\((i\gamma^\mu \partial_\mu - m)\psi = 0\)
Robbert Dijkgraaf’s Thesis Frontispiece
INVESTIGATORS SEE NO EXOTICS IN PURE SU(N) GAUGE THEORY
Use of Motives Cited
By E. Diaconescu, et. al.

RUTGERS – An application of results on the motivic structure of quiver moduli spaces has led to a proof of a conjecture of GMN. p.A12

Semiclassical, but Framed, BPS States
By G. Moore, A. Royston, and D. Van den Bleeken

RUTGERS – Semiclassical, framed BPS states have been constructed as structures familiar from String Field Theory. LG models are nearly trivial in

Operadic Structures Found in Infrared Limit of 2D LG Models

NOVEL CONSTRUCTION OF D ON INTERVAL

Hope Expressed for Categorical WCF
By D. Gaiotto, G. Moore, and E. Witten

PRINCETON - A Morse-theoretic formulation of LG models has revealed structures familiar from String Field Theory. LG models are nearly trivial in
Goal Of Our Project

Recently there has been some nice progress in understanding BPS states in d=4, N=2 supersymmetric field theory:

No Exotics Theorem & Wall-Crossing Formulae

What can we learn about the differential geometry of monopole moduli spaces from these results?
Papers 3 & 4 ``almost done''
Let $G$ be a compact simple Lie group with Lie algebra $\mathfrak{g}$.

$X \in \mathfrak{g}$ is **regular** if $Z(X)$ has minimal dimension.

Then $Z(X) = \mathfrak{t}$ is a Cartan subalgebra.

$T = \exp[2\pi \mathfrak{t}]$ is a Cartan subgroup.

\[
\Lambda^\vee_G := \text{Hom}(T, U(1)) \quad \text{character lattice}
\]

\[
\Lambda_G := \text{Hom}(U(1), T) \quad \exp(2\pi X) = 1
\]

\[
\Lambda_{rt} \subset \Lambda^\vee_G \subset \Lambda_{wt} \subset \mathfrak{t}^\vee
\]

\[
\Lambda_{cr} \subset \Lambda_G \subset \Lambda_{mw} \subset \mathfrak{t}
\]
Moreover, a regular element $X$ determines a set of simple roots $\alpha_I \in t^\vee$ and simple coroots $H_I \in t$

\[ \Lambda_{rt} = \bigoplus_I \mathbb{Z} \alpha_I \subset t^\vee \]

\[ \Lambda_{cr} = \bigoplus_I \mathbb{Z} H_I \subset t \]
Examples:

\[ H_1 = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad h^1 = -\frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ H_1 = -i \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad h^1 = -\frac{i}{3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \]

\[ H_2 = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad h^2 = -\frac{i}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \]
Nonabelian Monopoles

Yang-Mills-Higgs system for compact simple $G$

$$(A, X) \quad \int_{\mathbb{R}^4} \text{Tr}(F \ast F + DX \ast DX)$$

$F = \ast DX \quad \text{on } \mathbb{R}^3$

$F = \gamma_m \text{vol}(S^2) + \cdots \quad X \rightarrow X_\infty - \frac{\gamma_m}{2r} + \cdots$

$X_\infty \in \mathfrak{g} \quad \text{regular} \quad \rightarrow \quad t \alpha_I H_I$

$\gamma_m \in \Lambda_{cr} \subset t \subset \mathfrak{g}$

$\gamma_m = \sum_{I=1}^{r} n^I_m H_I \quad n^I_m \in \mathbb{Z}$
Monopole Moduli Space

\[ \mathcal{M}(\gamma_m; X_\infty) \]

SOLUTIONS/GAUGE TRANSFORMATIONS

Gauge transformations: \( g(x) \rightarrow 1 \) for \( r \rightarrow \infty \)

If \( \mathcal{M} \) is nonempty then [Callias; E. Weinberg]:

\[ \dim \mathcal{M}(\gamma_m; X_\infty) = 4 \sum_I n^I_m \]

Known: \( \mathcal{M} \) is nonempty iff all magnetic charges nonnegative and \textit{at least one} is positive (so \( 4 \leq \dim \mathcal{M} \))

\( \mathcal{M} \) has a hyperkahler metric. Group of isometries with Lie algebra:

\[ \mathbb{R}^3 \oplus \mathfrak{so}(3) \oplus \mathfrak{t} \]

Translations Rotations Global gauge transformations
Action Of Global Gauge Transformations

\[ H \in t \quad \rightarrow \quad G(H) \quad \text{Killing vector field on } \mathcal{M} \]

\[ \hat{A} = A_i dx^i + X dx^4 \quad \hat{F} = *\hat{F} \]

Directional derivative along \( G(H) \) at \( \hat{A} \) in \( \mathcal{M} \):

\[ \frac{d\hat{A}}{ds} = -\hat{D}\epsilon \]

