)

$\mathcal{M}$ has a hyperkahler metric. Group of isometries with Lie algebra:

$$\mathbb{R}^3 \oplus \mathfrak{so}(3) \oplus \mathfrak{t}$$

Translations Rotations Global gauge transformations

Action Of Global Gauge Transformations

$H \in \mathfrak{t}$ ⟹ $G(H)$ Killing vector field on $\mathcal{M}$

$$\hat{A} = A_i dx^i + X dx^4 \qquad \hat{F} = * \hat{F}$$

Directional derivative along G(H) at $[\hat{A}] \in \mathcal{M}$ $\quad \frac{d\hat{A}}{ds} = -\hat{D}\epsilon$

$$\epsilon : \mathbb{R}^3 \rightarrow \mathfrak{g}$$

$$\lim_{x \rightarrow \infty} \epsilon(x) = H \qquad \hat{D}^2 \epsilon = 0$$

Strongly Centered Moduli Space

$$\widetilde{\mathcal{M}}(\gamma_m; X_\infty) = \mathbb{R}^3 \times \mathbb{R} \times \mathcal{M}_0$$

Orbits of translations

Orbits of $G(X_\infty)$

$$\mathcal{M}(\gamma_m; X_\infty) = \mathbb{R}^3 \times \frac{\mathbb{R} \times \mathcal{M}_0}{\mathbb{Z}}$$

Higher rank is different!

$$\mathcal{M}(\gamma_m; X_\infty) \neq \mathbb{R}^3 \times \frac{S^1 \times \mathcal{M}_0}{\mathbb{Z}_r}$$

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Singular Monopoles

$$F = \gamma_m \mathrm{vol}(S^2) + \cdots \qquad X \to X_\infty - \frac{\gamma_m}{2r} + \cdots$$

$$\vec{x} \to \infty$$

<u>AND</u>

$$F = P \mathrm{vol}(S^2) + \cdots \qquad X \to -\frac{P}{2r} + \mathcal{O}(r^{-1/2})$$

$$\vec{x} \to 0$$

Use: construction of ‘t Hooft line defects (``line operators’’)

Kapustin; Kapustin-Witten

Example: A Singular Nonabelian SU(2) Monopole

$$X = \frac{1}{2}h(r)H_\alpha \quad A = \frac{1}{2}(\pm 1 - \cos\theta)d\phi H_\alpha$$

$$+\frac{1}{2}f(r)\left[e^{\pm i\phi}(-d\theta - i\sin\theta d\phi)E_+ + c.c.\right]$$

Bogomolnyi eqs:

$$f'(r) + f(r)h(r) = 0$$

$$r^2h'(r) + f(r)^2 - 1 = 0$$

$$h(r) = m_W \coth(m_W r + c) - \frac{1}{r} \qquad f(r) = \frac{m_W r}{\sinh(m_W r + c)}$$

(‘t Hooft; Polyakov; Prasad & Sommerfield took c = 0)

c > 0 is the singular monopole: Physical interpretation?

Singular Monopole Moduli Space

$\overline{\underline{\mathcal{M}}}(P;\gamma_m;X_\infty)$ SOLUTIONS/GAUGE TRANSFORMATIONS

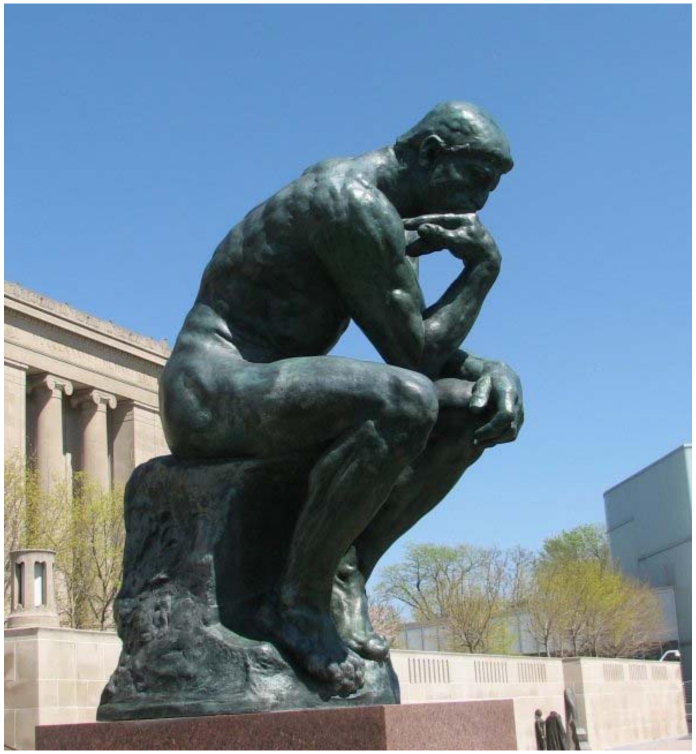
When is it nonempty?

What is the dimension?

If $P = \gamma_m$ is P screened or not ?

Is the dimension zero?

or not?



Dimension Formula

Assuming the moduli space is nonempty repeat computation of Callias; E. Weinberg to find:

$$\dim \overline{\mathcal{M}} = 2\mathrm{ind}(L) = \lim_{\epsilon\to 0^+} \mathrm{Tr}\left(\frac{\epsilon}{L^\dagger L+\epsilon} - \frac{\epsilon}{LL^\dagger+\epsilon}\right)$$

For a general 3-manifold we find:

$$\dim \overline{\mathcal{M}} = \int_{M_3 - \mathcal{S}} dJ^{(\epsilon)} = 4\sum_I \tilde{n}_m^I$$

Relative magnetic charges.

