WALL-CROSSING FORMULA FOR
BPS STATES & SOME APPLICATIONS

TRIESTE SPRING SCHOOL LECTURE IV
APRIL 4, 2008

BASED ON WORK DONE WITH
F. DENEFF (*hep-th/0702146*)

AND FURTHER RESULTS WITH
E. DIACONESCU (*0706.3193*)
E. ANDRIYASH

M. AGANAGIC + D. JAFFERIS

J. MANSCHOT (*0712.0573*)
1. INTRODUCTION

The "space of BPS states" has been a central concept in SUSY gauge theory & string theory for almost 30 years.

Today I'll focus on recent progress in understanding phenomena associated to marginal stability.

1. INTRODUCTION
2. WALL-CROSSING FORMULAE:
3. PHYSICAL DERIVATION
4. D6-D2-DO SYSTEM
5. D4D2DO SYSTEM: MODULAR GEN. FUNCTIONS
6. ROUTE TO OSV: ENTROPY ENIGMA & DEGENERACY DICHOTOMY
7. KONTSEVICH-SOIBELMAN FORMULA
8. OPEN PROBLEMS
A. DEFINING THE "SPACE OF BPS STATES"

For definiteness, we focus on theories with $d=4, \, N=2$ SUSY in (asymptotic) Minkowski space $M_4$.

HILBERT SPACE OF ONE-PARTICLE STATES, $\mathcal{H}$, IS A REP. OF THE $d=4, \, N=2$ ALGEBRA.

$\hat{Z} :$ CENTRAL CHARGE OPERATOR

\[
\{ \hat{Q}_{i\alpha}, \hat{Q}_{j\beta} \} = \delta_{ij} (C^n)_{\alpha\beta} \hat{P}_\mu + \epsilon_{ij} C_{\alpha\beta} \hat{Z}
\]

DECOMPOSE $\mathcal{H} = \bigoplus_{z \in \mathbb{C}} \mathcal{H}_{\hat{Z} = z}$
**Lemma:** $E \geq |\mathcal{Z}|$ on $\mathcal{H}_Z$

**Proof:** $N = 2 \implies$

\[
\{ Q_i \alpha, Q_j \beta \} = \delta_{ij} (C_{\alpha \beta})_{\alpha \beta} \gamma^\mu P_\mu + \epsilon_{ij} C_{\alpha \beta} \mathcal{Z}
\]

This is a 6D SUSY algebra $Q_{\alpha}$,

\[
\{ Q_A, Q_B \} = (C T^M)_{AB} \gamma^M
\]

with $P_4 + i P_5 = \mathcal{Z}$. But

\[
M^2 = E^2 - P^2 - |\mathcal{Z}|^2 \geq 0.
\]

**Def'n:** $\mathcal{H}_{\text{BPS}}$ is the subspace of $\mathcal{H}$ where $E = |\mathcal{Z}|$. 
Now specialize to Type II string theory on $M_4 \times X$.

- $M_4$ is noncompact $\Rightarrow$ to define the Hilbert space as a rep. of $W=2$ we must specify boundary cond's for the massless fields:

$$\lim_{x \to \infty} (g_{\mu\nu}, \phi, B_{\mu\nu}, RR) = \mathcal{M}_\infty \in \mathcal{M}$$

$$\mathcal{H}_{\mathcal{M}_\infty} : \text{1-particle Hilbert space depends on } \mathcal{M}_\infty$$

- Generalized Maxwell theory $\Rightarrow$ $\mathcal{H}_{\mathcal{M}_\infty}$ is graded by electric/magnetic charge sectors:

$$\mathcal{H}_{\mathcal{M}_\infty} = \bigoplus_{\Gamma} \mathcal{H}_{\mathcal{M}_\infty}^\Gamma$$

$\Gamma \in (\text{twisted}) K$-theory$(X)$
K-THEORY TO COHOMOLOGY

Physicists usually work with cohomology

\[ E \in K^0(X) \rightarrow \text{ch}(E) \sqrt{\Delta} \in H^e(X, \mathbb{Q}) \]

D-branes are sources:

\[
\begin{array}{cccccc}
D6 & D4 & D2 & D0 & p^0 & p^1 \\
H_6 & H_4 & H_2 & H_0 & Q & Q_0 \\
H^6 & H^4 & H^2 & H^0 & & \\
\end{array}
\]

Often identify \( H^6(X, \mathbb{Z}) \cong \mathbb{Z} \)
$K^0(x)/\text{torsion} = \text{LATTICE \wedge}$

$\text{Ch}(\mathcal{E}) \sqrt{\mathcal{A}} \Rightarrow \text{corresponding LATTICE in } H^\nu(x,\mathbb{Q})$

$\wedge \text{ has a \mathbb{Z} SYMPLECTIC FORM}$

$\langle \mathcal{E}_1, \mathcal{E}_2 \rangle = \text{Index } \bigoplus_{\mathcal{E}_1 \otimes \mathcal{E}_2}
= \int (\text{ch}\mathcal{E}_1 \sqrt{\mathcal{A}}) \wedge (\text{ch}\mathcal{E}_2 \sqrt{\mathcal{A}})$

\text{IN TERMS OF COHOMOLOGY}

$\langle \mathcal{\Pi}, \mathcal{\Pi}' \rangle = \int -p^0 q^0 + pq' - qp' + q_0 p_0$

\text{ PHYSICALLY: DIRAC-SCHWINERG-ZWAIRZ...}

\text{DUALITY INVT. PRODUCT OF ELECTRIC AND MAGNETIC CHARGES.}
Now we put these things together.

Consider IIA strings with

1. $X = \text{static, compact, CY 3-fold}$
2. Flat B-field: $B \in H^2(X, \mathbb{R})$
3. Flat RR fields

$\Rightarrow \quad \mathcal{N} = 2, \ d = 4 \quad \text{SUGRA}$

- Each $\mathcal{H}_{\Phi_\infty}^{\Gamma}$ is a rep of $\mathcal{N} = 2$

- Central charge $Z = Z(\Gamma; \Phi_\infty)$

So, we study the BPS spectrum

$$\mathcal{H}_{\text{BPS}} = \bigoplus_{\Gamma \in \mathcal{K}^0(X)} \mathcal{H}_{\Phi_\infty, \text{BRS}}^{\Gamma}$$

Finitely dimensional
B. DEPENDENCE ON MODULI

THE SPACES $\Phi_{\infty, \text{BPS}}$ ARE LOCALY CONSTANT BUT NOT GLOBALLY CONSTANT AS FUNCTIONS OF $\Phi_{\infty}$.