Map \( \epsilon : \mathbb{R}^3 \rightarrow g \):

\[ \lim_{x \to \infty} \epsilon(x) = H \quad \hat{D}^2 \epsilon = 0 \]
Strongly Centered Moduli Space

\[ \tilde{\mathcal{M}}(\gamma_m; X_\infty) = \mathbb{R}^3 \times \mathbb{R} \times \mathcal{M}_0 \]

Orbits of translations

\[ \mathcal{M}(\gamma_m; X_\infty) = \mathbb{R}^3 \times \frac{\mathbb{R} \times \mathcal{M}_0}{\mathbb{Z}} \]

Higher rank is different!

\[ \mathcal{M}(\gamma_m; X_\infty) \neq \mathbb{R}^3 \times \frac{S^1 \times \mathcal{M}_0}{\mathbb{Z}_r} \]
Introduction

Monopoles & Monopole Moduli Space

Singular Monopoles

Singular Monopole Moduli: Dimension & Existence

Semiclassical N=2 d=4 SYM: Collective Coordinates

Semiclassical (Framed) BPS States

Application 1: No Exotics & Generalized Sen Conjecture

Application 2: Wall-crossing & Fredholm Property

Future Directions
Singular Monopoles

\[ F = \gamma_m \text{vol}(S^2) + \cdots \quad X \rightarrow X_\infty - \frac{\gamma_m}{2r} + \cdots \]
\[ \vec{x} \rightarrow \infty \]

**AND**

\[ F = P \text{vol}(S^2) + \cdots \quad X \rightarrow -\frac{P}{2r} + \mathcal{O}(r^{-1/2}) \]
\[ \vec{x} \rightarrow 0 \]

Use: construction of ‘t Hooft line defects (``line operators'')
Where Does The ‘t Hooft Charge $P$ Live?

$P \in \mathfrak{t} \quad \gamma_m \in \Lambda_{cr} + P$

Example: Rank 1

$H_1 = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

SU(2) Gauge Theory: Minimal $P$ \quad $P = \pm H_1$

SO(3) Gauge Theory: Minimal $P$ \quad $P = \pm \frac{1}{2} H_1 = \pm h^1$
Example: A Singular Nonabelian SU(2) Monopole

\[ X = \frac{1}{2} h(r) H \quad A = \frac{1}{2} (\pm 1 - \cos \theta) d\phi H \]

\[ + \frac{1}{2} f(r) \left[ e^{\pm i\phi} (-d\theta - i \sin \theta d\phi) E_+ + c.c. \right] \]

Bogomolnyi eqs:

\[ f'(r) + f(r) h(r) = 0 \]
\[ r^2 h'(r) + f(r)^2 - 1 = 0 \]

\[ h(r) = m_W \coth(m_W r + c) - \frac{1}{r} \quad f(r) = \frac{m_W r}{\sinh(m_W r + c)} \]

(‘t Hooft; Polyakov; Prasad & Sommerfield took \( c = 0 \))

c > 0 is the singular monopole: Physical interpretation?
1 Introduction
2 Monopoles & Monopole Moduli Space
3 Singular Monopoles
4 Singular Monopole Moduli: Dimension & Existence
5 Semiclassical N=2 d=4 SYM: Collective Coordinates
6 Semiclassical (Framed) BPS States
7 Application 1: No Exotics & Generalized Sen Conjecture
8 Application 2: Wall-crossing & Fredholm Property
9 Future Directions
Singular Monopole Moduli Space

\( \overline{\mathcal{M}}(P; \gamma_m; X_\infty) \) SOLUTIONS/GAUGE TRANSFORMATIONS

Now \( g(x) \) must commute with \( P \) for \( x \to 0 \).

When is it nonempty?

What is the dimension?

If \( P = \gamma_m \) is \( P \) screened or not?

Is the dimension zero?

or not?
Dimension Formula

Assuming the moduli space is nonempty repeat computation of Callias; E. Weinberg to find:

$$\dim \overline{M} = 2 \text{ind}(L) = \lim_{\epsilon \to 0^+} \text{Tr} \left( \frac{\epsilon}{L^\dagger L + \epsilon} - \frac{\epsilon}{LL^\dagger + \epsilon} \right)$$

For a general 3-manifold we find:

$$\dim \overline{M} = \int_{M_3 - \Sigma} dJ^{(\epsilon)} = 4 \sum_I \tilde{n}^I_m$$

Relative magnetic charges.
Dimension Formula

\[ \dim \overline{M} = 4 \sum_I \tilde{n}_m^I \]

\[ \sum_I \tilde{n}_m^I H_I = \gamma_m - P^- \]

\( \gamma_m \) from \( r \to \infty \) and \( -P^- \) from \( r \to 0 \)

\( P^- : \) Weyl group image such that \( \langle \alpha_I, P^- \rangle \leq 0 \)

(Positive chamber determined by \( X_\infty \) )
Existence

Conjecture:

\( \overline{M}(P; \gamma_m; X_\infty) \neq \emptyset \quad \iff \quad \forall I, \tilde{n}_m^I \geq 0 \)

