ASC Lecture I

Outline

1. Brief Remarks On Physical Motivation
2. States, Projectors, and Wigner's Theorem
3. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ - Vector Spaces
4. Rep^{n} Theory Of $\mathbb{Z}_2$-Graded Groups
5. Quantum Automorphisms Compatible With Dynamics
6. Dyson's 3-Fold Way
7. Ensembles of Hamiltonians w/ A Gap
8. Some Physical Systems With $\chi \neq 1$
9. $\mathbb{Z}_2$-Graded (Super-) Linear Algebra
10. Real & Complex Clifford Algebras
11. Generalized Dyson Problem & 10-Fold Way
12. The 10 CT-Groups & Morita Classes Of Clifford Algebras
13. Free Fermion Dyson Problem & 10 Classical Cartan Symm. Spaces
14. Matching Symmetric Spaces To Superdivision Algebras
15. Bott Periodicity & Morse Theory

Lecture I (items 1–4)

Lecture II (items 4–9)

Lecture III (items 10–11)

Notes: Go to PiTP homepage:

pitp2015.ias.edu

scroll down to paragraph on pre-reading list

Or go to my homepage: talk # 61

① Brief Remarks on Physical Motivation

The theme of the three lectures I'll be giving is classifying various ensembles of Hamiltonians.

Some of the mathematics used to describe random matrix ensembles found useful application in the so-called periodic table of topological insulators and superconductors.

My goal will be to explain some of the relevant mathematics, especially the 10-fold way and the related Altland-Zirnbauer classific. of free fermion Hamiltonians.

These ideas first entered physics in the study of random matrices in the 1950's. Experiments showed that scattering slow neutrons off heavy nucleii, like thorium and uranium led to a complicated pattern of thousands of resonance lines.

Wigner had the idea that one should model the statistics of these lines using ~~p~~ a probability measure on an ensemble of Hamiltonians - hence "random" Hamiltonians

Dyson took up this problem in a series of papers and, in 1962, wrote a famous paper "The Threefold Way" – pointing out that basic symmetry principles profoundly alter the statistics of eigenvalues.

2A ~~Much of this lecture (+ The next) is aimed at explaining Dyson's paper, in such a way that the generalization to the 10-fold way will~~ be easy and natural.

Quick Overview

(2) Start with basic idea that probability amplitudes should be preserved and end with Wigner's theorem: all symmetries are realized by unitary and antiunitary op's on Hilbert space.

Then, we want to represent these, but to understand that properly we need to review some linear algebra not usually covered in physics courses:

(3) $\mathbb{R}, \mathbb{C}, \mathbb{H}$ vector spaces and their relations.

Usually in physics we start with a Hamiltonian and find its symmetries:

$$H \rightsquigarrow \text{Symmetries of } H$$

Dyson reverses the logic: He postulated a symmetry ~~and asks for the compatible Hamiltonians:~~ – more precisely a rep. of a group – and asks for the ensemble, or set, of compatible Hamiltonians.

Much of lectures 1+2 are aimed at explaining Dyson's paper, in such a way that the generalization to the 10-fold way will be easy and natural.

(4) Then we describe rep^{n} theory of groups of unitary + anti-unitary

(5) Then we add the constraint that the groups of operators are compatible with dynamics and derive a simple constraint for that: Crucially all groups are $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded

(6) Then Dyson's 3-fold way is a simple corollary - having set things up right. That is our first example of ensembles

(7) Then we see how things change when we add a gap condition to H.

(8) Then we talk about physical systems where the second grading is nontrivial.

(9) This motivates a quick review of super-algebra

(10) In particular we describe in some detail the superalgebras known as $\mathbb{R}, \mathbb{C}$ Clifford

(11) Then we can state the generalized Dyson problem and its solution: The 10-fold way

(12) The rest makes contact with the literature on free fermions + AZ.

