ASC Lecture II

- Put up the outline again for review.
- Answer question about 1stini Berry phase.
- Review briefly lecture 1 esp. 4A.
The 4 boxed spaces are examples of Cartan symmetric spaces, and their tangent spaces at the origin will define 4 of the 10 ensembles of free fermion Hamiltonians in the AZ scheme.

4) Representation Theory For \( \mathbb{Z}_2 \)-Graded Groups

Now recall what we said about \( \mathbb{Z}_2 \)-graded groups.

Let \((G, \phi)\) be a \(\mathbb{Z}_2\)-graded group.

**Def. 1.** A \((G, \phi)\)-rep, denoted \((V, \rho)\) is a C-v.s. \(V\) with
\[
\rho : G \to \text{End}(V_R)\]
\[
\rho(g) = \begin{cases} 
\text{C-linear } & \phi(g) = +1 \\
\text{C-antilinear } & \phi(g) = -1
\end{cases}
\]

2. An **intertwiner** \(\phi : (V, \rho_1) \to (V, \rho_2)\) is a C-linear map:
\[
\phi(g) \quad \rho_2(g)
\]
\[
(V, \rho_1) \to (V, \rho_2)
\]

\[
\text{Hom}_G^\phi(V_1, V_2) = \text{R-v.s. of intertwiners}
\]

(Note: If \(\phi\neq 1\) then \(\phi \) intertwiner \(\Rightarrow i\phi \) not an intertwiner)
Expl. $(M^\pm_2, \phi)$ reps

$M^+_2$: $\rho(T)$ C-antilinear, $\rho(T)^2 = 1$

\[ \therefore \text{an } (M^+_2, \phi) \text{ rep is a complex vector-space w/ real structure.} \]

$M^-_2$: $\rho(T)$ C-antilinear, $\rho(T)^2 = -1$

\[ \therefore \text{an } (M^-_2, \phi) \text{ rep is a quaternionic structure } (\implies V \cong \mathbb{C}^{2n} \text{ has even dimension } \implies \text{Kramers' Theorem}) \]

4.B. Most of the rep. theory of $(G, \phi)$ is just like ordinary rep. theory: unitary, reducible etc. One big difference is in Schur’s lemma. To state it nicely we need:

\[ \text{Def: A } \mathfrak{A} \text{ division algebra } \mathfrak{A} \text{ is such that } \forall a \neq 0, \ a^{-1} \text{ exists.} \]

Thm (Schur) 1. $\text{Hom}_G (V_1, V_2) = 0$ for irreps $V_1 \neq V_2$

2. $\mathbb{Z}(V, \rho) = \text{Hom}_G$
Thm (Schur)
1. An intertwiner between two irreps is either zero, or an isomorphism.

2. If \((V, \rho)\) is a \((G, \phi)\) irrep then
\[
\text{End}_c^G(V) = \text{Hom}_c^G(V, V) = \text{Real division algebra}
\]

Pf: 

So: What are the real division algebras?

Thm (Frobenius): \(\exists\) precisely 3 associative real division algebras: \(\mathbb{R}, \mathbb{C}, \mathbb{H}\)

That's the "three" in the "three-fold way"

**END ASC LECTURE I**

**Expl 1** \((G, \phi) = M_2^+\)
Imrep \(V = \mathbb{C}\), \(\rho(T) = C\).

\[
\text{Hom}_c^G(V, V) = \left\{ \psi : z \rightarrow rz \mid r \in \mathbb{R} \right\} \cong \mathbb{R}
\]

**Expl 2** : \((G, \phi) = M_2^-\) ; \(V = \mathbb{C}^2\)

\[
\rho(e^{i\theta})(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}) = \begin{pmatrix} e^{i\theta}z_1 \\ e^{i\theta}z_2 \end{pmatrix} \quad \text{and} \quad \rho(T)(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}) = \begin{pmatrix} -\overline{z}_2 \\ \overline{z}_1 \end{pmatrix}
\]
How shall we compute the vector space of intertwiners?

Note that the operators \( p(e^{i\theta}) \) and \( p(T) \) generate an algebra of operators, and the intertwiners must be \( \mathbb{C} \)-linear operators commuting with this algebra.