Intuition for relative charges comes from D-branes. Example: Singular SU(2) monopoles from D1-D3 system

\[
X = \begin{pmatrix}
x_1 & 0 \\
0 & x_2
\end{pmatrix}
\]

\[
\gamma_m = P = (p_1^1 - p_2^2) \frac{1}{2} H
\]
\[ p^1 < p^2 \quad \Rightarrow \quad \gamma_m = P^- \]

\[ \dim \mathcal{M} = 0 \]

\[ p^1 > p^2 \quad \Rightarrow \quad \gamma_m = -P^- \]

\[ \dim \mathcal{M} = 4(p^1 - p^2) \]
Application: Meaning Of The Singular ‘t Hooft-Polyakov Ansatz

\[ X = (m_W \coth(m_W r + c) - \frac{1}{r}) \frac{1}{2} H \]

\[ \gamma_m = P = H \Rightarrow \tilde{n}_m = 2 \]

\[ \Rightarrow \text{dim} \mathcal{M} = 8 \]

Two smooth monopoles in the presence of minimal SU(2) singular monopole.

They sit on top of the singular monopole but have a relative phase: \[ e^{-c} = \sin(\psi/2) \]

Two D6-branes on an O6^- plane;

Moduli space of d=3 N=4 SYM with two massless HM
Properties of $\mathcal{M}$

Hyperkähler (with singular loci - monopole bubbling)

[Kapustin-Witten]
Isometries of $\overline{\mathcal{M}}$

$\overline{\mathcal{M}}$ has an action of $\mathfrak{so}(3) \oplus \mathfrak{t}$

$\mathfrak{so}(3)$: spatial rotations

$t$-action: global gauge transformations commuting with $X_\infty$

$H \in \mathfrak{t}$ \quad $\xrightarrow{G}$ \quad $G(H) \in \text{VECT}(\overline{\mathcal{M}})$
\( \mathcal{N}=2 \) Super-Yang-Mills

Second real adjoint scalar \( Y \)

Vacuum requires \([X_\infty, Y_\infty]=0\).

\[ \zeta^{-1} \varphi = Y + iX \]

Meaning of \( \zeta \): BPS equations on \( \mathbb{R}^3 \) for preserving

\[ Q + \zeta^{-1} \bar{Q} \]

\[ F = B = *DX \quad E = DY \]
ζ And BPS States

Framed case: Phase ζ is part of the data describing ‘t Hooft line defect L

$$\overline{H}^{\text{BPS}}(L, \gamma; u) \quad u \in \mathcal{M}_{\text{Coulomb}}$$

Smooth case: Phase ζ will be related to central charge of BPS state

$$\mathcal{H}^{\text{BPS}}(\gamma; u) \quad \zeta = -Z_\gamma(u)/|Z_\gamma(u)|$$
Semiclassical Regime

Definition: Series expansions for $a_D(a; \Lambda)$ converges: $|\langle \alpha, a \rangle| > c|\Lambda|$

Local system of charges has natural duality frame:

$$\Gamma = \Lambda_{rt} \oplus \Lambda_{mw}$$

$$\gamma = \gamma^e \oplus \gamma_m$$

(Trivialized after choices of cuts in logs for $a_D$.)

$$\Lambda(t) = e^{-\pi t/\hbar} \Lambda_0 \lim_{t \to +\infty} \mathcal{H}^{BPS}(\gamma; u_t)$$

In this regime there is a well-known semiclassical approach to describing BPS states.
Collective Coordinate Quantization

At weak coupling BPS monopoles with magnetic charge $\gamma_m$ are heavy: Study quantum fluctuations using quantum mechanics on monopole moduli space

The semiclassical states at $(u, \zeta)$ with electromagnetic charge $\gamma^e \oplus \gamma_m$ are described in terms of supersymmetric quantum mechanics on

$$\overline{M}(P, \gamma_m; X_{\infty}) \quad \text{OR} \quad M(\gamma_m; X_{\infty})$$

What sort of SQM? How is $(u, \zeta)$ related to $X_{\infty}$?

How does $\gamma^e$ have anything to do with it?
What Sort Of SQM?

(N = 4 SQM on $\mathcal{M}(\gamma_m, X_\infty)$ with a potential:

$$S = \int (\| \dot{z} \|^2 - \| G(\mathcal{Y}_\infty^{cl}) \|^2 + \cdots)$$

$$\mathcal{Y}_\infty^{cl} := \frac{4\pi}{g_0^2} Y_\infty + \frac{\theta_0}{2\pi} X_\infty$$

$\{Q, z^\mu\} \sim \chi^\mu$

States are spinors on $\mathcal{M}$

$$Q_4 = \chi^\mu (D + G(\mathcal{Y}_\infty^{cl}))_{\mu} := D$$

(Sethi, Stern, Zaslow; Gauntlett & Harvey; Tong; Gauntlett, Kim, Park, Yi; Gauntlett, Kim, Lee, Yi; Bak, Lee, Yi; Bak, Lee, Lee, Yi; Stern & Yi)

What Sort Of SQM?

