Lecture III: Extremal W=2 CFT's

1. Summary of key points from lectures I+II.

2. Quantum gravity in 2+1 dimensions.

3. Defining extremal W=2 CFT

4. Counting polar polynomials

5. Counting weak Jacobi forms

6. Search for the extremal E.G.

7. Near-extremal W=2 CFT

8. Conclude: two open problems

Sections 3-7 are unpublished results with

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1. SUMMARY OF KEY POINTS FROM LECTURES I & II.

A. A VECTOR-VALUED NEARLY HOLomorphic MODULAR FORM OF WEIGHT W < 0 IS DETERMINED BY ITS POLAR PART

B. THE ELLIPTIC GENUS OF AN $\mathcal{W} = 2$ THEORY WITH INTEGRAL $U(1)$ CHARGES AND $C = 6m$ IS A WEAK JACOBI FORM $X(\tau, z; C) \in \tilde{J}_{0, m}$

C. JACOBI FORMS ARE EQUIVALENT TO $V$-$V$-$N$-$H$ MOD. FORMS

\[ X(\tau, z) = \sum_{n,l} C(n,l) q^n y^l \]
\[ = \sum_{\mu \mod 2m} h_\mu(\tau) \Theta_{\mu,m}(\tau, z) \]
D. THE POLAR TERMS IN $h_\mu(x)$ ARE THE TERMS $c(n,l)$ WITH

$$p = 4mn - l^2 < 0$$

AFTER SPECTRAL FLOW $i$ CHARGE CONJUGATION THE INDEPENDENT

$C(n,l)$ HAVE $(l,n) \in P$

E. IF $C$ HAS AdS$_3$ DUAL, BTZ BLACK HOLES ONLY CONTRIBUTE TO THE NON-POLAR PART OF $X$. 
2. Quantum Gravity in 2+1 Dim's

Recently, Witten revived an old (1986) proposal that Quantum Gravity in 2+1 Dim's is exactly soluble.

Focus on the case $\Lambda < 0$

What is the holographic dual of 2+1 Q.G.? 

To motivate Witten's answer

Let us recall why people think that 2+1 Q.G. is exactly soluble:
THE ACTION

\[ S = \frac{1}{16\pi G} \int d^3x \sqrt{g} \left( R + \frac{2}{\ell^2} \right) + \frac{k}{4\pi} \int \text{Tr}(\omega d\omega + \frac{2}{3} \omega^3) \]

\( \ell - \text{AdS LENGTH} \)

\( \omega - \text{SPIN CONNECTION} \)

\( k - \text{QUANTIZED} \)

IS CLASSICALLY EQUIV. TO:

\[ S = \frac{k_+}{4\pi} \int \text{Tr} \left( A_+ dA_+ + \frac{2}{3} A_+^3 \right) - \frac{k_-}{4\pi} \int \text{Tr} \left( A_- dA_- + \frac{2}{3} A_-^3 \right) \]

\[ k_+ = \frac{2}{16G} \pm \frac{k}{2\ell}, \quad A_\pm = \omega \mp \ast e / \ell \]

WITTEN SUGGESTS THAT "THEREFORE"

THE HOLOGRAPHICALLY DUAL PARTITION FUNCTION IS FACTORIZED:

\[ Z(\tau, \bar{\tau}) = Z_{k_+}(\tau) \overline{Z_{k_-}(\tau)} \]
On a compact space C.S. has no local degrees of freedom. Neither does 2+1 gravity: no grav. waves. Locally solutions are just AdS

Brown & Henneaux: edge states = Virasoro algebra descendants

\[ C_L + C_R = \frac{3\mathcal{L}}{G} = 24(k_+ + k_-) \]

\[ C_L - C_R = 24(k_+ - k_-) \]

\[ C_L = 24k_+ \quad C_R = 24k_- \]

\[ Z_\kappa(\tau) = \chi_{\text{vac}} = q^{-k} \sum_{n=2}^{\infty} \frac{1}{1-q^n} \]

(Note that \( L_{-1, 10} \) is a null state)

\[ \chi_{\text{vac}}(\tau) = q^{-k+\frac{1}{2}}(1-q)^{1 \eta(\tau)} \]
EVIDENTLY, THIS IS NOT MODULAR

BUT WE EXPECT MODULARITY IN A DIFF - INV THEROY.