② States, Projectors, and Wigner's Theorem

In our courses on QM we are often taught that "states" (meaning pure states) are normalizable vectors in Hilbert space

$$\psi \in \mathcal{H}$$

Then we are told that we should normalize the wave vector: $\|\psi\| = 1$.

Then we are told that states related by

$$\psi_1 = z\,\psi_2 \qquad \text{where } |z| = 1 \text{ pure phase}$$

represent the same physical system.

I will now rephrase this in a more geometrical and conceptual way:

To every normalizable vector we assign a <u>rank 1 projection operator</u>:

$$P_\psi := |\psi\rangle \frac{1}{\langle\psi|\psi\rangle} \langle\psi|$$

So $P_\psi^2 = P_\psi$ and $\operatorname{Im} P_\psi \subset \mathcal{H}$ is a 1-dimensional subspace, a <u>line</u>:

$$\ell_\psi = \{ z\cdot\psi \mid z \in \mathbb{C} \} \subset \mathcal{H}.$$

So, there is a 1-1 correspondence

$$\mathbb{P}\mathcal{H} = \{\text{lines} \subset \mathcal{H}\} \Longleftrightarrow \left\{\begin{matrix}\text{rank } 1\\ \text{projectors}\end{matrix}\right\} \Longleftrightarrow \left\{\begin{matrix}\text{normalizable}\\ \text{vectors}\\ \psi \in \mathcal{H}\end{matrix}\right\} / \sim$$

Indeed, notice $\quad P_{z\psi} = P_\psi \quad$ if $\quad z \in \mathbb{C}^*$

So call the space $\mathbb{P}\mathcal{H}$.

In Q.M.'s we speak of probabilities, and these ultimately come down to <u>overlaps</u>:

$$\frac{|\langle\psi_1|\psi_2\rangle|^2}{\langle\psi_1|\psi_1\rangle\langle\psi_2|\psi_2\rangle} = \mathrm{Tr}_{\mathcal{H}}(P_{\psi_1} P_{\psi_2})$$

This defines a function

$$\vartheta : \mathbb{P}\mathcal{H} \times \mathbb{P}\mathcal{H} \longrightarrow [0,1].$$

Now $\mathbb{P}\mathcal{H}$ can be given a Riemannian metric s.t.

$$\vartheta(P_1, P_2) = \cos^2\left(\frac{d(P_1, P_2)}{2}\right)$$

Rather than prove this in detail, let's work out an important example:

~~<u>Expl</u>: $\mathcal{H} = \mathbb{C}^2 \ni \psi = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$~~

~~$\|\psi\|^2 = |z_1|^2 + |z_2|^2 \neq 0$~~

~~$S_1\mathcal{H} = \{\psi \mid \|\psi\|^2 = 1\} \cong S^3$~~

Expl: $\mathcal{H} = \mathbb{C}^2$

P = 2×2 rk 1 projector

---

1. P is Hermitian :

$$\mathrm{Herm}_2(\mathbb{C}) = \{ a\mathbb{1} + \vec{b}\cdot\vec{\sigma} \mid a, \vec{b} \text{ real} \}$$

2. $P \geq 0$: $\quad a \pm |\vec{b}| \geq 0.$

3. $\mathrm{Tr}\, P = 1$: $\quad a = \frac{1}{2}$

$$\boxed{1,2,3 \Rightarrow P = \frac{1}{2}(1 + \vec{x}\cdot\vec{\sigma}) \qquad \|\vec{x}\| \leq 1}$$

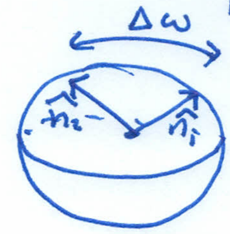
4. $P^2 = P$: $\Rightarrow \|\vec{x}\|^2 = 1$

since $P^2 = \frac{1}{2}\left(\frac{1+\vec{x}^2}{2} + \vec{x}\cdot\vec{\sigma}\right) = P$

~~∴ $\mathbb{P}\mathcal{H} \cong S^2$ ... $\{P = \frac{1}{2}(1+\hat{n}\cdot\vec{\sigma}) \mid \hat{n} \in S^2\}$~~

$$\therefore \quad \mathbb{P}\mathcal{H} = \{ P_{\hat{n}} = \tfrac{1}{2}(1 + \hat{n}\cdot\vec{\sigma}) \mid \hat{n}^2 = 1 \} \cong S^2$$

Indeed this <u>does</u> have a $\natural$ metric – the unit sphere metric.