In fact, quite generally, for a \((G, \phi)\)-rep \( V \), let

\[
A = \text{algebra of operators gen. by } p(g), I \in \text{End}(V_R)
\]

\[
\text{End}_G^\phi(V) = A' = \{ O : V_R \to V_R \mid O a = a O \forall a \in A \}
\]

= "commutant of \( A \) in \( \text{End}(V_R) \)"

This is itself an algebra, and \( A'' = A \).

Now let's get back to our example.

We use a trick:

\[
\begin{align*}
1^2 & \sim \#1 \\
\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} & \mapsto z_1 + z_2 j \\
& = x_1 + y_1 i + x_2 j + y_2 k
\end{align*}
\]
\( \rho (e^{i\theta}) = c(\theta) + s(\theta) L(\bar{z}) \quad \forall \theta \)
\[ \rho (T) = L(f) \]

Together these generate the full algebra
\[ \{ L(q) \} \cong \mathbb{H} \]

Now recall that \( \{ R(q) \} \cong \mathbb{H}^{opp} \)
is in the commutant and since
\[ \mathbb{H} \otimes \mathbb{H}^{opp} \cong \text{End}(\mathbb{R}^d) \]
this is the full commutant.

So, this gives an example of the third case of Schur's lemma.
Complete Reducibility

Thm: Suppose $G$ is compact. Any $(G, \phi)$-rep is completely reducible:

- $\{ (V_\lambda, \rho_\lambda) \}$ complete list of distinct irreps

$$V = \bigoplus_{\lambda \in \mathcal{U}} S_\lambda \otimes V_\lambda$$

$S_\lambda =$ real vector space of degeneracies

If we choose a basis this is just block diagonalization

$$\rho(g) = \begin{bmatrix}
\mathbf{1}_{S_\lambda} \otimes \rho_\lambda(g)
\end{bmatrix}$$

$$1_{S_\lambda} \otimes \rho_\lambda(g) = \begin{bmatrix}
\rho(g) \\
\rho_\lambda(g)
\end{bmatrix}$$

$S_\lambda$ blocks
So the operators commuting with \( \rho(g) \) act on the blocks, with entries in \( \text{End}_C^G(V_\lambda) \):

\[
\text{End}_C^G(V) \cong \bigoplus \lambda \text{ End}(S_\lambda) \otimes \text{End}_C^G(V_\lambda) \\
\cong \bigoplus \lambda \text{ Mat}_{s_\lambda}(D_\lambda)
\]

**Def:** An algebra consisting of a direct sum of matrix algebras over \( D = \mathbb{R}, \mathbb{C}, \mathbb{H} \) is called a semisimple algebra and the list of \( D' \)'s which occur is the Wedderburn type.

Put differently:

\[
A = \text{ algebra gen. by } \rho(g), I \in \text{End}(V_{1R}) \\
\text{End}_C^G(V) = \text{ Commutant of } A = A' \text{ in } \text{End}(V_{1R})
\]

Now introduce some notation:

If \( K \) is any algebra let

\[
K(m) := \text{Mat}_m(K) \\
mK := \{ (k \ldots k) | k \in K \} \subset K(m) \\
\cong K
\]
Thm (Weyl). \( G \) cpt

\[ d_\lambda = \dim_{\mathbb{C}} V_{\lambda}, \quad \tau_\lambda = \begin{cases} 2d_\lambda & D_\lambda = \mathbb{R} \\ d_\lambda & = \mathbb{C} \\ \frac{1}{2} d_\lambda & = \mathbb{H} \end{cases} \]

\[ A = \bigoplus_\lambda S_\lambda D_\lambda(\tau_\lambda) \]
\[ A' = \bigoplus_\lambda \tau_\lambda D_\lambda^{opp}(S_\lambda) \]

Remarks: (1) N.B. The Wedderburn type of \( A' \) is the same - we'll make use in (2)

(1) Complete reducible not true for general \( G \)
(2) Nevertheless it does apply to many examples of nonept groups, notably crystallographic groups.