WHAT TO DO?

WITTEN PROPOSES THAT THE P.F. SHOULD BE AS CLOSE TO THE VIRA SORO CHARACTER AS POSSIBLE:

\[
Z_k(z) = \left[ \sum_{n=2}^{\infty} \frac{1}{1 - q^n} \right] + O(q^0)
\]

AS I EXPLAINED IN LECTURE 1 THIS MEANS:

\[
Z_k(z) = a_k j^k + \cdots + a_0
\]

THAT IS - BY ADDING NONPolar TERMS WE CAN RENDER \( Z(z) \) MODULAR
Witten interprets these terms as the contribution of BTZ black holes.

This fits in perfectly with the Fareytail story of Lecture II.

Obviously the reasoning is far from air-tight. Whether you like it or not, Witten makes a sharp proposal:

Let \( Z_k(t) \) be the unique mod. invt. function such that:

\[
Z_k(t) = \left[ q^{-\sum_{n=2}^{\infty} \frac{1}{1-q^n}} \right] + o(q) \]
**DEF:** An extremal CFT of level $K$ is a CFT with partition function $Z_k(z)$.

**Witten's Proposal:** The holographic dual of pure 2+1 gravity is $\mathcal{C}_k \otimes \overline{\mathcal{C}}_k$ where $\mathcal{C}_k$ is an ECFT$_k$.

- **But do ECFT's exist?**
- **Yes for $K = 1$**
- **Controversial for $K > 1$**

This leads to a question: Maybe extremal superconformal theories are easier to find...

The story for $W = 1$ is discussed in Witten's paper and is similar to $W = 0$. 

3. DEFINING EXTREMAL $W=2$ CFT

I will now describe some work in progress with
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Something qualitatively new happens in the $W=2$ case.

We will use modularity of the elliptic genus to put constraints on the spectrum of $W=2$ primary fields.

How should we define an "extremal $W=2$ CFT"?

It's up to us.

Let's follow Witten's lead
AND TAKE IT TO BE SUCH THAT THE PARTITION FUNCTION

\[ Z(z, \tau; \bar{z}, \bar{\tau}) \] DEFINED LAST TIME IS "AS CLOSE AS POSSIBLE" TO THE VACUUM CHARACTER.

VACUUM CHARACTER FOR \( |h, q> \equiv 10, 0> \) AND \( c = 6m \):

\[ X_{Vac}(z, \bar{z}) = q^{-m/4} \frac{\prod_{j=1}^{\infty} (1 + g^j q^{j+1/2}) (1 + \bar{g}^{-1} q^{j+1/2})}{\prod_{j=1}^{8} (1 - q^j) \prod_{j=2}^{8} (1 - q^{-j})} \]
This is not spectral flow invar & does not have good modular properties – so we force it to have these prop's:

**DEF:** An \( W=(2,2) \) extremal CFT is a hypothetical theory with

\[
Z_{NSNS} = \left| \sum_{\Theta \in \mathbb{Z}^2} S_{\Theta} \chi_{\text{vac}} \right|^2 + \sum_{\rho > 0} \sum_{\tilde{\rho} > 0} c(n, l; \tilde{n}, \tilde{l}) q^n y^l \tilde{q}^{\tilde{n}} \tilde{y}^{\tilde{l}}
\]

We could have formulated this in the Ramond or NS sector, but the def seems best motivated in the NS-sector.
This definition implies that an $\mathcal{N}=2$ ECFT must have elliptic genus

$$\chi_{\text{ext}}(\tau, z) = 2(-1)^m \sum_{\Theta \in \mathbb{Z} + \frac{1}{2}} SF_\Theta \chi_v + \text{nonpolar}$$

$$= 2(-1)^m \left\{ (1-y^m) \frac{\eta}{\Delta} \left( 1 - y q^{m+1} \right) \left( 1 - y^{-1} q^m \right) \frac{(1-q^m)^2}{(1-q^m)^2} + (y \rightarrow y^{-1}) + \text{nonpolar} \right\}$$