(N = 4 SQM on $\mathcal{M}(\gamma_m, X_\infty)$ with a potential:

$$S = \int (\| \dot{z} \|^2 - \| G(\mathcal{Y}_\infty^{cl}) \|^2 + \cdots)$$

$$\mathcal{Y}_\infty^{cl} := \frac{4\pi}{g_0^2} Y_\infty + \frac{\theta_0}{2\pi} X_\infty$$

$\{Q, z^\mu\} \sim \chi^\mu$

States are spinors on $\mathcal{M}$

$$Q_4 = \chi^\mu (D + G(\mathcal{Y}_\infty^{cl}))_{\mu} := D$$

(Sethi, Stern, Zaslow; Gauntlett & Harvey; Tong; Gauntlett, Kim, Park, Yi; Gauntlett, Kim, Lee, Yi; Bak, Lee, Yi; Bak, Lee, Lee, Yi; Stern & Yi)
How is \((u, \zeta)\) related to \(X_{\infty}\)?

Need to write \(X_{\infty}, \mathcal{Y}_{\infty}\) as functions on the Coulomb branch

\[
X_{\infty} := \text{Im}(\zeta^{-1} a(u)) := X
\]
\[
\mathcal{Y}_{\infty} := \text{Im}(\zeta^{-1} a_D(u; \Lambda)) := \mathcal{Y}
\]

Framed case: Phase \(\zeta\): data describing \('t Hooft line defect \(L\)

Smooth: Phase \(\zeta\) will be related to central charge of BPS state
What’s New Here?

Include singular monopoles: Extra boundary terms in the original action to regularize divergences: Requires a long and careful treatment.

Include effect of theta-term: Leads to nontrivial terms in the collective coordinate action

Consistency requires we properly include one-loop effects:

Essential if one is going to see semiclassical wall-crossing. (failure to do so lead to past mistakes...)
We incorporate one-loop effects, (up to some reasonable conjectures): Use the above map to $X, Y$.

Moreover, we propose that all the quantum effects relevant to BPS wall-crossing (in particular going beyond the small $Y_\infty$ approximation) are captured by the ansatz:

\[
X_\infty := \text{Im}(\zeta^{-1}a(u)) \\
Y_\infty := \text{Im}(\zeta^{-1}a_D(u; \Lambda))
\]
\[ H_{c.c.} = M_{\gamma_m}^{\text{cl}} + \frac{g_0^2}{8\pi} \left\{ \pi_m g^{mn} \pi_n + g_{mn} G(\mathcal{Y}_\infty)^m G(\mathcal{Y}_\infty)^n + \frac{4\pi i}{g_0^2} \chi^m \chi^n \nabla_m G(\mathcal{Y}_\infty)^n \right\} + \\
+ i\tilde{\theta}_0 \left( i G(X_\infty)^m \pi_m + \frac{2\pi}{g_0^2} \chi^m \chi^n \nabla_m G(X_\infty)^n \right) + O(g_0^2). \]
Introduction

Monopoles & Monopole Moduli Space

Singular Monopoles

Singular Monopole Moduli: Dimension & Existence

Semiclassical N=2 d=4 SYM: Collective Coordinates

Semiclassical (Framed) BPS States

Application 1: No Exotics & Generalized Sen Conjecture

Application 2: Wall-crossing & Fredholm Property

Future Directions
Semiclassical BPS States: Overview

\[ Q_4 = \chi^\mu (D + G(\mathcal{Y}))_\mu := D \]

Semiclassical framed or smooth BPS states with magnetic charge \( \gamma_m \) will be:

a Dirac spinor \( \Psi \) on \( \mathcal{M}(\gamma_m) \) or \( \overline{\mathcal{M}}(\gamma_m) \)

\[ D\Psi = 0 \]

Must be suitably normalizable:

\[ \ker_{L^2} D \]

Must be suitably equivariant...

Many devils in the details....
States Of Definite Electric Charge

$\overline{\mathcal{M}}$ has a t-action: $G(H)$ commutes with $D$

$$\exp[2\pi G(H)] \cdot \Psi = \exp[2\pi i \langle \gamma^e, H \rangle] \Psi$$

$$\gamma^e \in \mathfrak{t}^\vee$$

Cartan torus $T$ of adjoint group acts on $\overline{\mathcal{M}}$

$$T = \mathfrak{t}/\Lambda_{mw} \quad \Rightarrow \quad \gamma^e \in \Lambda_{rt} \subset \mathfrak{t}^\vee$$

Organize $L^2$-harmonic spinors by $T$-representation:

$$\ker_{L^2} D = \bigoplus \gamma^e \ker_{L^2} \gamma^e D$$
Geometric Framed BPS States

\[ \ker_{L^2} D = \bigoplus_{\gamma^e \in \Lambda_{rt}} \ker_{L^2} \gamma^e D \]

\[ \overline{\mathcal{H}}^{BPS} (P; \gamma; X, Y) := \ker_{L^2} \gamma^e D \]

\[ \overline{\mathcal{H}}^{BPS} (L, \gamma; u) = \overline{\mathcal{H}}^{BPS} (P; \gamma; X, Y) \]

\[ X = \text{Im}(\zeta^{-1} a(u)) \]

\[ Y = \text{Im}(\zeta^{-1} a_D(u; \Lambda)) \]
BPS States From Smooth Monopoles
- The Electric Charge -