For example if \( G = \) lattice in Euclidean space then we have Bloch's thm:

\[ \mathcal{H} = \int dk \mathcal{H}_k \]

Brill.

thus

(That's "Bloch" with an "h"!)
Quantum Aut's Compatible With Dynamics

We saw that a group preserving quantum probabilities is $\mathbb{Z}_2$-graded.

In physical situations there is also a second grading:

$$\tau : G \rightarrow \mathbb{Z}_2$$

telling whether the symmetry preserves or reverses the orientation of time.

Suppose (for simplicity) that our system is time-translation invariant. Then unitary evolution is generated by a self-adjoint Hamiltonian:

$$U(t) = \exp\left(-\frac{it}{\hbar}H\right)$$

Def: A $(G,\phi)$ rep $\rho$ is compatible with dynamics if

$$\rho(g) U(t) \rho(g)^{-1} = U(\tau(g) t) \quad \forall g \in G$$

Ex: Show this is equivalent to

$$\rho(g) H \rho(g)^{-1} = \chi(g) H \quad \forall g \in G$$

where $\chi(g) = \phi(g) \tau(g)$.
Note: \( \psi \cdot x = 1 \)

Which two of the three are given a priori depends on how we formulate the problem.

**Remarks:**

\[ x + x(g) = 1 \text{ for some } g \]

General principles of QM. \( \Rightarrow \)
A "Symmetry group" always comes with a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-grading

\[(\phi, x): G \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \]

**Remarks:**

1. If \( x(g) = -1 \) for some \( g \in G \)
   Then \( \text{Spec}(H) \) is symmetric around \( E = 0 \)

\[
\begin{array}{c}
E \uparrow \\downarrow \\
\downarrow \uparrow \quad \rho(g) \quad \uparrow \downarrow \\
-E \uparrow \\downarrow
\end{array}
\]

In general, there is no distinguished zero of energy, but here there is.
2.) So, if $H$ is bounded from below, but not above then $X = 1$.
   In fact all QM textbooks say $X = 1$, i.e. they all say $\phi = \pi$.

3.) Nevertheless there are physically interesting examples with $X \neq 1$.
    That is, with anti-linear "symmetries" that preserve time orientation.
    I will give a list later in Lecture II, but Charlie Kane already mentioned
    "chiral symmetries" in polyacetylene.
    Henceforth in this lecture we take $X = 1$.
    Then in the next we allow $X \neq 1$ and that will lead to the 10-fold way.

4.) $I \to \ker c \to G \xrightarrow{c} \mathbb{Z}_2 \to I$
    In general this is NOT split, i.e.
    $G \neq G_0 \times \mathbb{Z}_2$ nor $G_0 \times \mathbb{Z}_2$
    "nonsymmetric magnetic crystallographic groups".

   230 3D crystallographic groups but 1651 magnetic crystal groups!
5.) (For experts)  

When discussing band structure with Magnetic Crystallographic symmetry you need the double-cover.

Should the magnetic point group be a subgroup of \( \text{Pin}(3) \) or \( \text{Pin}^{-1}(3) \)?

The rep. theory is different so there is a physical difference. I claim it is in \( \text{Pin}^+(3) \) and I can explain afterwards.

Of course a crystal breaks Lorentz invariance: \( O(1,3) \rightarrow \mathbb{Z}_2 \times O(3) \)

Nevertheless we know that the rel. corrections to band structure such as spin orbit and Darwin terms can be obtained from the Dirac Hamiltonian

\[
(C \alpha \cdot p + \beta m c^2 + V(\vec{r})) \psi = (mc^2 + \varepsilon) \psi
\]

But now we are constrained in our choice of Dirac rep. If we take

\[
\alpha = \left( \begin{array}{cc} 0 & \sigma^2 \\ \sigma^1 & 0 \end{array} \right), \quad \beta = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \quad \psi = (\psi_1)_{\text{large}}(\psi_2)_{\text{small}}
\]

and we get Schröd + Rel corrections for \( \psi_1 \).
So let's recall the $PT$ invariance of the Dirac eq.

\[(i\gamma^\mu \partial_\mu - mc - \gamma^0 V)\psi = 0\]

**Time reversal:**

\[(T\psi)(x) = \Sigma C\psi^*(Tx)\]

\[C(\gamma^i)^*C^{-1} = -\gamma^i\]

\[C(\gamma^0)^*C^{-1} = \gamma^0\]

\[T^2 = -1\] for any phase.