Now let us check the overlap formula:

$$\mathrm{Tr}\left(P_{\hat{n}_1} P_{\hat{n}_2}\right) = \frac{1}{2}\left(1 + \hat{n}_1 \cdot \hat{n}_2\right).$$

$$= \cos^2\left(\frac{\Delta\omega}{2}\right) = \cos^2\left(\frac{d(P_{\hat{n}_1}, P_{\hat{n}_2})}{2}\right)$$

Now, before moving on to quantum automorphisms and symmetry let me make ~~a few~~ two remarks raised by the example (c. 8 minutes)

(1) In general, "states" in quantum mechanics are <u>density matrices</u>

<u>Def:</u> Hermitian, Nonnegative, $\mathrm{Tr}\,\rho = 1$

Our discussion above shows that the space of all density matrices on $\mathcal{H} = \mathbb{C}^2$ is

$$\left\{ \rho = \frac{1}{2}\left(\mathbb{1} + \vec{x} \cdot \vec{\sigma}\right) \;\middle|\; \|\vec{x}\| \leq 1 \right\}$$

This is a 3-dim'l ball. Note

- It is a convex space
- The set of extremal points is the space of pure states, in this case S^2

These two features are quite general.

(2) There is some beautiful topology here. For $\mathcal{H} = \mathbb{C}^2$ consider the set of unit normalizable states $\psi = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

$$S_1\mathcal{H} = \{ \psi \mid \|\psi\|^2 = 1 \}$$

$$= \{ x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1 \} = S^3 \subset \mathbb{R}^4$$

For every $\psi \in S_1\mathcal{H} \cong S^3$ we get a projector $P_\psi \in \mathbb{P}\mathcal{H} \cong S^2$ so we have a map:

$$\pi : S_1\mathcal{H} \longrightarrow \mathbb{P}\mathcal{H}$$

$$S^3 \longrightarrow S^2$$

Fiber is a copy of $U(1)$.

It is called the Hopf fibration.

Very important in both physics + math.

If $\psi = u \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$P_\psi = u \, \tfrac{1}{2}(1+\sigma^3)\, u^+ = \tfrac{1}{2}(1+\hat{n}\cdot\vec{\sigma})$$

$$u = e^{-i\frac{\phi}{2}\sigma^3} e^{-i\frac{\theta}{2}\sigma^2} e^{-i\frac{\psi}{2}\sigma^3} \implies$$

$$\hat{n} = \hat{n}(\theta, \phi)$$

In general, if $\mathcal{H} = \mathbb{C}^{N+1}$ then $\mathbb{P}\mathcal{H} = \mathbb{C}\mathbb{P}^N$ has a similar fibration

$$\begin{array}{ccccc} U(1) & \longrightarrow & S_1\mathcal{H} & \xrightarrow{\pi} & \mathbb{P}\mathcal{H} \\ & & \cong & & \cong \\ & & S^{2N+1} & \longrightarrow & \mathbb{C}\mathbb{P}^N \end{array}$$

These are examples of topological spaces known as fiber bundles – from this structure one can learn about the homotopy groups of $\mathbb{C}\mathbb{P}^N$:

First:

$$\pi_j(U(1)) = \begin{cases} \mathbb{Z} & j=1 \\ 0 & \text{else} \end{cases}$$

Next:

$$\pi_j(S^{2N+1}) = \begin{cases} 0 & j \leq 2N \\ \mathbb{Z} & j = 2N+1 \\ !!! & j > 2N+1 \end{cases}$$