Dimension Formula

$$\dim \overline{\mathcal{M}} = 4 \sum_I \tilde{n}_m^I$$

$$\sum_I \tilde{n}_m^I H_I = \gamma_m - P^-$$

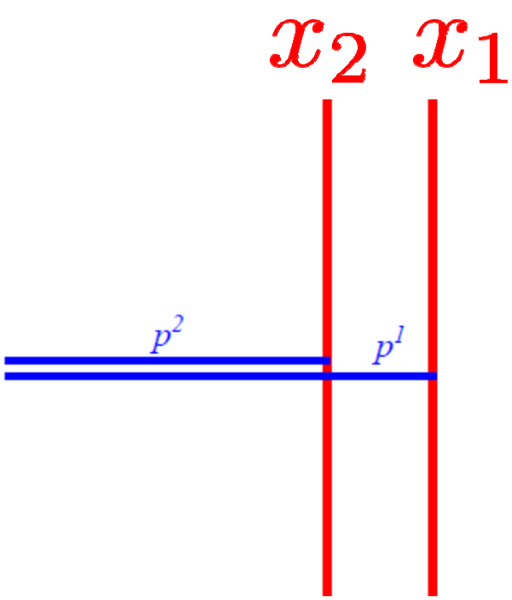
γ_m from $r \longrightarrow \infty$ and $-P^-$ from $r \longrightarrow 0$

P^- : Weyl group image such that $\langle \alpha_I, P^- \rangle \leq 0$

(Positive chamber determined by X_∞)

D-Brane Interpretation

Intuition for relative charges comes from D-branes. Example: Singular SU(2) monopoles from D1-D3 system

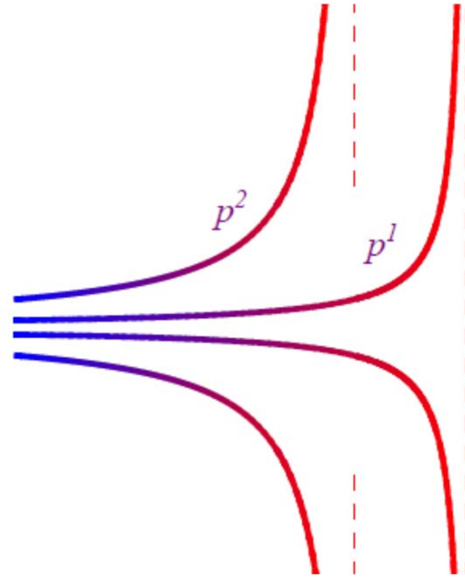


$$X = \begin{pmatrix} x_1 - \frac{p_1}{2r} & 0 \\ 0 & x_2 - \frac{p_2}{2r} \end{pmatrix}$$

$$\gamma_m = P = (p^1 - p^2) H_\alpha$$

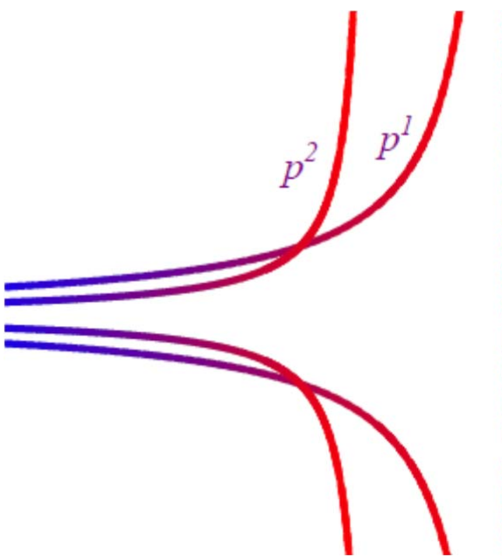
$$p^1 < p^2 \Rightarrow \gamma_m = P^-$$

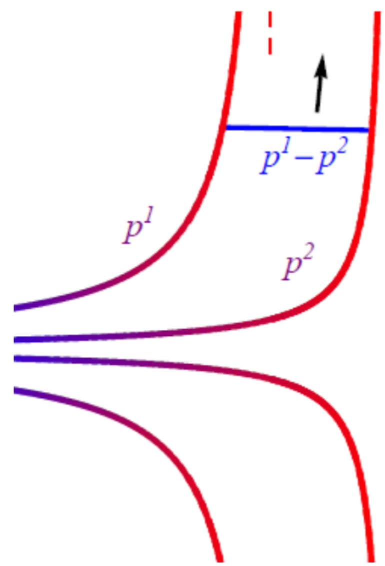
$$\Rightarrow \dim \overline{\underline{\mathcal{M}}} = 0$$



$$p^1 > p^2 \Rightarrow \gamma_m = -P^-$$

$$\Rightarrow \dim \overline{\underline{\mathcal{M}}} = 4(p^1 - p^2)$$





Application: Meaning Of The Singular 't Hooft-Polyakov Ansatz

$$X = (m_W \coth(m_W r + c) - \frac{1}{r})\frac{1}{2}H$$

$$\gamma_m = P = H \Rightarrow \tilde{n}_m = 2$$

$$\Rightarrow \dim \underline{\overline{\mathcal{M}}} = 8$$

Two smooth monopoles in the presence of minimal SU(2) singular monopole.