But \[(P\hat{n}\psi)(x) = \Sigma \hat{n} \cdot (\gamma\hat{n}\cdot \gamma) \chi_5 \psi P(x)\]

\[\Sigma \hat{n} = \pm 1 ?\]

Equation doesn't tell you but we should if we turn off $V$ we should be representing $Pin (1-, 3+)$ and not some other signature-by rel. invariance.

Now if we do have $V$ note that in this basis

\[C = \begin{pmatrix} 0 & -\eta \cdot \sigma \\ -\sigma \cdot \eta & 0 \end{pmatrix}\]

\[\hat{n} \cdot \hat{\sigma} \otimes \chi_5 = -i \begin{pmatrix} \hat{n} \cdot \sigma \\ -\hat{n} \cdot \hat{\sigma} \end{pmatrix}\]

**ARE BLOCK DIAGONAL.**
Now we have

\[ 1 \to \mathbb{Z}_2 \to \tilde{O}_m \to O(3) \times \mathbb{Z}_2 \]

If we pull back the 4D $\phi$-impep of $\text{Pin}(1,3+)$ it becomes reducible, a sum of 2 2D imps for the large and small components.

Moreover the correct cover of the magnetic point group is the pullback

\[ 1 \to \mathbb{Z}_2 \to \tilde{P}(C) \to P(C) \to 1 \]

\[ 1 \to \mathbb{Z}_2 \to \tilde{O}_m \to O(3) \times \mathbb{Z}_2 \to 1 \]

Finally the correct double cover space group for ball structure is:

\[ 1 \to L(C) \to \tilde{G}(C) \to \tilde{P}(C) \to 1 \]

\[ 1 \to L(C) \to G(C) \to P(C) \to 1 \]
6) Dyson's 3-Fold Way

Now finally we return to the Dyson problem.

Usually we start with $H$ and ask for the symmetries. Instead, Dyson turned it around:

**DP:** Given a $(G, \phi)$-rep $H$, what is the ensemble of compatible Hamiltonians (with $\chi = 1$)?

\[ H = \bigoplus_{\lambda} S_{\lambda} \otimes V_{\lambda} \]

\[ Z(H, \rho) = \bigoplus_{\lambda} \text{Mat}_{s_{\lambda}}(D_{\chi}) \]

\[ E = \bigoplus_{\lambda} \text{Herm}_{s_{\lambda}}(D_{\chi}) \]

So we look at $\text{Herm}(D)$.

Each ensemble has a $U(N, D)$ invariant measure with statistically indpt matrix elements: $D = \mathbb{C}$:

\[ d\mu = \frac{1}{Z} \prod_{i} T dH_{ii} \prod_{i < j} T dH_{ij} e^{-N \text{Tr} H^2} \]
This becomes a prob. distribution on the eigenvalues:

\[ d\mu(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z} \prod_{i \neq j} |\lambda_i - \lambda_j|^{\beta} e^{-\sum_i \lambda_i^2} d\lambda_i \]

\[ \beta = \begin{cases} 
\frac{1}{2} & \text{if } D \in \mathbb{R} \\
\frac{1}{4} & \text{if } D \in \mathbb{H} 
\end{cases} = \dim_{\mathbb{R}} D \]

Now, I'd like to put this into some context:

As mentioned, RMT originated in nuclear physics.
Ensembles of Hamiltonians With A Gap

We can restrict our ensembles above by requiring the existence of a "gap" in the spectrum.

Often this means a lower bound on the energy gap between the ground state and the first excited state.

But more generally we could choose an energy level $E_0$ and require all the Hamiltonians in our ensemble to sit inside

$$E_N(E_0) = \{ H \in \text{Herm}_N(\mathbb{C}) \mid (H - E_0)^- \text{ exists} \}$$

Now we can describe the homotopy type of this ensemble easily.