A simple topological argument shows that this implies

$$\pi_j(\mathbb{CP}^N) = \begin{cases} \mathbb{Z} & j=2 \\ 0 & j \neq 2 \quad 0 \leq j \leq 2N \\ !!! & j > 2N \end{cases}$$

Now one can take an $N \to \infty$ limit based on inclusions of Hilbert spaces

$$\cdots \hookrightarrow \mathbb{C}^N \hookrightarrow \mathbb{C}^{N+1} \hookrightarrow \cdots$$

and

$$\pi_j(\mathbb{CP}^\infty) = \begin{cases} \mathbb{Z} & j=2 \\ 0 & \text{else} \end{cases}$$

This is an example of an "Eilenberg-Maclane space" $K(\mathbb{Z},2)$ - very important in topology.

In Kitaev's classification of SRE, invertible quantum groundstates in d dimensions, they form a space $\mathfrak{B}_d$. For the case of $d=0$, <u>with no locality imposed</u>, $\mathfrak{B}_0 = \mathbb{CP}^\infty$.

This is the first step in a series of spaces known as a <u>spectrum</u>. Hope to return to this @ end.

Now we return to our main theme,
We want to talk about <u>symmetry</u>

<u>Def:</u> The group of <u>quantum automorphisms</u> is the set of transformations:

$$\mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H}) = \{ F : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H} \text{ preserving } \theta \}$$

<u>Expl:</u> $\mathcal{H} = \mathbb{C}^2$, $\mathbb{P}\mathcal{H} = S^2$, $\mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H}) = O(3)$

N.B. These are the <u>potential</u> symmetries of any quantum system with Hilbert space $\mathcal{H}$.

Now, it is easier to work with the <u>linear</u> space $\mathcal{H}$ than $\mathbb{P}\mathcal{H}$, so we have the crucial result:

> Thm (Wigner): Every quantum automorphism is induced by a unitary or antiunitary operator on $\mathcal{H}$.

Meaning:

if $u \in U(\mathcal{H})$ then it induces

$\bar{f}_u : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}$ defined by

$\bar{f}_u : P \to u P u^{+}$ } clearly rk 1 projector, preserves ϑ.

so $\bar{f}_u \in Aut_{qtm}(\mathbb{P}\mathcal{H})$

Note: $\pi : u \to \bar{f}_u$ a homom.

Next:

Def: a) A map $\Theta : \mathcal{H} \to \mathcal{H}$ is <u>$\mathbb{C}$-antilinear</u> if

$$\Theta(\psi_1 + \psi_2) = \Theta(\psi_1) + \Theta(\psi_2)$$
$$\Theta(z\psi) = z^* \Theta(\psi)$$

b.) It is <u>antiunitary</u> if $\|\Theta(\psi)\| = \|\psi\|$

<u>Ex:</u> Define the adjoint of a $\mathbb{C}$-antilinear operator by

$$(\psi_1, \Theta\psi_2) = (\psi_2, \Theta^{+}\psi_1)$$

Show that Θ is antiunitary iff

$$\Theta\Theta^{+} = \Theta^{+}\Theta = 1$$

Now $a \xrightarrow{\pi} \bar{f}_a : P \to a P a^{+}$ gives induced QA.

~~Ex: Define $(\psi_1, \mathcal{O}\psi_2) = (\psi_2, \mathcal{O}^\dagger \psi_1)$ for $\mathcal{O}$ antilinear. Show that if $\mathcal{O}$ is antiunitary then $\mathcal{O}\mathcal{O}^\dagger = \mathcal{O}^\dagger\mathcal{O} = 1$.~~

$a \overset{\pi}{\Rightarrow} F_a : P \to aPa^\dagger$ is a QA.