Spinors and \( \mathbf{D} \) live on universal cover: \( \mathcal{M}^\sim \)

\( T \) acts on \( \mathcal{M} \), so \( t \) acts on \( \mathcal{M}^\sim \)

\[
T = t / \Lambda_{\text{mw}}
\]

States \( \Psi \) of definite electric charge transform in a definite character of \( t \): (``momentum’’)

In order to have a \( T \)-action the character must act trivially on \( \Lambda_{\text{mw}} \)

\[
\gamma^e \in \Lambda^\vee_{\text{mw}} \cong \Lambda_{rt}
\]
Smooth Monopoles – Separating The COM

\[ \widetilde{\mathcal{M}}(\gamma_m; X_\infty) = \mathbb{R}^3 \times \mathbb{R} \times \mathcal{M}_0 \]

No \( L^2 \) harmonic spinors on \( \mathbb{R}^4 \).
Only \text{"plane-wave-normalizable"} in \( \mathbb{R}^4 \)

\[
\mathbf{D} = \mathbf{D}_{\text{com}} + \mathbf{D}_0
\]

\[
\Psi = \Psi_{\text{com}} \otimes \Psi_0
\]

\[
\mathbf{D}_{\text{com}} \Psi_{\text{com}} = 0 \quad \mathbf{D}_0 \Psi_0 = 0
\]
Smooth Monopoles – Separating The COM

\[ D = \chi^\mu (D + G(\mathcal{Y}))_\mu = \mathbf{D}_{\text{com}} + \mathbf{D}_0 \]

Need orthogonal projection of \( G(\mathcal{Y}) \) along \( G(X_\infty) \).

\[ (G(X_\infty), G(H))_{\text{metric}} = (\gamma_m, H)_{\text{Killing}} \]

\( X_\infty \): generic, irrational direction in \( t \)

A remarkable formula!

\( \gamma_m \) is a rational direction in \( t \)

Flow along \( \gamma_m \) in \( T = t/\Lambda_{\text{mw}} \) will close.

Not so for flow along \( X_\infty \)
Smooth Monopoles – Separating The COM

\[ D_{\text{com}} = \sum_{i=1}^{3} \chi^i \frac{\partial}{\partial x^i} + \chi^4 \left( \frac{\partial}{\partial x^4} - \frac{(\mathcal{U}, \gamma_m)}{(X, \gamma_m)} \right) \]

\[ \Psi_{\text{com}} = e^{iqx^4} s_{\text{com}} \quad q = (\mathcal{U}, \gamma_m)/(X, \gamma_m) \]

But for states of definite electric charge \( \gamma^e \):

\[ q = -\langle \gamma^e, X \rangle/(X, \gamma_m) \]

\[ \langle \gamma^e, X \rangle + (\gamma_m, \mathcal{U}) = 0 \]
Dirac Zeromode $\Psi_0$

$\Psi_0$ with magnetic charge $\gamma_m \in \ker_{L^2} D_0$

Note: The $L^2$ condition is crucial!
We do not want ``extra'' internal d.o.f.

Contrast this with the hypothetical ``instanton particle'' of 5D SYM.

Organize $L^2$-harmonic spinors by $t^\perp$-representation:

$$\ker_{L^2} D_0 = \bigoplus \gamma_e \ker_{L^2} \gamma_e^\perp D_0$$

$$\gamma_e^\perp \in (\Lambda_{mw} \cap \gamma_m^\perp) \vee \subset t^\vee$$
Semiclassical Smooth BPS States

\[ \mathcal{H}^{\text{BPS}}(\gamma; u) = \ker^q(D_{\text{com}}) \otimes \ker^{\gamma^e_{L^2}} D_0 \]

\[ X = \text{Im}(\zeta^{-1} a(u)) \]

\[ \mathcal{V} = \text{Im}(\zeta^{-1} a_D(u; \Lambda)) \]

\[ \zeta = -Z_{\gamma}(u)/|Z_{\gamma}(u)| \]

\[ \langle \gamma^e, X \rangle + (\gamma_m, \mathcal{V}) = 0 \]
Tricky Subtlety: 1/2

Spinors must descend to \( \mathcal{M} = \tilde{\mathcal{M}} / \mathcal{D} \)

\( \mathcal{D} \cong \mathbb{Z} \) Generated by isometry \( \phi \)

Subtlety: Imposing electric charge quantization only imposes invariance under a proper subgroup of the Deck group:

\[
\exp[2\pi G(\lambda)] \Psi = \Psi \quad \lambda \in \Lambda_{m_w}
\]

\[
\exp[2\pi G(\lambda)] = \phi^\mu(\lambda)
\]
Tricky Subtlety: 2/2

Conjecture: \[ \mu(\lambda) = (\lambda, \gamma_m) \]

\[ \exp[2\pi G(\lambda)] = \phi^{\mu}(\lambda) \]

only generate a subgroup \( r \mathbb{Z} \), where \( r \) is, roughly speaking, the gcd(magnetic charges)