MODULI SPACE $\tilde{M}$ IS A PRODUCT:

HYPERMULTIPLETS $\times$ VECTORMULTIPLETS

$[\text{CPLX STR., } \phi, \text{ RR FIELDS}] \quad [\text{COMPLEXIFIED KÄHLER}]$

WE WORK AT A GENERIC HYPERMULTIPLE.

RECENT PROGRESS HAS BEEN CONCERNED WITH THE DEPENDENCE ON VECTORMULTIPLETS, IN THIS TALK,

$z = B + iJ$

- THE JUMPING LOCUS IS REAL CODIMENSION ONE
Define an index

\[ \Omega(\Gamma; t_\infty) = -\frac{1}{2} \text{Tr}_{\mathcal{H}(\Omega, t_\infty, \text{BPS})} (2J_3)^2 (-1)^{2J_3} \]

(compare A. Sen's talk: he had 6th helicity supertrace.)

**Technical Point:**

\[ \mathcal{H}(\Omega, t_\infty, \text{BPS}) = \mathcal{H}_{\frac{1}{2} \text{HM}} \otimes \mathcal{H}(\Gamma, t_\infty) \]

\[ \frac{1}{2} \text{ hyper } 2(0) + \left( \frac{1}{2} \right) \text{ as spin rep } \]

\[ \Omega(\Gamma; t_\infty) = \text{Tr}_{\mathcal{H}(\Gamma, t_\infty)} (-1)^F \]

Henceforth focus on \( \mathcal{H}(\Gamma; t_\infty) \)

**Key Point:** \( \Omega \) changes across walls of marginal stability
C. Why do we care?

**PHYSICS MOTIVATION**

1. The main motivation for recent work is the program, initiated by Strominger-Vafa (1995) of accounting for BH entropy via microstate counting. That goal is still not fully accomplished.

   We don't know BPS degeneracy for certain natural charge regimes, for example:

   $\Gamma \rightarrow 2\Gamma \quad \lambda \rightarrow \infty$

2. OSV Conjecture:

   Relation between

   $\Omega(\Gamma) \in \text{GW/DT/GV invariants}$

   $\Rightarrow$ NonPTVE Topological String?
1. Physical stability of BPS states is related to math. Stability in the bounded derived category of coherent sheaves on a C.Y.: Kontsevich, Douglas, Bridgeland, Thomas, Pandharipande... 

Physics $\Rightarrow$ Predictions/Constraints on what we expect should be true.

2. Many interesting connections to automorphic forms and analytic number theory; some relations to arithmetic C.Y.'s.

3. There are several other more speculative applications, e.g. BPS algebras: generalizing Nakajima's work and suggested by type II/het duality should be closely related.
2. WALL-CROSSING FORMULAE: STATEMENT

$N=2$, $d=4$ Algebra \[ \Rightarrow \]
- **Moduli of vacua $\mathcal{W}$**
- **Lattice of electric/magnetic charges $\Lambda$**
- **Central charge**: $Z: \Lambda \times \mathcal{W} \rightarrow \mathbb{C}$

**Walls where BPS might jump**

$$MS(\Gamma_i, \Gamma_j) := \{ t \mid Z(\Gamma_i, t) = \lambda Z(\Gamma_j, t), \lambda \in \mathbb{R}_+ \}$$

$$\left| Z_i + Z_j \right| = \left| Z_i \right| + \left| Z_j \right|$$

Cecotti, Intriligator, Yaafa; Seiberg & Witten: A boundstate of particles with charges $\Gamma_i, \Gamma_j$ can decay

We want to say how many states decay.
PRIMITIVE WALL-CROSSING FORMULA:

∧ HAS SYMPLECTIC FORM  \langle \cdot, \cdot \rangle

LET  \quad I_{12} := \langle \Gamma_1, \Gamma_2 \rangle

\text{If}  \quad I_{12} \text{Im}(z_1z_2^*) > 0
\quad \text{then} \quad \MS(\Gamma_1, \Gamma_2)

\text{If}  \quad I_{12} \text{Im}(z_1z_2^*) < 0

\Gamma_1, \Gamma_2 \quad \text{PRIMITIVE, } t_{ms} \quad \text{GENERIC} \Rightarrow

\mathcal{H}_+ - \mathcal{H}_- = (\mathbf{J}_{12}) \otimes \mathcal{H}(\Gamma_1; t_{ms}) \otimes \mathcal{H}(\Gamma_2; t_{ms})

\mathbf{J}_{12} = \frac{1}{2} (I_{12} - 1)

\Delta \Omega = (-1)^{I_{12} - 1} \quad I_{12} \quad \Omega(\Gamma_1; t_{ms}) \Omega(\Gamma_2; t_{ms})
**Semi-primitive wall-crossing formula**

In addition to \( \Gamma_1 + \Gamma_2 \) boundstates, we can also form \( N_1 \Gamma_1 + N_2 \Gamma_2 \) boundstates

\[
MS(\Gamma_1, \Gamma_2) = MS(N_1 \Gamma_1, N_2 \Gamma_2) \quad N_1, N_2 \in \mathbb{Z}_+
\]

**Consider** \( N_1 = 1, \quad N_2 \geq 1 \):

\[
\bigoplus_{N_2} u^{N_2} \Delta \mathcal{H}_{\Gamma_1 + N_2 \Gamma_2}
\]

**Claim:** This is a \( \mathbb{Z}_2 \)-graded Fock space

\[
\mathcal{H}(\Gamma_1; t_{m_5}) \bigotimes_{k=1}^{\infty} \mathcal{F} (u^k \left( J_{\Gamma_1, k\Gamma_2} \right)) \otimes \mathcal{H}(k\Gamma_2; t_{m_5})
\]

**Graded space of oscillators**

**In particular:**

\[
\Omega_1 + \sum_{N > 0} u^N \Delta \Omega(\Gamma_1 + N\Gamma_2) = \\
= \Omega(\Gamma_1) \prod_{k > 0} (1 - (-1)^{\langle \Gamma_1, k\Gamma_2 \rangle} u^k) ^{\langle \Gamma_1, k\Gamma_2 \rangle} \Omega(k\Gamma_2)
\]
D-BRANES ARE OBJECTS IN A CATEGORY IN TYPE IIA/CY, THE SUBCATEGORY OF SUSY BRANES IS PROBABLY THE BOUNDED DERIVED CATEGORY OF COHERENT SHEAVES.

BUT WE WANT TO DESCRIBE THE (PHYSICALLY) STABLE OBJECTS.