They sit on top of the singular monopole but have a relative phase: $e^{-c} = \sin(\psi/2)$

Two D6-branes on an O6$^-$ plane;

Moduli space of d=3 N=4 SYM with two massless HM

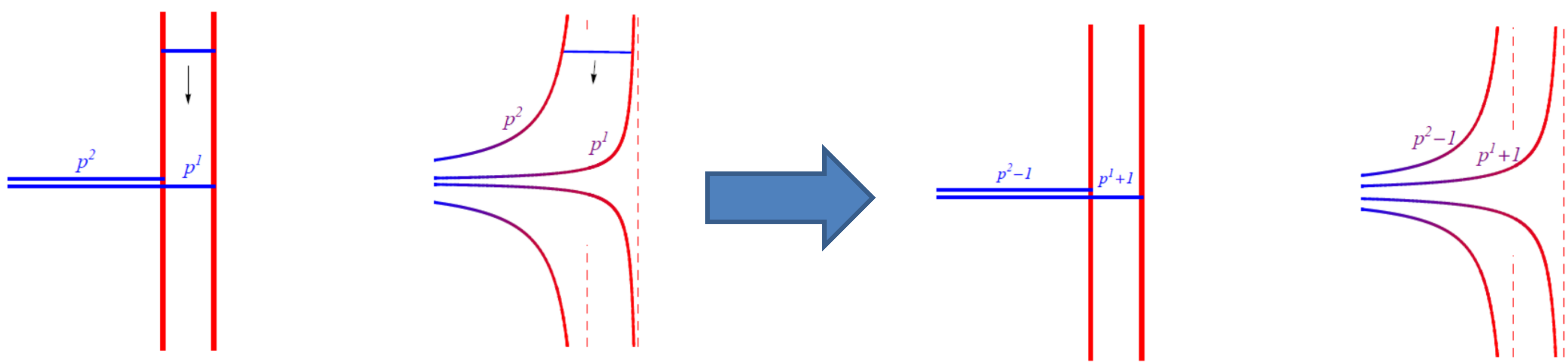
Properties of $\overline{\mathcal{M}}$

Conjecture:

$$\overline{\mathcal{M}}(P;\gamma_m;X_\infty) \neq \emptyset \quad \Longleftrightarrow \quad \forall I, \tilde{n}_m^I \geq 0$$

$\overline{\mathcal{M}}$ Hyperkähler (with singular loci - monopole bubbling)

[Kapustin-Witten]



Isometries as before, but without translation.

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$\mathcal{N}$=2 Super-Yang-Mills

Second real adjoint scalar Y

Vacuum requires $[X_\infty, Y_\infty]=0$.

$$\zeta^{-1}\varphi = Y + \mathrm{i}X$$

Meaning of ζ: BPS equations on $\mathbb{R}^3$ for preserving

$$Q + \zeta^{-1}\bar{Q}$$

$$F = B = *DX \qquad E = DY$$

ζ And BPS States

BPS bound: $$E \geq -\mathrm{Re}(\zeta^{-1} Z_\gamma) \qquad \gamma \in \Gamma$$

Definition: ***BPS states*** saturate the bound.

***Framed* BPS states**: Phase ζ is part of the data describing 't Hooft-Wilson line defect L

$$\underline{\overline{\mathcal{H}}}^{\mathrm{BPS}}(L, \gamma; u) \quad u \in \mathcal{M}_{\mathrm{Coulomb}}$$

Smooth/unframed/vanilla BPS states:

$$\zeta = -Z_\gamma(u)/|Z_\gamma(u)| \qquad \mathcal{H}^{\mathrm{BPS}}(\gamma; u)$$

Semiclassical Regime

Definition: Series expansions for
$a_D(a;\Lambda)$ converges: $|\langle\alpha, a\rangle| > c|\Lambda|$

Local system of charges has natural duality frame:

$$\Gamma \subset \Lambda_{wt} \oplus \Lambda_{mw}$$

$$\gamma = \gamma^e \oplus \gamma_m$$

(Trivialized after choices of cuts in logs for a_D.)

$$\Lambda(t) = e^{-\pi t/h^\vee}\Lambda_0 \quad \lim_{t\to+\infty}\mathcal{H}^{\mathrm{BPS}}(\gamma; u_t)$$

In this regime there is a well-known semiclassical approach to describing BPS states.

Collective Coordinate Quantization

At weak coupling BPS monopoles are heavy:
Use moduli space approximation

The semiclassical states at (u,ζ) with electromagnetic charge $\gamma^e \oplus \gamma_m$ should be described in terms of supersymmetric quantum mechanics on

$$\overline{\mathcal{M}}(P, \gamma_m; X_\infty) \quad \text{OR} \quad \mathcal{M}(\gamma_m; X_\infty)$$

What sort of SQM? *How is (u,ζ) related to X_∞?*

How does γ^e have anything to do with it?

What Sort Of SQM?

(Sethi, Stern, Zaslow; Gauntlett & Harvey ; Tong; Gauntlett, Kim, Park, Yi; Gauntlett, Kim, Lee, Yi; Bak, Lee, Yi; Bak, Lee, Lee, Yi; Stern & Yi)

N=4 SQM on $\mathcal{M}(\gamma_m, X_\infty)$ with a potential:

$$S = \int \left(\| \dot{z} \|^2 - \| G(\mathcal{Y}^{\rm cl}_\infty) \|^2 + \cdots \right)$$

$$\mathcal{Y}^{\rm cl}_\infty := \frac{4\pi}{g_0^2} Y_\infty + \frac{\theta_0}{2\pi} X_\infty$$

$$\{Q, z^\mu\} \sim \chi^\mu$$

→ States are spinors on $\mathcal{M}$

$$Q_4 = \chi^\mu (D + G(\mathcal{Y}^{\rm cl}_\infty))_\mu := \mathbf{D}$$

How is (u,ζ) related to X_∞?

Need to write X_∞, $\mathcal{Y}_\infty$ as functions on the Coulomb branch

$$X_\infty := \mathrm{Im}(\zeta^{-1} a(u)) := X$$

$$\mathcal{Y}_\infty := \mathrm{Im}(\zeta^{-1} a_D(u;\Lambda)) := \mathcal{Y}$$

Nicely encodes quantum corrections, e.g.

$$G(\mathcal{Y}_\infty^{\mathrm{cl}}) \to G(\mathcal{Y}_\infty)$$

goes beyond the weak-potential approximation.