Extra restriction to \( \mathbb{Z}/r\mathbb{Z} \) invariant subspace:

\[ \left( \ker^q(D_{\text{com}}) \otimes \ker^{\gamma_e^\perp}_{L^2} D_0 \right)^{\mathbb{Z}/r\mathbb{Z}} \]
Combine above picture with results on $N=2, d=4$:

No Exotics Theorem

Wall-Crossing

(higher rank is different)
1. Introduction
2. Monopoles & Monopole Moduli Space
3. Singular Monopoles
4. Singular Monopole Moduli: Dimension & Existence
5. Semiclassical N=2 d=4 SYM: Collective Coordinates
6. Semiclassical (Framed) BPS States
7. Application 1: No Exotics & Generalized Sen Conjecture
8. Application 2: Wall-crossing & Fredholm Property
9. Future Directions
**Exotic (Framed) BPS States**

\[
\overline{\mathcal{H}}^\text{BPS}_\gamma \quad \mathcal{H}^\text{BPS}_\gamma \quad \mathfrak{so}(3)_{\text{rot}} \oplus \mathfrak{su}(2)_R \text{-reps}
\]

**Smooth monopoles:**

\[
\mathcal{H}^\text{BPS}_\gamma = \rho_{hh} \otimes \mathfrak{h}(\gamma)
\]

**Half-Hyper from COM:**

\[
\rho_{hh} = \left( \frac{1}{2}; 0 \right) \oplus \left( 0; \frac{1}{2} \right)
\]

**Singular monopoles:** No HH factor:

\[
\overline{\mathcal{H}}^\text{BPS}_\gamma = \mathfrak{h}(\gamma)
\]

**Definition:**

*Exotic BPS states:* States in \( \mathfrak{h}(\gamma) \) transforming nontrivially under \( \mathfrak{su}(2)_R \)
No Exotics Conjecture/Theorem

Conjecture \cite{GMN}: $\mathfrak{su}(2)_R$ acts trivially on $\mathfrak{h}(\gamma)$: exotics don’t exist.

Theorem: It’s true!

Diaconescu et. al.: Pure SU(N) smooth and framed (for pure \textquoteleft{}t Hooft line defects)

Sen & del Zotto: Simply laced $G$ (smooth)

Cordova & Dumitrescu: Any theory with \textquoteleft\textquoteleft{}Sohnius’’ energy-momentum supermultiplet (smooth, so far...
Geometry Of The R-Symmetry

Geometrically, SU(2)_R is the commutant of the USp(2N) holonomy in SO(4N). It acts on sections of T\mathcal{M} rotating the 3 complex structures;

Collective coordinate expression for generators of \mathfrak{su}(2)_R

\[ I^r \sim \omega^{r}_{\mu\nu} \chi^\mu \chi^\nu \]

This defines a lift to the spin bundle.

Generators do not commute with Dirac, but do preserve kernel.
have \( \mathfrak{so}(3) \) action of rotations. Suitably defined, it commutes with \( \mathfrak{su}(2)_R \).

Again, the generators do not commute with \( D_0, D \), but do preserve the kernel.

\[
\mathcal{H}^{BPS}(P; \gamma; X, Y) := \ker \gamma^e_{L^2} D \quad \text{under} \quad \mathfrak{so}(3)_{\text{rot}} \oplus \mathfrak{su}(2)_R
\]

\[
\mathfrak{h}^{BPS}(\gamma; X, Y) := \ker \gamma^e_{L^2} D_0 \quad \text{under} \quad \mathfrak{so}(3)_{\text{rot}} \oplus \mathfrak{su}(2)_R
\]
Geometrical Interpretation Of The No-Exotics Theorem -1

\[ \rho : SU(2)_R \times USp(2N) \rightarrow \text{Spin}(4N) \]

\[ \rho : (-1, 1) \rightarrow \text{vol} \]

All spinors in the kernel have chirality +1

\[ \text{Ind} \mathbf{D}^+_0 = \dim \ker \mathbf{D}_0 \]
So, the **absolute** number of BPS states is the same as the BPS **index**!

This kind of question arises frequently in BPS theory...
Geometrical Interpretation Of The No-Exotics Theorem - 2

Choose any complex structure on \( \mathcal{M} \).

\[
\mathcal{S} \cong \Lambda^{0,*}(T\mathcal{M}) \otimes K^{-1/2}
\]

\[
Q_3 + iQ_4 \sim \bar{\partial} + G^{0,1}(\mathcal{V}) \wedge
\]

\( \mathfrak{su}(2)_R \) becomes ``Lefshetz \( \mathfrak{sl}(2) \)''

\[
I^3|_{\Lambda^{0,q}} = \frac{1}{2}(q - N) \mathbf{1}
\]

\[
I^+ \sim \omega^{0,2} \wedge \quad I^- \sim \iota(\omega^{2,0})
\]
Geometrical Interpretation Of The No-Exotics Theorem - 3

\[ H_{L^2}^{0,q}(\bar{\partial} + G^{0,1}(Y_\infty)) \]
vanishes except in the middle degree \(q = N\), and is primitive wrt ``Lefshetz \(\mathfrak{sl}(2)\).''
Adding Matter
(work with Daniel Brennan)

Add matter hypermultiplets in a quaternionic representation $R$ of $G$.