AT WEAK STRING COUPLING, AND $J \rightarrow \infty$ \[ \exists \ A \ BEAUTIFUL \ DESCRIPTION \ OF \ STABLE \ BPS \ STATES \ USING \ SUGRA. \]

IN THE SEMICLASSICAL LIMIT \[ \Psi \in \mathcal{H}_{\text{BPS}} \rightarrow \text{BPS SOLUTION OF SUGRA EQUATIONS} \]

* SUPERGRAVITY ALLOWS ONE TO IDENTIFY MANY "STABLE OBJECTS" THANKS TO THE ATTRACTOR MECHANISM.
ATTRACTOR MECHANISM: (F.K.S.; STROMINGER)
\[ \Gamma, \lambda \rightarrow \xi \]
SPHERICAL SYMMETRY
\[ \Rightarrow \exists \text{ AT MOST ONE BPS SOLUTION OF SUGRA.} \]

IF IT EXISTS …

SCALAR FIELDS \( t = t(\gamma) \), AND
EVOLUTION FROM \( \gamma = \infty \) TO \( \gamma = 0 \)
APPROACHES AN ATTRACTIVE FIXED POINT \( t_*(\Gamma) \):

\[ \tilde{\mathcal{M}}_{\nu m} \]

RADIAL MOTION TO HORIZON
Attractor flow = gradient flow for

\[ \log | Z(\Gamma; t) |^2 \]

\[ Z = \frac{\langle \Gamma, \omega \rangle}{\sqrt{\langle \omega, \omega^* \rangle}} \]

\[ \langle \Gamma, \Gamma' \rangle = \int (-p o q' + pq' - q p' + q_o p_o) \]

\[ \omega = \text{period vector} \]

In large radius approximation:

\[ \omega = -e^t = -e^{B+ij} \]

\[ Z \approx \frac{\frac{1}{6} p_o t^3 - \frac{1}{2} p t^2 + Q t - q_o}{\sqrt{(Im t)^3}} \]
BASIC TRICHOTOMY

1. \( \pi_*(\Gamma) \in \text{Interior}(\tilde{\mathcal{M}}) \)
   and \( \mathcal{Z}(\Gamma; t^*(\Gamma)) \neq 0 \)

"REGULAR ATTRACTOR POINT"

2. \( \exists \text{ NONEMPTY SUBVARIETY } \subseteq \tilde{\mathcal{M}} \)
   \( \mathcal{Z}(\Gamma; t) = 0 \)

3. \( \pi_*(\Gamma) \in \partial \tilde{\mathcal{M}} \)

(1.) \( \exists \text{ SPHERICALLY SYMMETRIC BPS BLACK HOLES IN } \mathcal{H}_{BPS}(\Gamma; t) \text{ FOR ALL } t \)

(2.) \( \mathcal{H}_{BPS}(\Gamma; t) = \emptyset \) IN AN OPEN REGION OF THE ZERO LOCUS.
\( \mathcal{H}_{BPS} \) MIGHT BE NONEMPTY FURTHER AWAY

(3.) CANNOT USE SUGRA TO ESTABLISH EXISTENCE; MUST USE MICROSCOPIC ARGUMENTS.
B. SPLIT ATTRACTION FLOWS

If \( \mathcal{Z}(\Gamma; t) = 0 \) has solutions in the interior of moduli space then use:

**Deneef's Rule:** \( \mathcal{H}(\Gamma; t) \neq 0 \iff \exists \) a split attractor flow (S.A.F.)

S.A.F.: A piecewise attractor flow, joined along walls of M.S., conserving charge at the vertices, terminating on R.A.P.'s:

\[ \Gamma = \Gamma_1 + \Gamma_2 \]

Diagram:

- \( \mathcal{AF}(\Gamma) \)
- \( \mathcal{AF}(\Gamma_1) \)
- \( \mathcal{AF}(\Gamma_2) \)
- \( \mathcal{AF}(\Gamma_2') \)
- \( \mathcal{AF}(\Gamma_2'') \)
- \( \mathcal{MS}(\Gamma_1, \Gamma_2) \)
- \( \mathcal{MS}(\Gamma_2', \Gamma_2'') \)
- If such attractor flow trees exist we can construct a corresponding solution of sugra.

- Spacetime picture:

  \( \mathbb{R}^3 \)

  \( (\vec{x}_i, \vec{n}_i) \)

  \( (\vec{x}_z, \vec{n}_z) \)

  \( (\vec{x}_j, \vec{n}_j) \)

  \( (\vec{x}_n, \vec{n}_n) \)

- Near each point \( \vec{x}_i \), the solution looks like the single-centered solution: "black-hole molecules"
MULTICENTERED SOLUTIONS:

GENERAL BPS EQUATIONS

(1.) \[ ds^2 = -e^{2u}(dt+\Theta)^2 + e^{-2u} d\vec{x}^2 \]
\[ u = u(\vec{x}), \quad \vec{x} \in \mathbb{R}^3 \]

(2.) CHOOSE A HARMONIC MAP
\[ H: \mathbb{R}^3 \rightarrow H^{ev}(x, \mathbb{R}) \]
\[ H(\vec{x}) = \sum_j \frac{\Gamma_j}{|\vec{x} - \vec{x}_j|} + H_\infty \]

\[ 2e^{\Gamma} \text{Im}(e^{-i\alpha \omega}) = -H(\vec{x}) \]

(a.) \( \pm(\vec{x}) \) completely fixed,

(b.) \( e^{-2u(\vec{x})} = S(H(\vec{x})) \)
(3.) \( \ast_3 d \Theta = \langle dH, H \rangle \)

\[ \sum_{j \neq i} \frac{\langle \Gamma_i, \Gamma_j \rangle}{|\bar{x}_i - \bar{x}_j|} = 2 \text{Im} (e^{-i \alpha \cdot \bar{z}_i}) \]

\[ \text{SUGRA SOLUTION EXISTS } \iff \forall \bar{x} \in \mathbb{R}^3: \]

\[ \| \bar{x} \| \leq \text{max}_{\Delta} \text{e}^{-2u(x)} \]

\[ \pi e^{-2u(x)} = S(H \bar{x}) \geq 0 \]

\( \text{(A VERY NONTRIVIAL CONDITION TO CHECK ... )} \)
SPLIT ATTRACTOR CONJECTURE (DENEFF)

(a.) (Components of moduli of) multicentered solutions are in $1 \leftrightarrow 1$ correspondence with S.A.F.’s.

(b.) For a fixed $(t_0, \Gamma)$ there are a finite number of S.A.F.’s.

- Useful because checking $S(H^{\infty}) > 0$ is difficult.

- $H_{BPS}$ is partitioned by split attractor flows.

- Some interesting open problems here....