- Is this compatible with modular invariance?
- Does there exist such a

$$\chi_{\text{ext}} \in \tilde{J}_{0,m}?$$
4. COUNTING POLAR POLYNOMIALS

AS I EXPLAINED LAST TIME, THE ELLIPTIC GENUS IS COMPLETELY DETERMINED BY ITS FOURIER COEFFICIENTS IN THE POLAR REGION $\Omega$.

\[ p = 4mn - l^2 \]
Def: If $\phi = \sum \hat{\phi}(n,l) q^n y^l$ is any Fourier series its polar polynomial is:

$$\text{Pol}(\phi) := \sum_{(n,l) \in \mathcal{P}} \hat{\phi}(n,l) q^n y^l$$

Let us count the dimension of $P_m := \text{vector space of polar poly's.}$

$$p(m) = \dim P_m = \sum_{l=1}^{m} \left\lfloor \frac{l^2}{4m} \right\rfloor$$

This is the number of lattice points in the shaded region $\mathcal{P}.$

It is not entirely trivial to evaluate, and we explain in the notes that
\[ P(m) = \frac{m^2}{12} + \frac{5m}{8} + A(m) \]

\[ \uparrow \quad \downarrow \]

AREA \quad LATTICE CORRECTION

\[ A(m) = \# - \text{THEORETIC EXPRESSION INVOLVING CLASS NUMBERS} \]

MAIN POINT FOR US IS THAT FOR LARGE \( m \) IT GROWS ROUGHLY LIKE \( O(m^{1/2}) \)
Proof (time permitting)

Introduce sawtooth function

\[ ((x)) = x - \frac{1}{2} (\lfloor x \rfloor + \lceil x \rceil) = \begin{cases} 
0 & x \in \mathbb{Z} \\
\alpha & x = n + \alpha \\
0 & 0 < \alpha < 1
\end{cases} \]

Write

\[ \sum \left[ \frac{r^2}{4m} \right] = \sum \frac{r^2}{4m} - \sum ((\frac{r^2}{4m})) + \frac{1}{2} \sum \left( \left[ \frac{r^2}{4m} \right] - \frac{r^2}{4m} \right) \]

(A) \quad (B) \quad (C)

(A) \quad \frac{m^2}{12} + \frac{m}{8} + \frac{1}{24}

(C) \quad \frac{m}{2} + \text{correction when } \frac{r^2}{4m} \in \mathbb{Z} \text{ happen } O(m^{1/2})

\[ = \frac{m}{2} - \left| \frac{b}{2} \right| \] \quad \text{ where } b = \text{largest integer } b^2 \mid m

(B) \quad \text{Roughly, a random walk between } -\frac{1}{2} \text{ and } +\frac{1}{2}. \text{ So } O(m^{1/2}). \text{ Can express exactly in terms of class numbers.}
5. Counting Weak Jacobi Forms

Now the elliptic genus must be a weak Jacobi form of weight zero and index $m$.

What is the dimension of $\hat{J}_{0,m}$?

**Thm [Eichler-Zagier]** The bigraded ring $\hat{J}_{*,*}$ of weak Jacobi forms is a polynomial ring:

$$\hat{J}_{*,*} = \mathbb{C}[E_4, E_6, \phi_{0,1}, \phi_{2,1}]$$

Note the similarity to the theorem we proved about $m_*$ in Lecture 1. The proof is not hard—but skip it.