Bundle of hypermultiplet fermion zeromodes defines a real rank $d$ vector bundle over $\mathcal{M}$: Structure group $SO(d)$

Associated bundle of spinors, $\mathcal{E}$, has hyperholomorphic connection.

(Manton & Schroers; Gauntlett & Harvey; Tong; Gauntlett, Kim, Park, Yi; Gauntlett, Kim, Lee, Yi; Bak, Lee, Yi)

\[
d = \sum_\mu \left[ \text{sign}(\langle \mu, X \rangle + 2m_I) \langle \mu, \gamma_m \rangle + |\langle \mu, P \rangle| \right]
\]

Sum over weights $\mu$ of $R$. \( \zeta^{-1}m = m_R + im_I \)
States are now $L^2$-sections of

$$S \otimes \mathcal{E} \rightarrow \mathcal{M}_0, \overline{\mathcal{M}}$$

$$H^{0,q}_{L^2} \left( \overline{\partial } \mathcal{E} + G^{0,1}(Y_\infty) ; \mathcal{E} \right)$$

vanishes except in the middle degree $q = N$, and is primitive wrt ``Lefshetz $\mathfrak{sl}(2)$''.

SU(2) $N=2^*$ $m \rightarrow 0$ recovers the famous Sen conjecture
Semiclassical Wall-Crossing: Overview

Easy fact: There are no $L^2$ harmonic spinors for ordinary Dirac operator on a noncompact hyperkähler manifold.

∃ Semiclassical chamber ($\mathcal{Y}_\infty = 0$) where all populated magnetic charges are just simple roots ($\mathcal{M}_0 = \text{pt}$)

Other semiclassical chambers have nonsimple magnetic charges filled.

Nontrivial semi-classical wall-crossing

(Higher rank is different.)

Interesting math predictions
Jumping Index

The $L^2$-kernel of $D$ jumps.

No exotics theorem $\Rightarrow$ Harmonic spinors have definite chirality $\Rightarrow$ $L^2$ index jumps! How?!

Along hyperplanes in $Y$-space zeromodes mix with continuum and $D^+$ fails to be Fredholm.

(Similar picture proposed by M. Stern & P. Yi in a special case.)
When Is $D_0$ Not Fredholm?

$D_0^\mathcal{Y}$ is a function of $\mathcal{Y}$:

Translating physical criteria for wall-crossing implies:

$\ker D_0^\mathcal{Y}$ on $\mathcal{M}(\gamma_m)$ only changes when

$$\exists \gamma_1, \gamma_2 \quad \langle \gamma_1, \gamma_2 \rangle \neq 0 \quad \mathcal{H}(\gamma_i; X, \mathcal{Y}) \neq 0$$

$$\gamma_{1,m} + \gamma_{2,m} = \gamma_m$$

$$(\gamma_{i,m}, \mathcal{Y}) + \langle \gamma_{i,e}, X \rangle = 0, \quad i = 1, 2$$

($D_0^\mathcal{Y}$ only depends on $\mathcal{Y}$ orthogonal to $\gamma_m$ so this is real codimension one wall.)
When Is $D$ on $\mathcal{M}$ Not Fredholm?

$D^\mathcal{Y}$ as a function of $\mathcal{Y}$ is not Fredholm if:

$$\exists \gamma_h \quad \mathcal{H}(\gamma_h; X, \mathcal{Y}) \neq 0$$

$$(\gamma_h, m, \mathcal{Y}) + \langle \gamma_h, e, X \rangle = 0$$

$\mathcal{H}(\gamma; X, \mathcal{Y})$ jumps across:

$$W(\gamma_h) := \{ \mathcal{Y} | \text{above conditions} \}$$
How Does The BPS Space Jump?

Unframed/smooth/vanilla:

Framed:

& &
Framed Wall-Crossing: 1/2

$$\overline{\Omega}(L, \gamma; X, \mathcal{Y}) = \text{Tr}_{\mathcal{H}} y^{2J_3}$$