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Semiclassical BPS States: Overview

$$Q_4 = \chi^{\mu}(D + G(\mathcal{Y}))_{\mu} := \mathbf{D}$$

Semiclassical framed or vanilla BPS states with magnetic charge $\gamma_{\rm m}$ will be:

a Dirac spinor Ψ on $\mathcal{M}(\gamma_{\rm m})$ or $\overline{\underline{\mathcal{M}}}(\gamma_{\rm m})$ $\mathbf{D}\Psi = 0$

Must be suitably normalizable: $\ker_{L^2} \mathbf{D}$

Must be suitably equivariant…

Many devils in the details….

States Of Definite Electric Charge

$\overline{\mathcal{M}}$ has a t-action: G(H) commutes with **D**

$$\exp[2\pi G(H)] \cdot \Psi = \exp[2\pi \mathrm{i} \langle \gamma^e, H \rangle] \Psi$$

$$\gamma^e \in \mathfrak{t}^\vee$$

Cartan torus T of adjoint group acts on $\overline{\mathcal{M}}$

$$T = \mathfrak{t}/\Lambda_{mw} \Longrightarrow \gamma^e \in \Lambda_{mw}^\vee \cong \Lambda_{rt}$$

Organize L^2-harmonic spinors by T-representation:

$$\ker_{L^2} \mathbf{D} = \oplus_{\gamma^e} \ker_{L^2}^{\gamma^e} \mathbf{D}$$

Framed BPS States: The Answer

L is an 't Hooft line defect of charge P and phase ζ

$$\underline{\overline{\mathcal{H}}}^{\rm BPS}(L,\gamma;u) = \ ???$$

$$= \ker^{\gamma^e}_{L^2} \mathbf{D}$$

$$X = {\rm Im}(\zeta^{-1} a(u)) \ \Rightarrow \ \underline{\overline{\mathcal{M}}}(P;\gamma_m;X)$$

$$\mathcal{Y} = {\rm Im}(\zeta^{-1} a_D(u;\Lambda)) \ \Rightarrow \mathbf{D}$$

Vanilla BPS States & Smooth Monopoles

Begin analysis on universal cover: $\tilde{\mathcal{M}}$

Physical states Ψ must descend to $\mathcal{M}$

-- Electric Charge --

T acts on $\mathcal{M}$, and $T = \mathfrak{t}/\Lambda_{mw}$

States Ψ of definite electric charge transform with a definite character of $\mathfrak{t}$:

$$\gamma^e \in \Lambda_{mw}^{\vee} \cong \Lambda_{rt}$$

Separating The Center Of Mass

$$\widetilde{\mathcal{M}}(\gamma_m; X_\infty) = \mathbb{R}^3 \times \mathbb{R} \times \mathcal{M}_0$$

$$\mathbf{D} = \mathbf{D}_{\mathbf{com}} + \mathbf{D}_0$$

$$\Psi = \Psi_{\mathbf{com}} \otimes \Psi_0$$

$$\mathbf{D}_{\mathbf{com}} \Psi_{\mathbf{com}} = 0 \qquad \mathbf{D}_0 \Psi_0 = 0$$

Separating the COM involves the nontrivial identity:

$$(G(X_\infty), G(H))_{\mathbf{metric}} = (\gamma_m, H)_{\mathbf{Killing}}$$

L^2 - Condition

$$\widetilde{\mathcal{M}}(\gamma_m; X_\infty) = \mathbb{R}^3 \times \mathbb{R} \times \mathcal{M}_0$$

No L^2 harmonic spinors on $\mathbb{R}^4$

Only ``plane-wave-normalizable'' in $\mathbb{R}^4$

$$\Psi_0 \in \ker_{L^2} \mathbf{D}_0$$

Note: The L^2 condition is crucial!

We do not want ``extra'' internal d.o.f.

Contrast this with the hypothetical ``instanton particle'' of 5D SYM.

Semiclassical Smooth BPS States

????

$$\mathcal{H}^{\rm BPS}(\gamma; u) = \left(\ker(\mathbf{D}_{\rm com}) \otimes \ker_{L^2}\mathbf{D}_0\right)^{\gamma^e}$$

$$X = \mathrm{Im}(\zeta^{-1} a(u))$$

$$\mathcal{Y} = \mathrm{Im}(\zeta^{-1} a_D(u;\Lambda))$$

$$\zeta = -Z_\gamma(u)/|Z_\gamma(u)|$$

Tricky Subtlety: 1/3

Spinors must descend to $\mathcal{M} = \widetilde{\mathcal{M}}/\mathbb{D}$

$\mathbb{D} \cong \mathbb{Z}$ Generated by isometry ϕ

Subtlety: Imposing electric charge quantization only imposes invariance under a <u>*proper*</u> subgroup of the Deck group:

$$\exp[2\pi G(\lambda)]\Psi = \Psi \qquad \lambda \in \Lambda_{mw}$$

$$\exp[2\pi G(\lambda)] = \phi^{\mu(\lambda)}$$

Put differently:

T acts on $\mathcal{M}$, so choosing a point $m_0 \in \mathcal{M}$

$$f : T \to T \cdot m_0 \subset \mathcal{M}$$

$$f_* : \pi_1(T, 1) \to \pi_1(\mathcal{M}, m_0)$$

$$f_* : \Lambda_{mw} \to \mathbb{Z}$$

$$f_* := \mu$$

Using the relation of $\mathcal{M}$ to rational maps from $\mathbb{P}^1$ to the flag variety we prove:

$$\boxed{\mu(\lambda) = (\lambda, \gamma_m)}$$

Tricky Subtlety: 3/3

$$\exp[2\pi G(\lambda)] = \phi^{\mu(\lambda)}$$

only generate a subgroup $r\,\mathbb{Z}$, where r is, roughly speaking, the gcd(magnetic charges)