  * Quantum mixing between different attractors
  * Useful existence criterion for scaling solutions.
C. DERIVATION OF PRIMITIVE WCF:

**Consider bound state of two primitive charges:**

\[ R = \frac{1}{2} \langle \Gamma_1, \Gamma_2 \rangle \frac{|Z_1 + Z_2|_{\infty}}{\text{Im} (Z_1 \overline{Z_2})_{\infty}} \]

**Note:** \( \langle \Gamma_1, \Gamma_2 \rangle \text{Im} (Z_1 \overline{Z_2})_{\infty} > 0 \)

**Note that by changing \( t \rightarrow \infty \), we can make \( \text{Im} (Z_1 \overline{Z_2})_{t \rightarrow \infty} \) while \( |Z_1 + Z_2|_{t \rightarrow \infty} \neq 0 \)

**Illustrates the key point of marginal stability:**
\[ MS(\Gamma_1, \Gamma_2) := \left\{ \pm \epsilon \mu \nu_M \mid \frac{\nu^{-1}}{\nu} \in TR^+ \right\} \]

\[ \text{NO } \Pi_1^+ + \Pi_2^- \text{ BOUND. STATE EXISTS HERE} \]

\[ \text{CHANGE BC'S} \]
\[ \text{@ } \Gamma = \infty \Rightarrow \]
\[ R_{12} \rightarrow \infty \]

**IF** \( Z(\Gamma; t) \) **HAS A ZERO THEN**

**THERE IS NO BOUNDSTATE OF TYPE** \( \Pi_1^+ + \Pi_2^- \)

**IN THE BLUE REGION.**

\[ Z(\Gamma; t_0) = 0 \]

\[ Z \rightarrow \times \]

\[ \text{ATTRACTION FLOW} \]
MACROSCOPIC ARGUMENT FOR WCF:

\[ R_{12} = \frac{1}{2} \langle \Gamma_1, \Gamma_2 \rangle \frac{|Z_1 + Z_2|_\infty}{\text{Im}(Z_1 \overline{Z}_2)_\infty} \]

\[ \Gamma_1 \quad t_{\infty} \quad t_2 \quad t_{\infty} \quad t_{\infty} \quad \text{MS}(\Gamma_1, \Gamma_2) \quad t_{\text{MS}} \quad \Gamma_2 \]

ELECTROMAGNETIC FIELD OF TWO DYONS

HAS SPIN:

\[ J_{12} = \frac{1}{2} \left( K_{\Gamma_1, \Gamma_2} \right)_{1 - 1} \]

\( \text{quantum correction} \)

LOCALITY \( \Rightarrow \) FOR \( \Gamma_1, \Gamma_2 \) PRIMITIVE:

STATES LOST FROM \( \mathcal{H}(\Gamma; t_{\infty}) \) ARE

\( (J_{12}) \otimes \mathcal{H}(\Gamma_1; t_{\text{MS}}) \otimes \mathcal{H}(\Gamma_2; t_{\text{MS}}) \)
MICROSCOPIC ARGUMENT FOR WCF:

WHEN \( \Theta = \arg \frac{Z_2}{Z_1} \rightarrow 0 \), MODEL LIGHT D.O.F. BY A QUIVER GAUGE THRT:

\[
\begin{array}{c}
\Gamma_1 \\
\bullet \\
\downarrow \text{d=1}
\end{array}
\quad \xleftarrow{\text{n+}} \quad
\begin{array}{c}
\text{d=1} \\
\bullet \\
\uparrow \text{n-}
\end{array}
\begin{array}{c}
\Gamma_2
\end{array}
\]

TRANSLATION TO SUPERGRAVITY:

STABILITY DATA: \((\Theta, -\Theta)\)

\(n_+ - n_- = I_{12}\)

GENERICALLY \(n_+ = 0\) or \(n_- = 0\).

SUPPOSE \(n_- = 0\):

\(\Theta > 0 \quad \mathcal{M} = \mathbb{C}P^{n_+-1}\)

\(\Theta < 0 \quad \mathcal{M} = \emptyset\)

\(\Delta \mathcal{H} = H^* \left( \mathbb{C}P^{n_+-1} \right)\)

\(\text{spin}(3) \cong \text{Lefschetz}\)
Quiver Quantum Mechanics

$0+1$ SUSY QED with

1 YM $(A_0, \bar{x}, \lambda)$

$n_\pm$ CM's $(\phi_\pm^+, \bar{\phi}_\pm)$ charge $\pm 1$

Small $|\langle \bar{x} \rangle | \Rightarrow$ Higgs branch = moduli of stable quiver reps

Large $|\langle \bar{x} \rangle | \Rightarrow$ integrate out $\phi_\pm \Rightarrow$

\[
\left( \begin{array}{c}
\text{Denef} \\
\text{QH} \\
\text{H} \\
\text{H} \end{array} \right)
\]

\[\frac{\theta^2}{2\mu} \]

\[V_{\text{eff}} = \frac{1}{2\mu} \left( \theta + \frac{n_+ - n_-}{2n} \right)^2 \]

$(n_+ - n_-)$ BPS states of spin $\frac{1}{2}(n_+ - n_- - 1)$

$\nu < 0$

$\nu > 0$

$n_+ \ \text{Higgs br.}$

$\nu = 0$

$n_+ \rightarrow \infty$

$n_+ \ \text{Higgs br.}$

BPS states

$\text{Coulomb br.}$
D. DERIVATION OF SEMI-PRIMITIVE WCF

**HALO STATES**

Suppose \( \langle \Gamma_1, \Gamma_2 \rangle \neq 0 \),

\[
\Gamma_j = \lambda_j \Gamma_2 \quad \lambda_j > 0, j = 2, \ldots, N
\]

are all mutually local integrability conditions say

\[
\begin{align*}
\text{for } j \geq 2: & \quad \frac{\langle \Gamma_j, \Gamma_1 \rangle}{|x_j - x_1|} = 2 \frac{\text{Im}(Z(\Gamma_j) \overline{Z(\Gamma_1)})}{|Z(\Gamma_1)|}\\
\Rightarrow & \quad \text{all } |x_j - x_1| \text{ are equal}
\end{align*}
\]

\[
\text{Cross } MS(\Gamma_1, \Gamma_2): \text{ HALO RADIUS } \rightarrow \infty
\]
The particles in the halo generate a Fock space with

\((J_{\Gamma_1, k\Gamma_2}) \otimes \mathcal{H}(k\Gamma_2; iTW)\) creation operators of charge \(k\Gamma_2\)

All walls \(W(\Gamma_1, N\Gamma_2)\) coincide \(\Rightarrow\) crossing a wall we lose entire Fock space:

\[\Omega(\Gamma_1) + \sum_{N \geq 1} \Delta \Omega (\Gamma_2 \rightarrow \Gamma_1 + N\Gamma_2) u^N\]

\[= \Omega(\Gamma_1) \prod_{k > 0} \left(1 - (-1)^k u^k\right)^{1\langle\Gamma_1, k\Gamma_2\rangle} |\Omega(k\Gamma_2)\rangle\]
4. D6D2D0 SYSTEM