``Protected spin characters’’

$$F(L) = \sum_{\gamma \in \Gamma} \overline{\Omega}(L, \gamma; X, \mathcal{Y}) V_{\gamma}$$

$$V_{\gamma_1} V_{\gamma_2} = y^{\langle \gamma_1, \gamma_2 \rangle} V_{\gamma_1 + \gamma_2}$$

$$F \rightarrow SFS^{-1} \quad S: \text{A product of quantum dilogs}$$
Framed Wall-Crossing: 2/2

\[ F(L) = \sum_{\gamma \in \Gamma} \overline{\Omega}(L, \gamma; X, Y) V_{\gamma} \]

\[ W(\gamma_h) := \{(X, Y) : (\gamma_{h,m}, Y) + \langle \gamma_{h,e}, X \rangle = 0\} \]

\[ F(L) \rightarrow SF(L)S^{-1} \]

\[ \Phi(z) = \prod_{k=1}^{\infty} (1 + y^{2^{k-1}} z)^{-1} \]

\[ S = \prod_{m} \Phi((-y)^{m} V_{\gamma_{h}})^{a_{m, \gamma_{h}}} \]

\[ \Omega(\gamma_{h}; u) = \sum_{m} a_{m, \gamma_{h}} (-y)^{m} \]
Example: Smooth SU(3) Wall-Crossing

[Gauntlett, Kim, Lee, Yi (2000)]

\[ g = su(3) \quad t \cong \mathbb{R}^2 \]

\[ \gamma_m = H_1 + H_2 = \gamma_{1,m} + \gamma_{2,m} \]

\[ \mathcal{V} = y_1 h^1 + y_2 h^2 \quad \Rightarrow \quad \mathcal{V}^\parallel = y_1 + y_2 \]

\[ \gamma^e = n_1 \alpha_1 + n_2 \alpha_2 = \gamma^e_1 + \gamma^e_2 \]

\[ M_0(X; \gamma_i, m) = pt \quad \mathcal{H}(\gamma_i; X, \mathcal{V}) \neq 0 \]

``Constituent BPS states exist’’
\[ \gamma_m = H_1 + H_2 \]

\[ \mathcal{M}_0(X; \gamma_m) = \text{Taub-NUT:} \]

Zeromodes of \( D_0 \) can be \textit{explicitly} computed [C. Pope, 1978]

\[ t^\perp \text{ orbits} = \text{orbits of standard HH U(1) isometry} \]

\[ G(\mathcal{Y}) = C(\mathcal{Y}^\perp) \frac{\partial}{\partial \psi} \]

\[ L \frac{\partial}{\partial \psi} \Psi_0 = i(n_1 - n_2) \Psi_0 \]

\[ = i\mu \Psi_0 \]
Just the primitive wall-crossing formula!
[Denef-Moore; Diaconescu-Moore]
\[ \Psi_0 \sim r^{(\mu - 1)/2} e^{-|C - \mu| r/2} \]

\[ r_{\text{max}} = \frac{\mu}{|C - \mu|} = r_{\text{Denef}} \]
Example: Singular SU(2) Wall-Crossing

\[ g = su(2) \quad t \cong \mathbb{R} \]

Well-known spectrum of smooth BPS states [Seiberg & Witten]:

\[ \gamma_n = n\alpha \oplus H \]

\[ W(\gamma_n) \quad W(\gamma_{n+1}) \]

\[ W(\gamma_h) := \{ V | (\gamma_h, m, V) + \langle \gamma_h, e, X \rangle = 0 \} \]

Line defect L: \[ P = \frac{p}{2} H \]

\[ F(L) = \sum_{\gamma \in \Gamma} \bar{\Omega}(L, \gamma; X, Y) V_{\gamma} \]
Explicit Generator Of PSC’s

\[ V_1 V_2 = y V_2 V_1 \]

\[ V_\gamma = V_{n^e \alpha + n_m H} = y^{-\frac{1}{2}} n^e n_m V_2 n^e V_1 n_m \]

\[ F(C_\ell) = \left[ y^{2\ell} V_1^{-1} V_2^{-\ell} (\mathcal{U}_\ell(f_\ell) - y^2 V_2^{-1} \mathcal{U}_{\ell-1}(f_\ell)) \right]^p \]

\[ \mathcal{U}_\ell(\cos \theta) := \frac{\sin((\ell+1)\theta)}{\sin \theta} \]

\[ f_\ell = \frac{1}{2} \left[ y^{-2} V_2 + y^2 V_2^{-1} (1 + y^{-1} V_1^2 V_2^{2\ell+2}) \right] \]

Predictions for \( \text{ker } D \) for infinitely many moduli spaces of arbitrarily high magnetic charge.
1. Introduction

2. Monopoles & Monopole Moduli Space

3. Singular Monopoles

4. Singular Monopole Moduli: Dimension & Existence

5. Semiclassical N=2 d=4 SYM: Collective Coordinates

6. Semiclassical (Framed) BPS States

7. Application 1: No Exotics & Generalized Sen Conjecture

8. Application 2: Wall-crossing & Fredholm Property

9. Future Directions
So, What Did He Say?

Recent new old

Recent results on N=2 d=4 imply new results about the differential geometry of old monopole moduli spaces.
Future Directions

Add matter and arbitrary Wilson-’t Hooft lines. (In progress with Daniel Brennan)

Understand better how Fredholm property fails by using asymptotic form of the monopole metric.

Combine with result of Okuda et. al. and Bullimore-Dimofte-Gaiotto to get an interesting $L^2$-index theorem on (noncompact!) monopole moduli spaces?