We therefore have a basis for $\hat{J}_{0,m}$.
\((\phi_{-2,1})^a (\phi_{0,1})^b E_4^c E_6^d\)\
\[
\left\{
\begin{array}{l}
a + b = m \\
-2a + 4c + 6d = 0
\end{array}
\right.
\quad a, b, c, d \in \mathbb{Z}_+
\]

\[j(m) := \dim \overline{J_{0,m}} \text{ IS THE # OF SOLUTIONS TO } (*)\]

A STRAIGHTFORWARD COMPUTATION SHOWS THAT

\[j(m) = \frac{m^2}{12} + \frac{m}{2} + \tilde{A}(m)\]

\[\tilde{A}(m) = \left( \delta_{5,0} + \frac{s}{2} - \frac{s^2}{12} \right); \quad m = s \mod 6 \quad 0 \leq s \leq 5\]

BUT \(P(m) = \frac{m^2}{12} + \frac{5m}{8} + O(m^{1/2})\) !

\[P(m) > j(m) \text{ FOR } m \geq 5\]

\[P(m) - j(m) = \frac{m}{8} + O(m^{1/2})\]
RECALL THAT WHEN WE DISCUSSED THE RECONSTRUCTION FORMULA IN LECTURE II

\[ f(z) = \frac{1}{2} \sum_{\mathfrak{p}} \frac{\langle y, z \rangle R}{S^2} + \text{REG} \]

I SAID THAT IN GENERAL YOU CANNOT TAKE AN ARBITRARY F- AND GET A MODULAR FORM.

THE OBSTRUCTION IS MEASURED BY A SPACE OF CUSP FORMS

\[
0 \to \tilde{J}_{0,m} \xrightarrow{\text{Pol}} P_m \xrightarrow{\text{deg}} S_{5/2}^1(\Gamma, M)
\]

AS DESCRIBED IN PAPER WITH MANSCHOT.

RECENTLY J. MANSCHOT COMPUTED THE DIMENSION OF THE OBSTRUCTION SPACE AND REPRODUCES THE FORMULA FOR \( P(M) - J(M) \). [TO APPEAR.]
6. **Search for the Extremal Elliptic Genus**

Now return to our hypothetical extremal $\mathcal{N}=2$ CFT with

$$ \chi_{\text{Ext}} = SF_{1/2} X_V + SF_{-1/2} X_V + \text{NP} $$

There is no guarantee that

$$ p^m_{\text{Ext}} := \text{Pol}(\chi_{\text{Ext}}) \in \text{Im}(\tilde{J}_{0,m}) $$

But maybe there is magic...

Choose a basis $\phi_i$ for $\tilde{J}_{0,m}$.

**If** the extremal $\mathcal{N}=2$ CFT exists then

$$ \exists \prod_{i=1}^{j(m)} x_i \text{ Pol}(\phi_i) = p^m_{\text{Ext}} \quad \bigtriangleup $$
Even if we find solutions there is a further test since

$$\sum_{i} x_i \phi_i = \sum c(n,e) q^n y^e$$

must have $c(n,e) \in \mathbb{Z}$

Of course, these are necessary, not sufficient conditions for the existence of the extremal theory.

To analyze equation, introduce a polarity-ordered basis for $P_m$,

$$q_i^{n(a)} y^e(l(a)) \quad a = 1, \ldots, (2^m - 1)$$
Polarity \uparrow \text{ as } a \uparrow

For \( a = 1 \), \( y^m \) has \( p = -m^2 \),

Define:

\[
\text{Pol } \phi_i := \sum_{a=1}^{P(m)} N_{ia} \, g^{n(a)} \, l(a) \]

\[
P_{\text{Ext}} := \sum_{a=1}^{P(m)} d_a \, g^{n(a)} \, l(a) \]

Equation \( \otimes \) is:

\[
\sum_{i=1}^{j(m)} x_i \, N_{ia} = d_a \quad a=1, \ldots, P(m) \]
RECALL: \( P(m) > J(m) \quad m \geq 5 \)

**COMPUTER:** \( 1 \leq m \leq 36 \)

SOLN'S \( x_i \) EXIST FOR: \( 1 \leq m \leq 5, 7, 8, 11, 13 \)

\( \not\exists \ x_i \) FOR \( m = 6, 9, 10, 12, 14 \leq m \leq 36 \)

**MOREOVER:** IN THE CASES WHERE SOLUTIONS EXIST THE \( C(n,2) \) SEEM TO BE INTEGRAL.

(THIS IS NONTRIVIAL. E.G. \( m = 2 \)
\[ \frac{1}{6} \phi_{0,1}^2 + \frac{5}{6} \phi_{2,1}^2 E_4 \])
Recently, we found an analytic argument:

For \( m \) suff. large, solutions \( X_i \) do not exist.