An important and useful example is the system of 1 D6 brane wrapping X, bound to D2 & D0 branes in X.

\[ H^0 \oplus H^2 \oplus H^4 \oplus H^6 \cong \Gamma = (\rho^0, \rho, Q, q_0) \]

\[ D6 \quad D4 \quad D2 \quad D0 \]

Consider: \[ \Gamma(\beta, n) := \Gamma = (1, 0, -\beta, n) \]

\[ \beta = \text{P.D.}[\sigma] \quad \sigma \subset X \quad \text{holomorphic curve} \]

Charge of (the dual of) an ideal sheaf:

\[ \text{ch} \int \sqrt{A} = 1 - \beta + ndV \]

Consider binding these to D2D0 particles with charge:

\[ \Gamma_h = (0, 0, -\beta_h, n_h) \]
Plot Marginal Stability Curve

\[ Z(\Gamma(\beta, n); t) = \lambda Z(\Gamma_n; t) \quad \lambda \in \mathbb{R}_+ \]

\[ Z(\Gamma, t) = \frac{\langle \mathbf{T}, \omega \rangle}{\sqrt{\langle \omega, \omega^* \rangle}} \]

Sugra regime: \[ \Omega = -e^t \]

\[ t = B + iJ \]

\[ \frac{t^3}{6} - \beta \cdot t - n = \lambda (-\beta_n \cdot t - n_n) \quad \lambda \in \mathbb{R}_+ \]
These walls extend to $\infty$ in the Kähler cone!

Set $z = z^P$, $P \in \mathbb{R}$

$z = x + iy$

$X = \frac{n_h}{2P \cdot \beta_h}$, $n_h < 0$

$X = \frac{n_h}{2P \cdot \beta_h}$, $n_h > 0$
Consider the halo boundstates with central particle \( \Pi(\beta, n) \) as we increase the B-field

\[ B = \times P \times \text{increases} \]

halos of D2DO particles \((0, 0, -\beta n, n)\). Appear \& disappear.

For \( x > 0 \)

all \( n_k < 0 \) states have decayed.

As \( x \to +\infty \) we move into the stable region for all \( n_k > 0 \), and ever larger "atoms" become stable

General picture: Bohr model
When $\beta_n = 0$ walls look different

\[ \Gamma = 1 + q_0 \frac{dV}{\Gamma_1} \quad \text{and} \quad Z = \frac{t^3}{6} - q_0. \]

Set $t = (x + iy)\tau \implies $ zero @ $z = \left(\frac{6q_0}{\tau^3}\right)^{\frac{1}{3}}$.

$q_0 > 0$

$q_0 < 0$
**INTRODUCE GENERATING FUNCTION**

\[ Z_{D6D2D0}(u,v; t) := \sum_{n, \beta} \Omega(\Gamma(\beta, n); t) u^n v^\beta \]

**SEMI-PRIMITIVE WALL-CROSSING FORMULA:**

**CONTRIBUTION OF FOCK SPACE GENERATED BY**

\[ T_h = -\beta_h + n_h dV \] **CROSSING INTO STABLE REGION:**

\[ Z_{D6D2D0} \rightarrow \left(1 - (-u)^{n_h} V^{\beta_h}\right)^{n_{h^*}} n_{\beta_h}^\circ Z_{D6D2D0} \]

\[ \Omega(-\beta_h + n_h dV) = \sum_{m_L, m_R} (-1)^{2m_L + 2m_R} N_{\beta_h}^{m_L m_R} \]

\[ = n_{\beta_h}^\circ \]

"SPIN ZERO GV INVARIANT" \( (\beta_h \neq 0) \)
EXAMPLE: $D6D0$

$$Z_{D6D0}(u) = \sum \Omega \left( l + q_0 dV : t \right) u^{q_0}$$

For $q_0 < 0$:

$$\Omega \left( q_0 dV \right) = -\chi(x)$$

$$Z_{D6D0}(u) = \begin{cases} 
(M(-u))\chi(x) & \text{arg } z < \frac{\pi}{3} \\
1 & \frac{\pi}{3} < \text{arg } z < \frac{2\pi}{3} \\
(M(-\bar{u}^{-1}))\chi(x) & \frac{2\pi}{3} < \text{arg } z
\end{cases}$$

$$M(u) := \prod_{k>1} (1 - u^k)^{-k}$$
Similarly, wall-crossings for the full $Z_{D6D2D0}$ as $x \to \infty$ build up an infinite product similar to the infinite product form of $Z_{DT}(u, v)$.

On the other hand, an argument from M-theory [Dijkgraaf, Verlinde, Vafa; Denef, Moore] implies:

\[ \lim_{x \to +\infty} Z_{D6D2D0}(u, v; z^p) = Z_{DT}(u, v) \]

\[ \lim_{x \to -\infty} Z_{D6D2D0}(u, v; z^p) = Z_{DT}(\bar{u}^1, v) \]
\textbf{STATES IN CORE REGION ARE COMPLICATED BOUND STATES}

\textbf{PRODUCT OF WALL-CROSSINGS} $$\Rightarrow$$

$$Z_{\text{DT}}^{1, r=0}(u, v) = \prod_{\beta > 0, \kappa > 0} (1 - (-u)^{\kappa} v^\beta)^{\kappa n_\beta^0}$$

\textbf{LIMIT FOR} $$x \to +\infty$$:

$$Z_{\text{DT}}^{1}(u, v) = \frac{Z_{\text{DT}}^{1, r=0}(u, v)}{\text{HALOS}} \cdot \frac{Z_{\text{DT}}^{1, r>0}(u, v)}{\text{CORES}}$$

$$Z_{\text{DT}}^{1, r>0}(u, v) = \prod_{\beta > 0, \kappa > 0} \prod_{l=0}^{2r-2} \left(1 - (-u)^{r-2-l} v^\beta\right)^{l + \frac{r+l}{2} \left(2r-2\right) n_\beta^r}$$
5. THE D4-D2-DO SYSTEM: MODULARITY

Now consider $p^0 = 0$

$$\Gamma = p + q + q_0 dv$$

**Regular attractor point:**

**P in Kähler cone if $\hat{q}_0 < 0$$

$$\hat{q}_0 = q_0 - \frac{1}{2} (\mathcal{D}_{ABC} P^C)^{-1} q_A q_B$$

**These are black holes:**

**Horizon area** $= 4 S(\Gamma) = 4\pi \left| z_*(\Gamma) \right|^2$

$$S(\Gamma) = \frac{2\pi}{\sqrt{6}} \sqrt{-\hat{q}_0} \chi(p)$$

$$\chi(p) := p^3 + c_2 \cdot p > 0 \text{ for } p \in \text{Kähler cone}$$

Expect: $\log \Omega(\Gamma; t) \sim S(\Gamma)$ for "large" $\Gamma$ and "large" $Im t$
A. Rough microscopic description