The set $\left\{ \begin{matrix} \text{Unitary and} \\ \text{antiunitary} \end{matrix} \right\}$ op's form a group:

middle of board! and save

$$1 \to U(1) \to Aut_{\mathbb{R}}(\mathcal{H}) \xrightarrow{\pi} Aut_{qtm}(\mathbb{P}\mathcal{H}) \to 1$$

↑ Wigner: <u>onto</u>

The map π is a <u>homomorphism</u>.
The kernel of π is the subgroup of unitary operators:

$$\mathcal{O}_z : \psi \to z\psi, \quad |z| = 1$$

Henceforth: just denote $\mathcal{O}_z$ by z.

We say there is an <u>exact</u> <u>sequence</u>

In our example of $\mathcal{H} = \mathbb{C}^2$

$$\begin{array}{ccc} \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) & \xrightarrow{\ \pi\ } & \mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H}) \\ \parallel & & \parallel \\ U(2) \amalg U(2)\cdot \mathcal{C} & & SO(3) \amalg SO(3)\cdot \underline{P} \end{array}$$

$\mathcal{C}\psi = \psi^*$
(does not have 2×2 matrix rep)

~~[illegible]~~

$$\mathcal{O}\, \tfrac{1}{2}(1+\hat{n}\cdot\vec{\sigma})\,\mathcal{O}^{-1} = \tfrac{1}{2}(1+\hat{n}'\cdot\vec{\sigma})$$

$\hat{n} \longrightarrow \hat{n}'$ $\begin{cases} SO(3) \text{ rotation for } \mathcal{O} \text{ unitary} \\ \underline{P} = \text{reflection in plane } \perp \hat{y} \text{ for } \mathcal{O} = \mathcal{C}. \end{cases}$

~~This~~ The discussion of components in this example generalizes:
$\mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$ and $\mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H})$ each have precisely two connected components:

$$1 \to U(1) \to \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \xrightarrow{\ \pi\ } \mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H}) \to 1$$

$$\mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \xrightarrow{\ \phi\ } \mathbb{Z}_2 = \{\pm 1\} \xleftarrow{\ \phi'\ } \mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H})$$

$\phi(u) = +1, \ \phi(a) = -1$

ϕ, ϕ' just measure which conn. component.

Note that for all $\mathcal{O} \in \mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$

$$\mathcal{O} z = z^{\phi(\mathcal{O})} \mathcal{O} = \begin{cases} z\mathcal{O} & \mathcal{O} = u \\ \bar{z}\mathcal{O} & \mathcal{O} = a \end{cases}$$

Now suppose a group $\bar{G}$ acts on a quantum system preserving overlaps

$$\begin{array}{ccccccccc} 1 & \rightarrow & U(1) & \longrightarrow & G & \xrightarrow{\pi'} & \bar{G} & & \\ & & \| & & \downarrow \lambda & & \downarrow \bar{\lambda} & & \\ 1 & \rightarrow & U(1) & \longrightarrow & \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) & \xrightarrow{\pi} & \mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H}) & \rightarrow & 1 \\ & & & & \downarrow \phi & \swarrow \phi' & & & \\ & & & & \mathbb{Z}_2 = \{\pm 1\} & & & & \end{array}$$

Then there is a tautological construction ("the pullback") of a group G such that

$$G = \{ (\bar{g}, \mathcal{O}) \mid \bar{\lambda}(\bar{g}) = \pi(\mathcal{O}) \} \subset \bar{G} \times \mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$$

By composition all 4 groups have homom's $\phi \rightarrow \mathbb{Z}_2$
Just call them all ϕ. Note for G:

$$\boxed{g z = z^{\phi(g)} g}$$

So we make an abstract def$^{\underline{n}}$

① A Group with a homom. to $\mathbb{Z}_2$ is a "$\mathbb{Z}_2$-graded group"

② $$1 \to U(1) \to G \xrightarrow{\pi} \bar{G} \to 1$$

$$\phi : G \to \{\pm 1\}, \qquad \bar{\phi} : \bar{G} \to \{\pm 1\}$$

[illegible] • ~~(G,ϕ) is a $\mathbb{Z}_2$-graded group~~

• $g z = z^{\phi(g)} g$ "ϕ-twised central extension by $U(1)$"

or just a "ϕ-twisted extension" for short.