Thus, extremal \( N=2 \) theories at best exist for a finite "sporadic" set of \( m \).
7. NEAR EXTREMAL $W=2$ CFT

Perhaps our definition was too restrictive...

Maybe there are quantum corrections to the cosmic censorship bound...

Let us define a $\beta$-extremal $W=2$ CFT

If we only demand agreement with the vacuum character up to polarity $-\beta$
Now we only try to match the polar degeneracies in $p_\beta$. Only descendents of the vacuum.
So we increase $\beta$ until we are solving

$$\sum_{i=1}^{j(m)} x_i N_{ia} = d_a \quad a = b, \ldots, j(m)$$

This happens for

$$\beta = \frac{m}{2} + O(m^{1/2})$$

**Computer:**

- **Solutions $x_i$ do exist!**
  $$1 \leq m \leq 36$$
- **And**
  $$\sum_{i} x_i \phi_i = \sum c(n,l) q^n y^l$$

Do have $c(n,l) \in \mathbb{Z}$ ...
EXCEPT FOR \( m = 17 \)!

QUI IN ITALIA IL DICIASSETTE PORTA SFIGA!

FOR \( m = 17 \), WE MUST LOWER THE POLARITY CUTOFF STILL FURTHER....
ON THE OTHER HAND,

THERE IS SOME $\beta_*(m)$ SO THAT WE CAN MATCH $\chi_{\text{vac}}$ FOR ALL STATES OF POLARITY LESS THAN $-\beta_*(m)$

THE COMPUTER EVIDENCE SUGGESTS

$$\beta_*(m) \geq \frac{m}{2} + \Theta(m^{1/2})$$

FOR ALL $m$.

IF TRUE, THEN FOR ANY $W=(2,2)$ CFT THERE MUST BE AN $W=2$ PRIMARY WITH

$$4m \left( h^2 - \frac{c}{24} \right) - \ell^2 > -\beta_*(m)$$
Thus we formulate the conjecture:

Any $W=(2,2)$ CFT with integral $U(1)$ charges must have a state $\Psi_L \otimes \bar{\Psi}_R$ where $\Psi_L$ is an $W=2$ primary with:

$$h^{NS} > \frac{m}{4} + \frac{(J_0^m)^2}{4m} - \frac{1}{8} + O(m^{-1/2})$$

On the other hand — using a different method — it is possible to show how to construct a $X$ with only contributions from descendants with

$$h^{NS} \leq \frac{5m}{16}$$
NOW—IT IS ABOUT TIME I REVEAL MY REAL MOTIVATION FOR PURSUING THIS PROBLEM:

IN RECENT YEARS THERE HAS BEEN MUCH ACTIVITY IN FLUX COMPACTIFICATION AND MODULI STABILIZATION. (C.f. L. McAllister talks.) Shamit Kachru has been a driving force in this development.

With many choices for flux there are, famously, many possible compactifications we expect there are some with
\[ \text{AdS}_3 \times \mathbb{R} \]

\[ \Rightarrow \exists \text{ Holographic dual CFT}_2 \]

\[ 0 < - \frac{\Lambda}{M_{Pl}^2} = + \left( \frac{1}{2M_{Pl}} \right)^2 \ll 1 \]

\[ \Rightarrow c = \frac{3 \beta}{2G} \gg 1 \]

\[ \text{KK Lengthscale of } \mathbb{R} \text{ is order } c \text{ in AdS units} \]

\[ \Rightarrow \text{"Most" Primaries have a large gap from the vacuum.} \]

The spectrum resembles an extremal CFT!
8. CONCLUSION

I WILL LEAVE YOU WITH TWO OBVIOUS OPEN PROBLEMS

- PROVE THE BOUND ON $h^+$
- DOES THIS BOUND PUT ANY INTERESTING CONSTRAINTS ON $\text{AdS}_3$ FLUX COMPACTIFICATIONS?