For large $J$: Single D4 wraps $\Sigma \in |P|$

$\chi(P) = P^3 + c_2 \cdot P = \text{Euler Character of } \Sigma$

Flux $F \in H^2(\Sigma, \mathbb{Z})$

And $N$ D6's

Compute induced RR charges:

$D_2$: $Q = (2\Sigma)_*(F)$

$D_0$: $\hat{Q}_0 = \frac{\chi(P)}{24} + \frac{1}{2}(F^-)^2 - N$

Susy $\Rightarrow$ $N \geq 0$, $F^{2,0} = 0 \Rightarrow (F^-)^2 \leq 0 \Rightarrow$

$\hat{Q}_0 \leq (\hat{Q}_0)_{\text{max}} = \frac{\chi(P)}{24}$
\[ \mathcal{M}(p, r, n) := \text{Moduli of such DH's} \]

\[ \text{Hilb}^n(\Sigma) \rightarrow \mathcal{M}(p, r, n) \]

\[ \Sigma \rightarrow \{ \Sigma \in \text{P} \mid F \in H^m(\Sigma) \} \]

\[ \text{Roughly:} \]

\[ \text{Moduli of stable objects } \Sigma \]

\[ \text{in the derived category} \]

\[ \text{with specified Chern classes} \]

\[ \text{ch } E \sqrt{A} = p + q + q_0 \] (★)

\[ = \bigcup_{F, N \text{ s.t. } \ast} \mathcal{M}(p, r, n) \]
B. INDEX OF BPS STATES

"\( \Omega(\Gamma)_{\infty} := \lim_{J \to \infty} \Omega(\Gamma; B + iJ) \)"

\[ d(F,N) := (-1)^{\dim U} \chi(U(P,F,N)) \]

\[ \Omega(\Gamma)_{\infty} = \text{FINITE SUM OF } d(F,N) \]

SURPRISE: WHEN \( h''(x) > 1 \) THERE ARE SPLITTINGS @ \( \infty \):

\[ \Gamma = P + Q + q_0 dV \]

\[ = (P' + Q' + q'_0 dV) + (P'' + Q'' + q''_0 dV) \]

WITH:

\[ \sqrt{-\hat{g}_0''(P'')^3} > \sqrt{-\hat{g}_0} P^3 \]

\[ \Rightarrow \text{EVEN THE LEADING ORDER ENTROPY IS CHAMBER DEPENDENT} \]

[ E. ANDRIYASH + G. M. ]
For $\Gamma = P + Q + q_* dV$, $P \in \text{Kähler cone}$, $\exists$ distinguished chamber:

$$\Omega(\Gamma)_\infty := \lim_{\lambda \to \infty} \Omega(\Gamma; B + i \lambda P)$$

Claim: Limit exists and is $B$-independent 
(Finiteness of attractor flow trees)

Henceforth work in this chamber.
C. Modularity

\[ \tau \in H_1 \quad \xi \quad \psi \in \mathcal{Z}^*(H^2(X, \mathbb{C})) \]

\[ Z(\tau, \overline{\tau}, \psi) := \sum_{F, N} d(F, N) \exp \left\{ -2\pi i \xi \hat{q}_0 - 2\pi i \xi \frac{1}{2}(F^+)^2 - 2\pi i F \cdot (\psi + \frac{1}{2}) \right\} \]

\textbf{Susy Partition Function of D3 Instanton}

\textbf{U-Duality} \implies \textbf{Susy Partition Function is a Jacobi Form} \implies \textbf{JACOBBI FORM}

\[ Z(\tau, \overline{\tau}, \psi) = \sum_{\mu \in L^* / L} H_{\mu}(\tau) \Theta_{\mu, L}(\tau, \overline{\tau}, \psi) \]

\textbf{Siegell–Narain}

\[ L := \mathcal{Z}^*(H^2(X, \mathbb{Z})) \subset H^2(\Sigma; \mathbb{Z}) \]

\textbf{Self-Dual}

\[ \overline{\xi} \in L \text{ is always in } H^{11}(\Sigma) \implies d(F + \xi, N) = d(F, N) \quad \forall \xi \in L \]
• $H_\mu(\tau)$ is a vector-valued nearly holomorphic modular form of weight $w = -1 - \frac{h_1(x)}{2}$ and multiplier system $M^*$ dual to that of $\Theta_{\mu, 1}$.

• $w < 0 \implies H_\mu$ is determined by its polar terms.

Suppress $\mu$-index for simplicity:

$$H(\tau) = \sum_{\hat{q}_0} \Omega(\Gamma)_\infty e^{-2\pi i \hat{q}_0 \tau}$$

$$= \sum_{0 < \hat{q}_0 \leq \frac{\chi(p)}{24}} (\ldots) + \sum_{-\infty < \hat{q}_0 \leq 0} (\ldots) \text{ POLAR } \text{ NONPOLAR}$$
D. MACROSCOPIC POLAR STATES

If \( \Gamma = (0, p, q, \theta_0) = P + Q + \theta_0 dv \)

is polar: \( 0 < \theta_0 \leq (\theta_0)_{\text{max}} \)

then \( Z(\Gamma; t) \) has a zero.

Indeed \( S(\Gamma) = \frac{2\pi}{\sqrt{6}} \cdot \sqrt{-\theta_0} \chi(p) \)

so no single-centered solution

but \( H(\tau) \) has \( w < 0 \) \( \Rightarrow \) some polar degeneracies are non-zero

\( \Rightarrow \) these must be realized as split attractor states.
**Simple Example**

**Pure D4**: \( \Gamma = P + q_0 dV \)

*With* \( q_0 = \hat{Q}_0 = (\hat{Q}_0)_{\text{max}} = \frac{\chi(P)}{24} \)

**Find only one splitting**

\[
\Gamma = P + q_0 dV = \Gamma_1 + \Gamma_2
\]

\[
= e^{S_1 \left(1 + \frac{c_2(x)}{24}\right)} - e^{S_2 \left(1 + \frac{c_2(x)}{24}\right)}
\]

1 D6 with flux = \( S_1 \)

1 \( \overline{D6} \) w/ flux \( S_2 \)

\( S_1 - S_2 = P \)
Moreover - you can compute the polar degeneracy:

\[
\Omega(\Gamma, t_\infty) = (-1)^{I_{12}^{-1}} |I_{12}| \Omega(\Gamma_1) \Omega(\Gamma_2) = (-1)^{I_{12}^{-1}} |I_{12}|
\]

\[
I_{12} = \langle \Gamma_1, \Gamma_2 \rangle = \frac{p^3}{6} + \frac{c_2(X) \cdot p}{12}
\]

Indeed = the correct answer for
\[
\chi(\text{moduli of pure D4}) = \chi(|P|)
\]
Describing the split attractor flows for \( 0 < \hat{\varphi}_0 < \frac{x(p)}{24} \) is much more complicated...