Example : $\bar{G} = \bar{M}_2 := \{1, \bar{T}\}$ with $\bar{T}^2 = 1$
$\cong \mathbb{Z}_2$

and we choose $\mathbb{Z}_2$-grading $\bar{\phi}(\bar{T}) = -1$.

What are the ϕ-twisted extensions?

There are precisely two:

$$1 \to U(1) \to M_2^{\pm} \xrightarrow{\pi} \bar{M}_2 \to 1$$

$$T \to \bar{T}$$

$$U(1) \ni z = T^2 \to 1$$

$$T^3 = T T^2 = T z = z^{-1} T$$
$$= T^2 T = z T \Rightarrow z = \pm 1$$

The lift is ambiguous: $T' = \mu T$, $|\mu| = 1$
is another lift.

But

$$(T')^2 = \mu T \mu T = \mu \mu^{-1} T^2 = T^2$$

So the sign of T^2 is well-defined.

So, we have:

$$M_2^{\pm} = \{ z T \mid T z = \bar{z}^{-1} T \ \&\ T^2 = \pm 1 \}$$

$$\phi(z) = +1, \quad \phi(T) = -1$$

Lesson: Groups in Q.M. preserving probabilities are $\mathbb{Z}_2$-graded and ~~come~~ Act on $\mathcal{H}$ via ϕ-twisted c.e.'s by $U(1)$.

Now we will take some time to cover another piece of mathematical background necessary to our story. Then we'll return to repn theory of (G, ϕ)

(3) $\mathbb{R}, \mathbb{C}, \mathbb{H}$ - vector spaces

3A Vector spaces are defined over a field k. For us, always $k = \mathbb{R}$ or $\mathbb{C}$, but the distinction is import. In a vector space over $k = \mathbb{R}$ we only know how to multiply vectors by real scalars. There is no sense to iV.

Def: Let V be a $\mathbb{R}$-v.s. A lin. trm.

$$I : V \to V \quad \text{s.t.} \quad I^2 = -1$$

is a complex structure

Given (V, I) we can define a ~~complex~~ $\mathbb{C}$-v.s. because now we know how to multiply vectors $v \in V$ by complex numbers $z = x + iy$:

$$z \cdot v = xv + y\, I(v).$$

For simplicity we assume V has a Eucl. metric and I is also an orthogonal transformation in what follows. (WLOG)

<u>Expl</u>: $V = \mathbb{R}^{2n}$ $\quad I_0 = \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}$

Note: If $S \in O(2n)$, I = any cplx str.,
then $S I S^{-1}$ also a cplx. str.
So {cplx structures} is has an $O(2n)$ action.

Can show: $\forall I \; \exists S \quad S I S^{-1} = I_0$

$$\Rightarrow \boxed{\mathbb{C}\,Str(\mathbb{R}^{2n}) = O(2n)/K = O(2n)/U(n)}$$

K = stabilizer of I_0

$$= \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \right\} \cong U(n)$$

because $\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = A \otimes \mathbb{1}_2 + i B \otimes \sigma^2 \sim \begin{pmatrix} A+iB & 0 \\ 0 & A-iB \end{pmatrix}$ (unitary)

⊗ ⟶

3B/// Now suppose V is a $\mathbb{C}$- v.s.
(Note change in notation!)

$V_{\mathbb{R}}$ = Same set of vectors, regarded as an $\mathbb{R}$-v.s. $\Rightarrow$ v, iv lin. indpt

$$\Rightarrow \dim_{\mathbb{R}} V_{\mathbb{R}} = 2 \dim_{\mathbb{C}} V$$

<u>Side Remark:</u> This space of complex structures is our first example of a Cartan symmetric space - that will play an important role later.