If we only care about the topology of the ensemble then we can use the technique of spectral flattening.

Set $E_0 = 0$.

Then for $H \in E_N(E_0)$ sgn $H$ makes sense and we can deform $H \rightarrow \text{sgn } H$.
Note:
1. $\text{sgn} H = \text{tr}(H)$ satisfies $\text{tr}(H)^2 = 1$

2. If $\varphi : H_1 \rightarrow H_2$ is a path in $E_N(0)$
Then $\varphi : \text{tr}(H_1) \rightarrow \text{tr}(H_2)$ is a cont. path
in $E_N(0)$

We can deformation retract $E_N(0)$ to the
Spectrally flattened Hamiltonians.

But now, if $\text{tr}^2 = 1$ then

$$P_\pm = \frac{1}{2}(1 \pm \text{tr}) = \text{pair of orthogonal projectors}$$

If $P_\pm$ has rank $k$, then it is in

$$Gr_{k,N}(C) = \{ \text{Space of rank } k \text{ projectors} \}$$
$$= \{ \text{Space of } k \text{-dim subspaces of } C^N \}$$
$$\cong U(N,C)/U(k) \times U(N-k)$$

$$E_N(E_8) \sim \bigoplus_{k=0}^{N} Gr_{k,N}(C)$$
As with the space of pure states, these spaces have interesting large \( k,N \) limits and define important "classifying spaces" in topology.

One can do exactly the same thing with \( \mathbb{R}, \mathbb{H} \)-Hermitian Hamiltonians and we just replace

\[
\begin{align*}
U(N, \mathbb{C}) &\rightarrow U(N, \mathbb{R}) = O(N) \\
U(N, \mathbb{H}) &\rightarrow Sp(N) \cong USp(2N)
\end{align*}
\]

This gives us 3 more series of Cartan symmetric spaces.
Let us now return to the general symmetry constraints. We saw that any group of symmetries in Q.M. is $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded, but that usually it is assumed that $X=1$.

To motivate the consideration of $X \neq 1$ let us return to the history of RMT. First entered physics in the 1980's in the subject of mesoscopic physics - physics on scales orders larger than atomic but quantum interference is still important.

(Bigger than a virus and smaller than a bacterium.)

Good example: Universal Conductance Fluctuations

Conductance of electrons in a disordered mesoscopic sample, gold ring or Si MOSFET exhibits reproducible fluctuations (say as a function of $B$) of order $e^2/h$, independent of sample size, disorder, and other things.

Explained by RMT (Beenakker review)
Later in 1970's

- Verbaarschot: Statistics of ev's of Dirac op.
  \[ \mathcal{D} = (D_{+}, D_{-}) \]
  in a random gauge field.
- Altland+Zirnbauer: e- transport in disordered hybrid systems of norm/sc metal
- Katz+Sarnak: Statistics of zeroes of L-functions in number theory - these are generalizations of the Riemann z-function.

All clearly showed the need for ensembles of Hamiltonians distinct from the 3 Dyson classes.

This culminated in the "10-fold way"
Classification of AZ, HTZ - describing the symmetry classes of free fermion Hamiltonians.

Around 2008 A. Ludwig et.al. and A. Kitaev applied the same mathematics (and some new elements, like adeling a gap and using K-theory)
to the classification of phases of matter in the context of gapped free fermion systems.

So the primary examples are ensembles of free fermion Hamiltonians

\[ H = \frac{i}{4} \sum A_{ij} e_i e_j \]

\[ e_i - Mandonna \]

We'll come back to this. But, are there other examples of physical systems with \( X \neq 1 \)?

Answer is "Yes." But the question is how natural these examples are.

1. Lattice systems on bipartite lattices can easily have such symmetries:

\[ \Lambda = \Lambda_0 \Lambda_1 \]

\[ H = \sum_{i,j} J \sigma_i^x \sigma_j^x + \sum_{i \in \Lambda_1} h \sigma_i^y \]

\[ g = \prod_{i \in \Lambda_1} \sigma_i^x \]

(Su Schnieffer-Hegge OR SSH MODEL OF POLYACETYLENE)
2. Photonic bandstructure (DeNittis+Lein, ?)