In general, polar states can be very complicated split attractors, realized in many different ways....

But in the limit \( p \to \infty \) we can say something
**EXTREME POLAR STATES**

\[
H^\text{POLAR}_{(T)} = |I_p| e^{-2\pi i \tau \frac{\chi(p)}{24}} + \cdots + O(e^{-1|p|})
\]

“EXTREME POLAR”  “BARELY POLAR”

**E.P.S. CONJECTURE:** \( \exists \epsilon < 1 \) so that

\[
\frac{\hat{q}_0^{\text{max}} - \hat{q}_0}{\hat{q}_0^{\text{max}}} < \epsilon \quad \Rightarrow
\]

**POLAR STATES SPLIT AS D6\overline{D6} + HALOS:**

\[
\Gamma_1 = e^{S_1(1 - \beta_1 + n_1 dV)} \\
\Gamma_2 = -e^{S_2(1 - \beta_2 + n_2 dV)}
\]
6. ROUTE TO THE OSV CONJECTURE

A. BY THE W.C.F. THE (EXTREME) POLAR DEGENERACIES GO LIKE

\[ \Sigma (\text{D6-D2-DO}) \times \Sigma (\text{D6-D2-DO}) \]

B. BUT BPS INVARIANTS OF THE D6-D2-DO SYSTEM ARE RELATED TO GROMOV-WITTEN INVARIANTS COUNTING WORLDSPHET INSTANTONS IN X
So, by the W.C.F. together with results on $Z_{D6D2D0}$, the extreme polar degeneracies are related

$$|Z^\text{top}|^2$$

suggesting a relation like the OSV conjecture

$$\Omega(\Gamma)_\infty = \int d\phi \ |Z^\text{top}(g^\text{top}, t)|^2 e^{-2\pi g\phi}.$$  

- \exists strong arguments for $|\hat{a}_0| \gg p^3$

- \exists potential counterexamples for $|\hat{a}_0| \lesssim p^3$: "entropy enigma"
In the charge regime

\[ g_{\text{top}} \sim \sqrt{-\frac{g}{a^*}} \lesssim O(1) \]

The derivation in Denef-Moore breaks down.

- Barely polar degeneracies become large
- Corrections to the Cardy formula become large.

There is a good physical reason the derivation breaks down...
**Entropy Enigma**

Now choose $q_0 < 0$, $P$ ample so

$$
\Gamma = (0, P, 0, q_0)
$$

has a regular attractor point

Nevertheless! we can choose $q_0$, $Q_A$ so that $\exists$ a two-centered solution with

$$
\Gamma = \Gamma_1 + \Gamma_2
$$

$$
\Gamma_1 = (r, \frac{1}{2}P, Q, \frac{1}{2}q_0) \quad \Gamma_2 = (-r, \frac{1}{2}P, -Q, \frac{1}{2}q_0)
$$

**Both solutions exist**
So... compare entropies

\[ S(\Gamma) \text{ vs. } S(\Gamma_1) + S(\Gamma_2) \]

In fact,

\[ \exists \text{ family of charges} \]

\[ \lambda \Gamma = \lambda (0, P, 0, q_0) = \Gamma_1^\lambda + \Gamma_2^\lambda \]

\[ \Gamma_1^\lambda = (\Gamma, \frac{\lambda}{2} P, \lambda^2 Q, \frac{\lambda}{2} q_0) \quad \Gamma_2^\lambda = (-\Gamma, \frac{\lambda}{2} P, -\lambda Q, \frac{\lambda}{2} q_0) \]

Scaling of entropies:

\[ S(\lambda \Gamma) = \lambda^2 S(\Gamma) \]

But!

\[ S(\Gamma_1^\lambda) = S(\Gamma_2^\lambda) \sim \left(\frac{\lambda P}{r}\right)^3 \sim \lambda^3 \]

\[ \Rightarrow \text{ many implications for physics & mathematics} \]
1. Construct a family of 2-centered
   \[ \tilde{\Gamma}_1^\lambda \triangleq (r, \frac{p}{2}, Q, \lambda^2 \frac{q_0}{2}) \]
   \[ \tilde{\Gamma}_2^\lambda \triangleq (-r, \frac{p}{2}, -Q, \lambda^2 \frac{q_0}{2}) \]
   \[ \tilde{\Gamma}_i^\lambda \text{ can be 1-centered BH's or can themselves be polar} \]

2. Attractor Formalism has a scaling symmetry under
   \[ T_2 (p^0, p, Q, q_0) = (p^0, \lambda p, \lambda^2 Q, \lambda^3 q_0) \]
   \[ S(T_\lambda \Gamma) = \lambda^3 S(\Gamma) \]

3. Apply to \[ T_\lambda \tilde{\Gamma}_1^\lambda + T_\lambda \tilde{\Gamma}_2^\lambda = \lambda \Gamma \]
Recently, Deboer et al. showed that if we split the D2-D0 charge asymmetrically between the two centers then the coefficient of the $\lambda^3$ growth can be increased:

\[ \lambda^3 \]

\[ \text{DOMINATES} \]

\[ \text{BUT BOTH CONTRIBUTIONS SCALE LIKE } \lambda^3. \]
DEGENERACY DICHOTOMY

- We have found contributions to $\Omega(\Delta \Gamma)_\infty$ growing like $e^{\lambda^3}$

- If indeed $\Omega(\Delta \Gamma)_\infty \sim e^{\lambda^3}$ then weak coupling OSV is wrong; since OSV $\Rightarrow \Omega(\Delta \Gamma)_\infty \sim e^{\lambda^2}$.

- But $\Omega(\Delta \Gamma)_\infty$ is an index. It is possible that $\Omega(\Delta \Gamma)_\infty = \sum \pm e^{\lambda^3} \sim e^{\lambda^2}$

- We argue that this is unlikely, but it is not excluded.
Suppose that there are “magical cancellations” and \( \Omega(\chi \pi) \propto e^{\chi^2} \).

- This raises the question of \( \dim \mathcal{H}(\Gamma; t) \) vs. \( \Omega(\Gamma; t) \).

- Physically: the dimension is relevant.

- But all tests of the Strominger-Vafa program use the index (with one exception).