Note that $I I^{tr} = 1$ and $I^2 = -1$ imply $I^{tr} = -I$ so I is antisymmetric.

It will turn out that $O(2n)/U(n)$ is the space of spectrally flattened free fermion Hamiltonians for $2n$ Majorana modes. This space has two components - measured by $\text{sign}(\text{Pfaff}(I))$ - that's one of the $\mathbb{Z}_2$ K-theory invariants.

But there is another way to associate a ~~real v.s.~~ $\mathbb{R}$-v.s. with a $\mathbb{C}$-v.s.:

<u>Def:</u> If V is a $\mathbb{C}$-v.s. then a $\mathbb{C}$-antilinear operator $\Theta : V \to V$ s.t. $\Theta^2 = +\mathbb{1}$ is called a <u>real</u> <u>structure</u>.

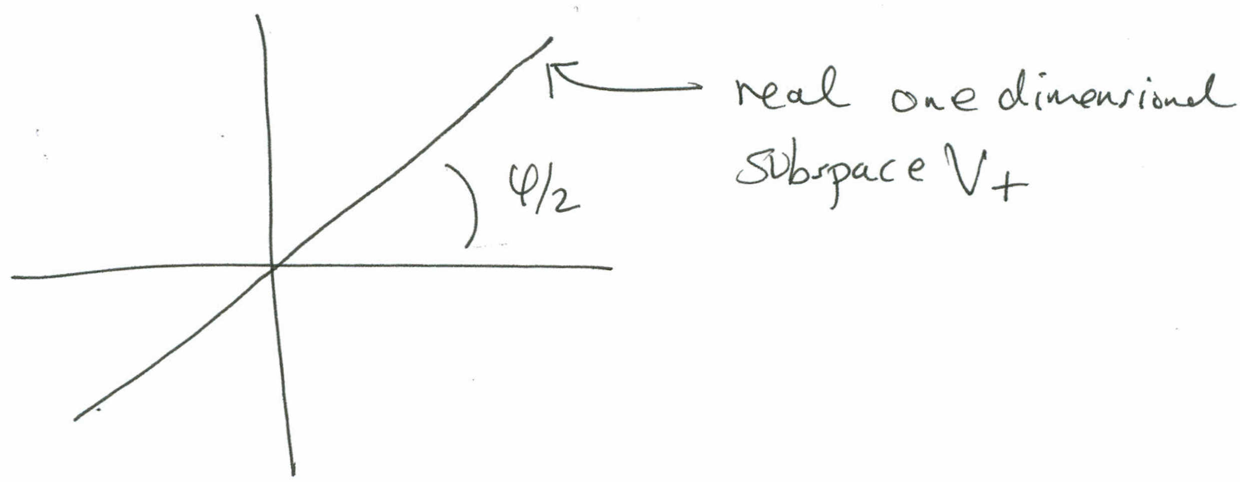
$$V_+ = \mathrm{Fix}(\Theta) = \{ v \mid \Theta(v) = v \}$$ real vector space

$$\Theta(v) = v \Rightarrow \Theta(iv) = -(iv)$$

<u>Exple:</u> $V = \mathbb{C}$, $V_{\mathbb{R}} = \mathbb{R}^2$

$$\Theta : x + iy \longrightarrow e^{i\varphi}(x - iy)$$ most general antiunitary

$V_{\mathbb{R}}$



If we <u>choose basis</u> $\{e_i\}$ for V, we get a canonical real structure

$$\Theta\left(\sum z_i e_i\right) = \sum z_i^* e_i$$

Conversely, any real structure is so obtained:

On $\mathbb{C}^n$ any (antiunitary) real structure is $S\mathcal{C}_0S^{-1}$ $\quad$ $\mathcal{C}_0$ = cplx conj. in standard basis, $S \in U(n)$

$$\Rightarrow \quad \boxed{\mathbb{R}\text{-}Str(\mathbb{C}^n) = U(n)/O(n)}$$

3C The quaternions

Def: An algebra $/k$ is a v.s. $/k$ with a k-bilinear mult:

$$\mu: \mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

- Usually simply denote $\mu(v_1, v_2) = v_1 v_2$

But note that given $\mathcal{A}$ there is a canonical $\mathcal{A}^{opp}$ with multiplication $\mu^{opp}(v_1, v_2) := \mu(v_2, v_1)$

- Our algebras will all be unital and associative.

Def: The quaternion algebra $\mathbb{H}$ is $\mathbb{R}^4$ with a basis $\underline{1}, \underline{i}, \underline{j}, \underline{k}$ s.t.

$$\underline{i}^2 = \underline{j}^2 = \underline{k}^2 = -1 \qquad \underline{i}\,\underline{j} + \underline{j}\,\underline{i} = 0 \quad \text{etc}$$

Associative, but not commutative.

$\mathbb{H}$ has a conjugation:

$$q = x_1 \underline{i} + x_2 \underline{j} + x_3 \underline{k} + x_4 \cdot \mathbb{1}$$

$$\bar{q} = -x_1 \underline{i} - x_2 \underline{j} - x_3 \underline{k} + x_4 \cdot \mathbb{1}$$

$$q\bar{q} = \bar{q}q = (x_\mu x_\mu) \cdot \mathbb{1}.$$

Many exercises in the notes are meant to familiarize you with $\mathbb{H}$.

In particular

① $U(n, \mathbb{H}) = \{ u \in Mat_n(\mathbb{H}) \mid u^+ u = 1 \}$ makes sense

$$\cong USp(2n) := \{ u \in U(2n, \mathbb{C}) \mid u^* = JuJ^{-1} \}$$

$$= \{ u \in U(2n, \mathbb{C}) \} \cap \{ u \mid u^{tr} J u = J \}$$

$$J = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$$

② $\mathbb{H}$ acts on itself by left and right multiplication:

$L(q): q' \longrightarrow qq'$ $\qquad \{L(q)\} \cong \mathbb{H}$

$R(q): q' \longrightarrow q'q$ $\qquad \{R(q)\} \cong \mathbb{H}^{opp} \cong \mathbb{H}$

Note that $L(q_1)$ and $R(q_2)$ obviously commute, and in fact

$$\mathbb{H} \otimes \mathbb{H}^{opp} \cong End(\mathbb{R}^4)$$

<u>Def</u>: An $\mathbb{H}$-v.s. is a $\mathbb{R}$-v.s. V together with a triplet of complex structures I, J, K satisfying:

$$I^2 = J^2 = K^2 = -1, \quad IJ + JI = 0 \text{ etc.}$$

<u>Exple</u> $\mathbb{H}^{\oplus n} \cong \mathbb{R}^{4n}$

I, J, K = componentwise $L(\underline{i}), L(\underline{j}), L(\underline{k})$.

<u>Def</u>: An $\mathbb{H}$-structure on a $\mathbb{C}$-v.s. V is a $\underline{\mathbb{C}\text{-antilinear map}}$: $J: V \longrightarrow V$ s.t. $J^2 = -1$

(Note: Then $V_{\mathbb{R}}$ is an $\mathbb{H}$ v.s. I = mult. by $\sqrt{-1}$, J is given and $K := IJ$ will then satisfy the required conditions.)

$$\boxed{\mathbb{H}\text{-Str}(\mathbb{C}^{2n}) = U(2n)/USp(2n)}$$

Finally, $\mathbb{C}$-structures on an $\mathbb{H}$-space:

$$\boxed{\mathbb{C}\text{-Str}(\mathbb{H}^n) = USp(2n)/U(n)}$$