Extra restriction to $\mathbb{Z}/r\mathbb{Z}$ invariant subspace:

$$\left((\ker(\mathbf{D}_{\mathrm{com}}) \otimes \ker_{L^2} \mathbf{D}_0)^{\gamma_e} \right)^{\mathbb{Z}/\mathrm{r}\mathbb{Z}}$$

Combine above picture with results on N=2,d=4:

No Exotics Theorem

Wall-Crossing

Exact Results On Line Defect VEV's

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Exotic (Framed) BPS States

$\overline{\underline{\mathcal{H}}}_{\gamma}^{\rm BPS}$ $\mathcal{H}_{\gamma}^{\rm BPS}$ $\mathfrak{so}(3)_{\rm rot} \oplus \mathfrak{su}(2)_R$ -reps

Vanilla BPS: $\mathcal{H}_{\gamma}^{\rm BPS} = \rho_{hh} \otimes \mathfrak{h}(\gamma)$

Half-Hyper from COM: $\rho_{hh} = (\frac{1}{2}; 0) \oplus (0; \frac{1}{2})$

Framed BPS: No HH factor:

$$\overline{\underline{\mathcal{H}}}_{\gamma}^{\rm BPS} = \mathfrak{h}(\gamma)$$

Definition: ***Exotic BPS states***: States in $\mathfrak{h}(\gamma)$ transforming nontrivially under $\mathfrak{su}(2)_R$

No Exotics Conjecture/Theorem

Conjecture [GMN]: $\mathfrak{su}(2)_R$ acts trivially on $\mathfrak{h}(\gamma)$: exotics don't exist.

Theorem: It's true!

Diaconescu et. al. : Pure SU(N) vanilla and framed (for pure 't Hooft line defects)

Sen & del Zotto: Simply laced G (vanilla)

Cordova & Dumitrescu: Any theory with ``Sohnius'' energy-momentum supermultiplet (vanilla, so far...)

Geometry Of The R-Symmetry

$$\dim_{\mathbb{R}} \mathcal{M} = 4N$$

Riemannian holonomy: SO(4N)

Hyperkähler holonomy: USp(2N)

$SU(2)_R$ is the commutant of USp(2N)

Collective coordinate expression for generators of $\mathfrak{su}(2)_R$

$$I^r \sim \omega^r_{\mu\nu} \chi^\mu \chi^\nu$$

This defines a lift to the spin bundle.

Generators do not commute with Dirac, but do preserve kernel.

$\underline{\overline{\mathcal{M}}}$ $\mathcal{M}$ have $\mathfrak{so}(3)$ action of rotations. Suitably defined, it commutes with $\mathfrak{su}(2)_R$.

Again, the generators do not commute with $\mathbf{D}_0$, $\mathbf{D}$, but do preserve the kernel.

$$\underline{\overline{\mathcal{H}}}^{\rm BPS}(L,\gamma,u) \cong \ker_{L^2}^{\gamma_e} \mathbf{D}$$

$$\mathfrak{h}^{\rm BPS}(\gamma;u) \cong \ker_{L^2}^{\gamma_0^e} \mathbf{D}_0$$

Equality of $\mathfrak{so}(3)_{\rm rot} \oplus \mathfrak{su}(2)_R$ - reps.

Geometrical Interpretation Of The No-Exotics Theorem -1

$$\rho : SU(2)_R \times USp(2N) \to \mathrm{Spin}(4N)$$

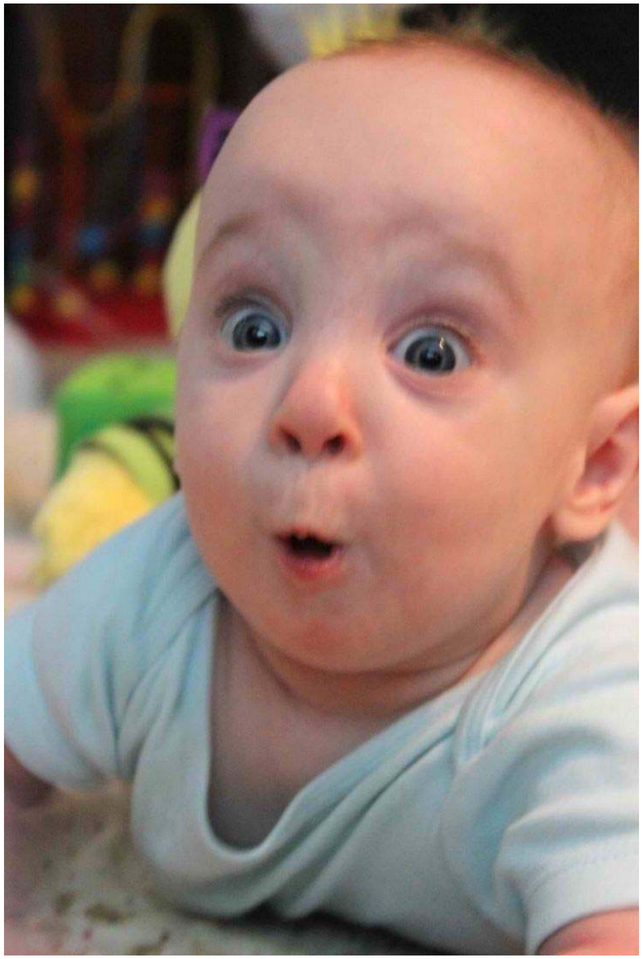
$$\rho : (-1, 1) \to \mathrm{vol} := \Gamma^1 \cdots \Gamma^{4N}$$

⇒ All spinors in the kernel have chirality +1

⇒ $\mathrm{Ind}\mathbf{D}_0^+ = \dim \ker \mathbf{D}_0$

So, the ***absolute*** number of BPS states is the same as the BPS ***index***!