- It is \& to suppose that in the exact theory, nonperturbative stringy effects give:

\[
\dim \mathcal{H}(\Gamma; t) = \Omega(\Gamma; t)
\]
IF WE GRANT THIS POINT, AND IF, MOREOVER, THERE ARE "MAGICAL CANCELLATIONS" SO THAT
\[ \log \Omega(0, \lambda P, 0, 190) \sim \chi^2 \]

THEN THE SPECTRUM OF NEAR-BPS STATES TAKES A REMARKABLE FORM:

\[ E^{-|z|} = 0 \sim e^{\chi^2} \text{ states} \]

\[ E^{-|z|} \sim e^{-1/9 s} \sim e^{\chi^3} \text{ states} \]
7. Kontsevich–Soibelman Formula

The KS formula is a relation between $\Omega(\Gamma; t^\pm)$ across MS walls with no restriction on primitivity of constituents.

- No physical derivation yet

Evidence that $k_\xi$'s $\Omega(\Gamma; t)$'s are the same as physical $\Sigma(\Gamma; t)$'s.

- Can recover primitive WCF

- Can recover semi-primitive WCF

- Nontrivial checks for $\text{SU}(2)$ Seiberg–Witten with $N_f = 0, 1, 2, 3$ hypermultiplets

(List two are results w/ Wu-yen Chuang)
The Kontsevich–Soibelman formula

For the lattice \( \Lambda \) of charges introduce a Lie algebra \( \mathbb{Z}[\Lambda] \) with one generator for each \( \gamma \in \Lambda \):

\[ [e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle \gamma, \gamma_2 \rangle} \langle \gamma_1, \gamma_2 \rangle e_{\gamma_1 + \gamma_2} \]

For fixed \( t \), \( \mathbb{Z} : \Lambda \rightarrow \mathbb{C} \), choose any convex angular sector \( V \)

\[
\prod_{\gamma \in \mathbb{Z}(V) \cap \Lambda} \left( \exp \sum_{n=1}^{\infty} \frac{e_{n\gamma}}{n^2} \right) \Omega(\gamma)
\]

Increasing slope

\[
\prod_{\gamma \in \mathbb{Z}(V) \cap \Lambda} \left( \exp \sum_{n=1}^{\infty} \frac{e_{n\gamma}}{n^2} \right) \Omega^+(\gamma)
\]

Decreasing slope
AT A GENERIC POINT $t \in MS(\Gamma_1, \Gamma_2)$

$Z(\Gamma; t) \parallel Z_1, Z_2 \implies$

$\Gamma = \Gamma_{a,b} = a\Gamma_1 + b\Gamma_2$

$(\Gamma_1, \Gamma_2$ primitive$)$

FOR SMALL CONE ANGLE ONLY THE LIE SUBALGEBRA $Z_1 \Gamma_1 + Z_2 \Gamma_2$ CONTRIBUTES:

$[e_{a,b}, e_{c,d}] = (-1)^{(ad-bc)}I_{12} (ad-bc)I_{12} e_{a+c,b+d}$

DEFINE:

$U_{a,b} := \exp\left(\sum_{m=1}^{\infty} \frac{e_{ma,mb}}{m^2}\right)$

$\prod_{\gamma \neq \gamma, \gamma \neq 0} U_{a_{\gamma},b_{\gamma}} = \prod_{\gamma \neq \gamma, \gamma \neq 0} U_{a_{\gamma},b_{\gamma}}$
LIE ALGEBRA IS FILTERED $\Rightarrow$

**Heisenberg Algebra**

\[
\left[ e_{0,1}, e_{1,0} \right] = (-1)^{I_2} I_{12} e_{1,1}
\]

$e_{1,1}$ **CENTRAL**

\[
\Omega^{-}(\Gamma_1) \quad \Omega^{-}(\Gamma_1 + \Gamma_2) \quad \Omega^{-}(\Gamma_2)
\]

\[
\Omega_{0,1} \quad \Omega_{1,1} \quad \Omega_{1,0}
\]

\[
\Omega^{+}(\Gamma_2) \quad \Omega^{+}(\Gamma_1 + \Gamma_2) \quad \Omega^{+}(\Gamma_1)
\]

\[
\Omega_{0,1} \quad \Omega_{1,1} \quad \Omega_{0,0}
\]

$U_{0,1} U_{1,0} = U_{1,1} \quad U_{1,0} U_{0,1}$ $\Rightarrow$

\[
\Omega^{+}(\Gamma_1 + \Gamma_2) - \Omega^{+}(\Gamma_1 + \Gamma_2)
\]

\[
\Omega_{1,1} \quad \Omega_{1,0}
\]

\[
\Omega^{-}(\Gamma_1) \quad \Omega(\Gamma_2)
\]

\[
\Omega_{0,1} \quad \Omega_{1,1} \quad \Omega_{0,0}
\]

$\quad I_{12} \quad \Omega(\Gamma_1) \quad \Omega(\Gamma_2)
\]

$\quad U_{1,1}$

**PRIMITIVE W.C. FORMULA!**
SU(2) SEIBERG-WITTEN THEORY

\[ \Gamma_1 = \text{MONPOLE} \]
\[ \Gamma_2 = \text{DYON} \]

\[ [e_{a,b}, e_{c,d}] = 2(bc-ad)e_{a+c,b+d} \]

**STRONG**: \( \pm (1,0), \pm (0,1) \) \( \Omega = +1 \) \( \text{HM} \)

**WEAK**: \( \pm (1,1) \) \( \Omega = -2 \) \( \text{VM} \)

\( \pm (n,n+1), \pm (n+1,n) \) \( \Omega = +1 \) \( \text{HM} \)

**STRONG**: \( U_{1,0} \cdot U_{0,1} \)

**WEAK**:

\( (U_{0,1} U_{1,2} U_{2,3} \cdots) \ U_{1,1}^{-2} (\cdots U_{3,2} U_{2,1} U_{1,0}) \)

**EQUALITY APPEARS TO BE TRUE!**

\( \exists \) **NEW IDENTITIES FOR** \( N_f = 1, 2, 3 \)
8. SOME OPEN PROBLEMS

a.) PHYSICAL DERIVATION OF THE KS FORMULA

b.) HOW TO COMPUTE POLAR DEGENERACIES EFFECTIVELY?

c.) RESOLVE THE QUESTION OF THE ENTROPY ENIGMA: ARE THERE CANCELLATIONS BRINGING $e^{\lambda^3} \rightarrow e^{\lambda^2}$?

d.) IS THERE AN OSV-LIKE RELATION FOR $\Omega(\Gamma, t_*(\Gamma))$? DO THESE ENJOY AUTOMORPHY PROPERTIES?