3. One can engineer phonon band structures
e.g. 1-d lattice with 2 atoms/unit cell

4. Cold atoms?

5. Hilbert-Polya Hamiltonian whose
spectrum gives nontrivial zeroes of \( \xi(s) \)
(if it exists)

So now we want to give the analog
of Dyson's 3-fold way, but allowing
for \( X \neq 1 \).

This will turn out to be a 10-fold way,
but to explain that I need to do some
more math.
**$\mathbb{Z}_2$-Graded (Super-) Linear Algebra**

**Def:** a) A $\mathbb{Z}_2$-Graded (Super-) Vector-Space $k$

$$V = V^0 \oplus V^1$$

- even
- odd

b) Homogeneous vectors are vectors entirely in $V^0, V^1$: \[ \deg V = \| V \| = 0, 1 \mod 2 \]

**Expl:** If $V$ has an operator $\Gamma$ with $\Gamma^2 = 1$, then use $\pm 1$ eigenspaces. For example $\Gamma = (-1)^F$ in theories with fermions.

**Expl 2:** If $V$ is a real vs. $\mathbb{C}$ператор complex str. I Then $\text{End}(V)$ has a $\mathbb{Z}_2$-Grading

\[ \Gamma: A \to \text{IAI}^{-1} \]

**Expl 3:** \( k = \mathbb{R}, \mathbb{C} \)

$$k^{n+1} = k_{\text{even}} \oplus k_{\text{odd}}$$

\((n+1)_{n-}\) is called the superdimension
Now if \( V, W \) are sv.s.

Then \( \text{Hom}(V, W) \) is a sv.s:

\[
\text{Hom}(V, W)^0 = \text{Hom}(V^0, W^0) \oplus \text{Hom}(V', W') \\
\text{Hom}(V, W)^1 = \text{Hom}(V^0, W') \oplus \text{Hom}(V', W^0)
\]

If we choose bases of homogen. vectors we have matrices:

\[
\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \text{ even op's } \quad \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \text{ odd op's}
\]

We can also speak of superalgebras:

**Def:** A superalgebra \( A \) is a sv.s/k with a \( k \)-bilinear even product

\( \begin{array}{c} A \times A \rightarrow A \\ \text{deg} (a a') = \text{deg}(a) + \text{deg}(a') \end{array} \)

* graded commute \( a a' = (-1)^{l(a)l(a')} a' a \)

(Convention: If I write a formula with "deg" in it— it is understood that the vectors are homogeneous.)
Super center \( \mathcal{Z}(A) = \{ a \mid a b = (-1)^{\lVert a \rVert \lVert b \rVert} b a, \forall b \in A \} \)

(might or might not be the same as the ungraded center)

Graded Tensor Product \( A_1 \hat{\otimes} A_2 \)

- SVS
- \( (a_1 \hat{\otimes} a_2)(b_1 \hat{\otimes} b_2) = (-1)^{\lVert a_2 \rVert \lVert b_1 \rVert} a_1 b_1 \hat{\otimes} a_2 b_2 \)

"Koszul sign rule" - can have dramatic consequences

Example: Matrix superalgebras:

\( V - \) SVS \( \Rightarrow \) \( \text{End} (V) = \text{Hom}(V, V) \)

is a superalgebra.

Not all superalgebras are matrix superalgebras - as well see.
Finally, we need the notion of "Morita equivalence" of two superalgebras. The proper math definition is that $A_1, A_2$ are Morita equivalent if there is an equivalence of their category of representations, but we won't go into that.

All we need is the following:

**Def**: $A_1 \sim_{\text{Morita}} A_2$ if $\exists v s s V$

$$A_1 \cong A_2 \hat{\otimes} \text{End}V \quad \text{(or vice versa)}$$

Denote Morita equiv. class by $[A]$, and note there is a product:

$$[A_1] \cdot [A_2] = [A_1 \hat{\otimes} A_2]$$

It might or might not have an inverse.