This kind of question arises frequently in BPS theory…



Geometrical Interpretation Of The No-Exotics Theorem - 2

Choose any complex structure on $\mathcal{M}$.

$$\mathcal{S} \cong \Lambda^{0,*}(T\mathcal{M}) \otimes K^{-1/2}$$

$$Q_3 + \mathrm{i}Q_4 \sim \bar{\partial} + G^{0,1}(\mathcal{Y})\wedge$$

$\mathfrak{su}(2)_R$ becomes ``Lefshetz $\mathfrak{sl}(2)$''

$$I^3|_{\Lambda^{0,q}} = \frac{1}{2}(q-N)\mathbf{1}$$

$$I^+ \sim \omega^{0,2}\wedge \qquad I^- \sim \iota(\omega^{2,0})$$

Geometrical Interpretation Of The No-Exotics Theorem - 3

$$H^{0,q}_{L^2}(\bar{\partial} + G^{0,1}(\mathcal{Y}))$$

vanishes except in the middle degree q =N, and is primitive wrt ``Lefshetz $\mathfrak{sl}(2)$''.

$$\forall \mathcal{Y} \in \mathfrak{t}$$

Adding Matter-1/2

(Manton & Schroers; Sethi, Stern & Zaslow; Gauntlett & Harvey ;
Tong; Gauntlett, Kim, Park, Yi; Gauntlett, Kim, Lee, Yi; Bak, Lee, Yi)

Add matter hypermultiplets in a quaternionic representation R of G.

Bundle of hypermultiplet fermion zeromodes defines a vector bundle $\mathcal{E}$ over $\mathcal{M}$:

$\mathcal{E} =$associated bundle to the universal bundle.

Universal connection is hyperholomophic

Adding Matter-2/2

(work with Daniel Brennan)

Real rank of $\mathcal{E}$

$$d = \sum_{\mu} \left[\mathrm{sign}(\langle \mu, X \rangle + m_I) \langle \mu, \gamma_m \rangle + |\langle \mu, P \rangle| \right]$$

Sum over weights μ of R. $m_I := \mathrm{Im}(\zeta^{-1} m)$

$$\mathcal{E} \otimes \mathbb{C} \cong \mathcal{W} \oplus \overline{\mathcal{W}}$$

States are now L^2-sections of

$$S \otimes \Lambda^* \mathcal{W} \to \mathcal{M}_0 \ , \underline{\overline{\mathcal{M}}}$$

Geometrical Interpretation Of The No-Exotics Theorem - 4

$$H^{0,q}_{L^2}(\bar{\partial} + G^{0,1}(\mathcal{Y}); \Lambda^* \mathcal{W})$$

vanishes except in the middle degree q =N, and is primitive wrt ``Lefshetz $\mathfrak{sl}(2)$''.

SU(2) N=2* m $\longrightarrow$ 0 recovers the famous Sen conjecture

Semiclassical Wall-Crossing: Overview

Easy fact: There are no L^2 harmonic spinors for ordinary Dirac operator on a noncompact hyperkähler manifold.

$\Rightarrow$ $\exists$ Semiclassical chamber ($\mathcal{Y}_\infty=0$) where all populated magnetic charges are just simple roots ($\mathcal{M}_0$ = pt)

Other semiclassical chambers have nonsimple magnetic charges filled.

$\Rightarrow$ Nontrivial semi-classical wall-crossing

(Higher rank is different.)

$\Rightarrow$ Interesting math predictions

Jumping Index

The L^2-kernel of D jumps.

No exotics theorem ⇒

Harmonic spinors have definite chirality

⇒ L^2 *index* jumps! How?!

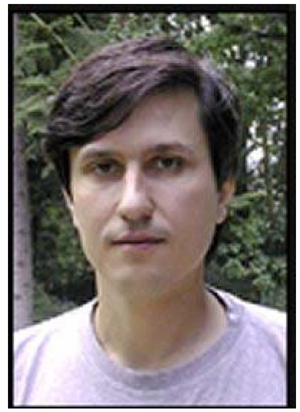
Along hyperplanes in $\mathcal{Y}$-space zeromodes mix with continuum and D^+ fails to be Fredholm.

(Similar picture proposed by M. Stern & P. Yi)

We give explicit formulae for these hyperplanes.

How Does The BPS Space Jump?

Unframed/ smooth/ vanilla:



&

Framed:

&

Framed Wall-Crossing: 1/2

$$\underline{\overline{\Omega}}(L,\gamma;X,\mathcal{Y}) = \mathrm{Tr}_{\underline{\overline{\mathcal{H}}}}\, y^{2J_3}$$

``Protected spin characters''

$$F(L) = \sum_{\gamma\in\Gamma} \underline{\overline{\Omega}}(L,\gamma;X,\mathcal{Y}) V_\gamma$$

Where does it jump?

$$\mathcal{W}(\gamma_h) := \{(X,\mathcal{Y}) : (\gamma_{h,m},\mathcal{Y}) + \langle\gamma_{h,e}, X\rangle = 0\}$$

$$\mathcal{H}^{\mathrm{BPS}}(\gamma_h, u) \neq 0$$

Framed Wall-Crossing: 2/2

$$F(L) = \sum_{\gamma \in \Gamma} \underline{\overline{\Omega}}(L, \gamma; X, \mathcal{Y}) V_\gamma$$

How does it jump across $\mathcal{W}(\gamma_h)$?

$$V_{\gamma_1} V_{\gamma_2} = y^{\langle \gamma_1, \gamma_2 \rangle} V_{\gamma_1 + \gamma_2}$$

$$F(L) \to S F(L) S^{-1}$$

S is an operator-valued function of V_{γ_h}

Example: Semiclassical Vanilla Wall Crossing

Does not exist for $\mathfrak{g} = \mathfrak{su}(2)$ (Seiberg & Witten 1994)

$$\mathfrak{g} = \mathfrak{su}(3)$$ [Gauntlett, Kim, Lee, Yi (2000)]

$$\gamma_m = H_1 + H_2 = \gamma_{1,m} + \gamma_{2,m}$$

$$\gamma^e = n_1\alpha_1 + n_2\alpha_2 = \gamma_1^e + \gamma_2^e$$

$$\mathcal{H}(\gamma_i; X, \mathcal{Y}) \neq 0$$

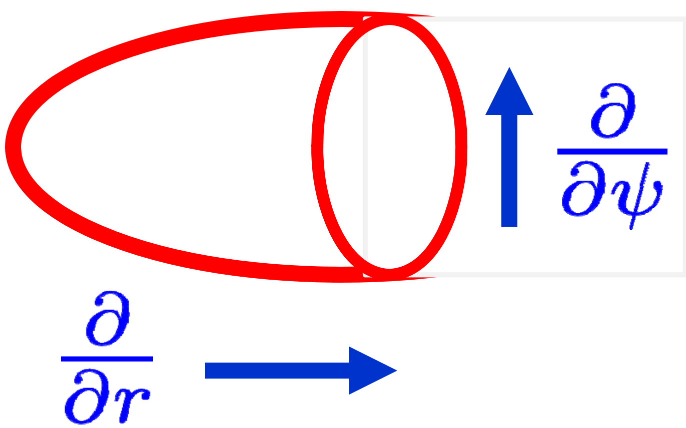
``Constituent BPS states exist''

Why choose γ_m = $H_1 + H_2$??

$\mathcal{M}_0(X;\gamma_m)$ = Taub-NUT:

Zeromodes of $\mathbf{D}_0$ can be ***explicitly*** computed [C. Pope, 1978]

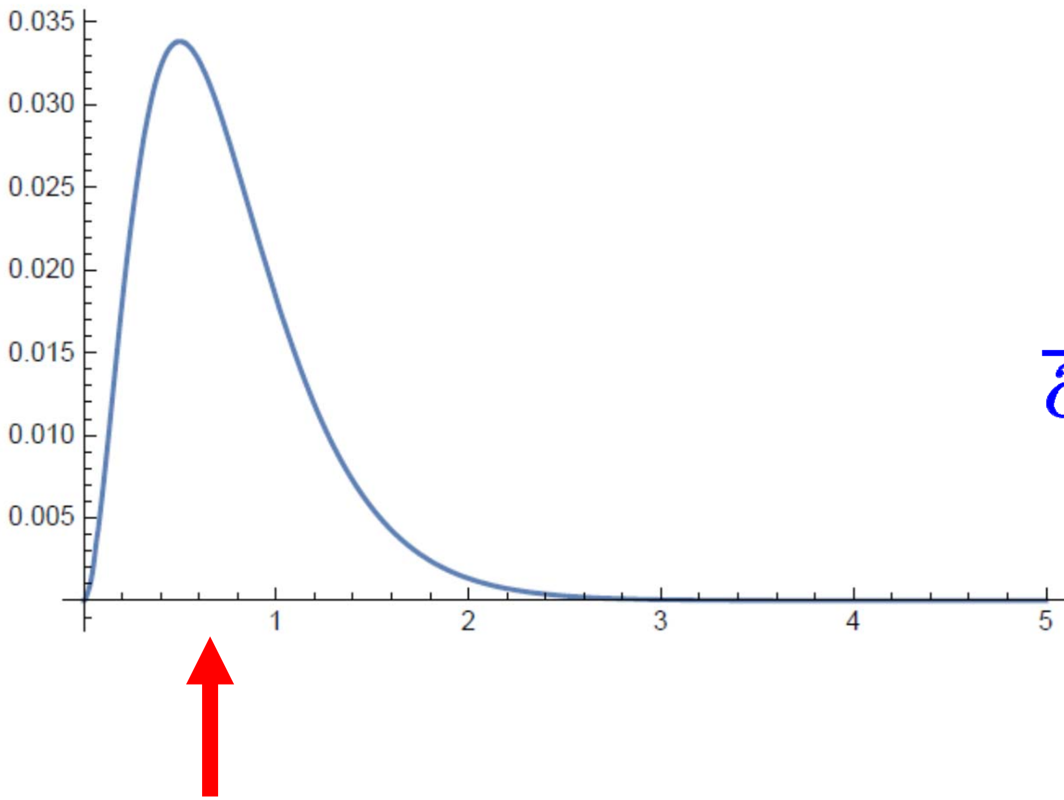
What do the zeromodes look like ??

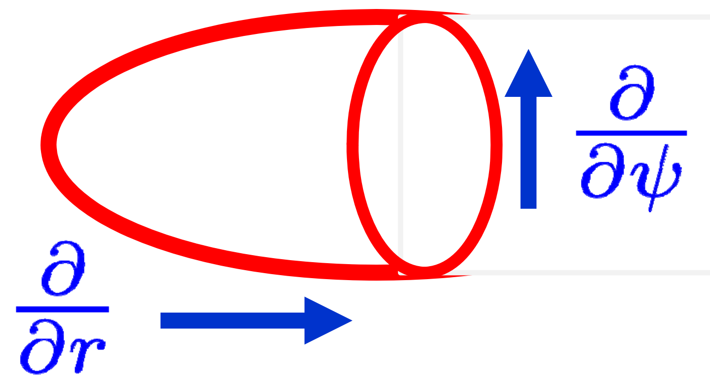


$$G(\mathcal{Y}) = C(\mathcal{Y})\frac{\partial}{\partial \psi}$$

$$L_{\frac{\partial}{\partial \psi}}\Psi_0 = \mathrm{i}(n_1 - n_2)\Psi_0$$
$$= \mathrm{i}\mu\Psi_0$$

$$\Psi_0 \sim r^{(\mu-1)/2} e^{-|C-\mu| r/2}$$





$$r_{\max} = \frac{\mu}{|C-\mu|} = r_{\mathrm{Denef}}$$

Example: Semiclassical Framed Wall-Crossing

$\mathfrak{g} = \mathfrak{su}(2) \quad \mathfrak{t} \cong \mathbb{R}$

Well-known spectrum of smooth BPS states [Seiberg & Witten]:

$$\gamma_n = n\alpha \oplus H_\alpha$$

$\mathcal{C}_n$

$\mathcal{W}(\gamma_n) \qquad \mathcal{W}(\gamma_{n+1})$

$$\mathcal{W}(\gamma_h) := \{\mathcal{Y} | (\gamma_{h,m}, \mathcal{Y}) + \langle \gamma_{h,e}, X \rangle = 0\}$$

Line defect L: $P = \frac{p}{2} H_\alpha$

$$F(L) = \sum_{\gamma \in \Gamma} \overline{\underline{\Omega}}(L, \gamma; X, \mathcal{Y}) V_\gamma$$

Explicit Generator Of PSC's

$$V_1 V_2 = y V_2 V_1$$

$$V_\gamma = V_{n^e \alpha + n_m H} = y^{-\frac{1}{2} n^e n_m} V_2^{n^e} V_1^{n_m}$$

$$F(\mathcal{C}_\ell) = \left[y^{2\ell} V_1^{-1} V_2^{-\ell} \left(\mathcal{U}_\ell(f_\ell) - y^2 V_2^{-1} \mathcal{U}_{\ell-1}(f_\ell) \right) \right]^p$$

$$\mathcal{U}_\ell(\cos\theta) := \frac{\sin((\ell+1)\theta)}{\sin\theta}$$

$$f_\ell = \frac{1}{2} \left[y^{-2} V_2 + y^2 V_2^{-1} \left(1 + y^{-1} V_1^2 V_2^{2\ell+2} \right) \right]$$

⇒ Predictions for ker **D** for infinitely many moduli spaces of arbitrarily high magnetic charge.

Future Directions -1/4

Add matter and arbitrary Wilson-'t Hooft lines. (In progress with Daniel Brennan)

Understand better how Fredholm property fails by using asymptotic form of the monopole metric.

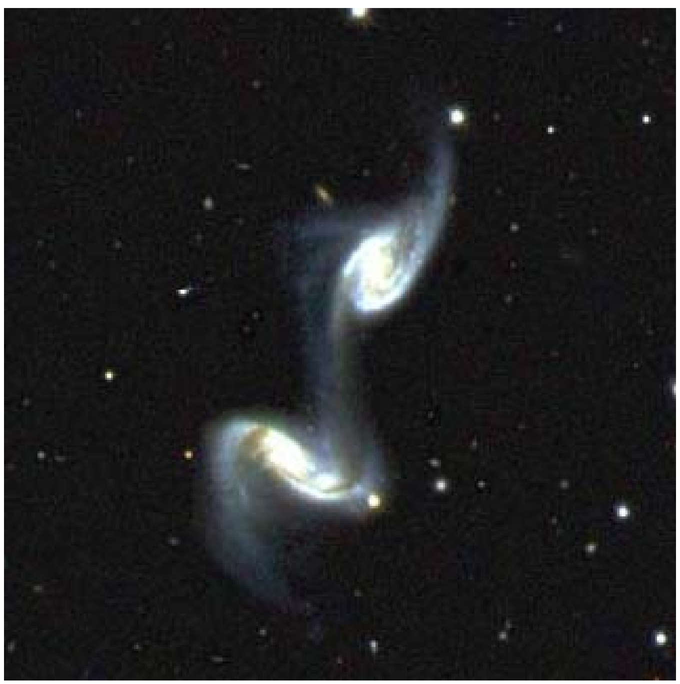
Combine the localization result of Ito, Okuda, Taki with ``Darboux expansion'' of GMN to get an interesting L^2-index theorem on (noncompact!) monopole moduli spaces ?

Future Directions -2/4

(In progress with Daniel Brennan & Andy Royston)

Understand better how Fredholm property of **D** fails:

Use asymptotic metric for moduli space for widely separated monopoles with charges $\in$ {simple roots}



$$\mathcal{M}_0^N$$

$$\Downarrow$$

$$\mathcal{M}_0^{N_1} \times \mathcal{M}_0^{N_2} \times TN$$

Future Directions -3/4

(In progress with Anindya Dey)

There are methods to compute vev's of susy line defects on $\mathbb{R}^3$ x S^1 exactly.

$$\langle L \rangle = \sum_{\gamma} \overline{\underline{\Omega}}(L,\gamma) \mathcal{Y}_{\gamma}$$ GMN-2010

$$L_{\zeta} = \mathrm{Tr}_{\mathbf{2}} \mathrm{Pexp} \int_{\mathbb{R}_t \times \vec{0}} \left(\zeta^{-1} \varphi + A + \zeta \bar{\varphi} \right)$$

Weak coupling expression + known nonperturbative corrections.

Surprising nonperturbative correction

$$\langle \mathrm{Tr}_2 L_{\zeta} \rangle = \sqrt{\mathcal{Y}_{\gamma_e}} + \frac{1}{\sqrt{\mathcal{Y}_{\gamma_e}}} + \sqrt{\mathcal{Y}_{\gamma_m + \gamma_e}}$$

Future Directions -4/4

Localization computations of the same quantities by Ito, Okuda, Taki (2011) give expressions like:

$$\langle L \rangle = \sum_v e^{2\pi \mathrm{i}(v,\mathfrak{b})} Z(P, v, \mathfrak{a})$$

$\mathfrak{a}, \mathfrak{b}$: complexified Fenchel-Nielsen coordinates

$Z(P, v, \mathfrak{a})$ Involves integrals over moduli spaces of singular monopoles of characteristic classes.

So, What Did He Say?

Recent new old

Recent results on N=2 d=4 imply new results about the differential geometry of old monopole moduli spaces.