N=2* SYM, Four Manifold Invariants, And Mock Modularity

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Project has taken a few years. I first spoke about it at Simons Foundation Conference, Sept. 2017 (run by Jeff and Shamit)
Preliminaries

$X$: $d=4$, Smooth, compact, oriented, $\partial X = \emptyset$.

$b_2^+(X)$: Odd & positive

We study a TQFT on $X$

$d=4$ $N=2^*$ SYM. $G = SU(2), SO(3)$

The partition function generalizes both the Donaldson invariants and the Vafa-Witten invariants, and interpolates between them.
Preliminaries

The theory depends on a choice of background spin-c structure $\mathfrak{s}$. Labastida-Marino [1995] noticed need to introduce $\mathfrak{s}$.

The detailed dependence has not previously been discussed. Including it turns out to be nontrivial. We believe we have solved the problem completely.

Long, long, ago, at the ITP in 1998....
General definition

Given a formal series \( s(z) = \sum_{n=0}^{\infty} a_n z^n \)
we say that a well-defined function \( f(z) \)

provides a non-potential function

\( f(z) \) if \( f(z) \) has an asymptotic

given by \( f(z) \approx \sum_{n=1}^{\infty} \frac{a_n}{n!} z^n \)

NP Ambiguity
Preliminary: $Spin^c$-structure

$Spin^c(4) := \{ (u_1, u_2) | \text{det}(u_1) = \text{det}(u_2) \} \subset U(2) \times U(2)$

$$1 \to U(1) \to Spin^c(4) \to SO(4) \to 1$$

Spin-c structure on $X$:
Reduction of structure group of $TX$ to $Spin^c(4)$

``Spinors'': Associated rank 2 bundles $W^\pm$

$c(\varsigma) := c_1(\text{det } W^\pm) \in H^2(X; \mathbb{Z})$

$$\ell = \frac{c(\varsigma)^2 - 2\chi - 3\sigma}{8} \in \mathbb{Z}$$
Preliminary: $Spin^c$ & ACS

An ACS $\mathcal{I}$ defines a canonical spin-c structure $\mathfrak{s}(\mathcal{I})$:

**Almost Complex Structure (ACS):**
Reduction of structure group of $TX$ to $U(2)$

$Spin^c(4) := \{ (u_1, u_2) | \det(u_1) = \det(u_2) \} \subset U(2) \times U(2)$

Use diagonal homomorphism $U(2) \to Spin^c(4)$.

For $c = c(\mathfrak{s})$ for $\mathfrak{s}$ an ACS we have

$$c^2 = 2 \chi + 3 \sigma \quad \ell = 0$$
1. Introduction & Preliminaries

2. Summary Of Main Claims

3. The N=2* Theory: UV Meaning Of Invariants

4. Remarks On S-Duality Orbits Of Partition Functions

5. Coulomb Branch Integral: Measure & Evaluation

6. LEET Near Cusps & Explicit Results
Data needed to formulate the partition function:

\[ \tau_{uv} \in \mathcal{H} \; ; \; q_{uv} := e^{2\pi i \tau_{uv}} \]

\[ m \in \mathbb{C} \quad \Lambda: \text{UV scale} \quad t := m/\Lambda \]

(\text{UV}) Spin-c structure: \( \varsigma \), \[ c_{uv} := c_1(\varsigma) \in H^2(X, \mathbb{Z}) \]

\[ \nu \in H^2(X; \mathbb{Z}/2\mathbb{Z}) \]
Path integral defines a "function"

\[ Z_\nu(\tau_{uv}, c_{uv}, t) : H_*(X; \mathbb{Z}) \to \mathbb{C} \]

\[ Z_\nu(x; \tau_{uv}, c_{uv}, t) = \sum_{k \geq 0} q^{k} \int_{\mathcal{M}_{Q,k}} e^{\mu(x)} \text{Eul}(\mathcal{E}_\varsigma; t) \]

\( \mathcal{M}_{Q,k} \): Moduli of nonabelian monopole connections on a principal \( SO(3) \) bundle \( P \to X \) with \( \nu = w_2(P) \) and instanton no. = \( k \)

\( \mu : H_*(X, \mathbb{Z}) \to H^{4-*}(\mathcal{M}_{Q,k}; \mathbb{Q}) \)

\( \mathcal{E}_\varsigma : U(1) \)-equivariant virtual bundle
Special cases were studied in [Moore & Witten 1997; Labastida & Lozano 1998]

Those studies were limited to spin manifolds with trivial spin-c structure.

Related work: Vafa-Witten & Dijkgraaf, Park, Schroers 1998 N=1 deformation of N=4 SYM, Kähler 4-folds with $b_2^+ \geq 3$ & no observables

Also related: Recent work of Göttsche, Kool, Nakajima, and Williams
Physical Mass Limits

\[ m \to 0 \]

\[ \{ N = 2^* \text{ SYM} \} \to \{ N = 4 \text{ SYM} \} \]

SW94:

\[ m \to \infty \& q_{uv} \to 0 \]

\[ \Lambda_0^4 = 4 \ m^4 q_{uv} \]

\[ \Rightarrow \text{pure SYM} \]
1A: For $\mathfrak{s} = \mathfrak{s}(J)$ and $t \to 0$

$$ Z_v(x, \tau_{uv}, c_{uv}, t) \to Z^\text{VW}_v(\tau_{uv}) $$

1B: For ANY spin-c structure, $m \to \infty$ & $q_{uv} \to 0$ with $\Lambda^4_0 := 4m^4q_{uv}$ fixed:

$$ Z^\text{renorm}_v(x, \tau_{uv}, c_{uv}, t) \to Z^\text{DW}_v(x) $$

What we mean by $Z^\text{renorm}_v$ is an interesting story best discussed later
Central Claim:
\[ Z_\nu \] can be computed by studying an integral over Coulomb Branch = Base of Hitchin system = (this case: modular curve \( \mathcal{H}/\Gamma(2) \cong \mathbb{C} - 3\text{pt} \))

2a: Writing a single-valued measure
⇒ implications for class S generalization

2b: \textbf{Integrand} is a total derivative of a mock Maass-Jacobi form.

2c: \textbf{Value} of the integral is a nonholomorphic completion of a mock modular form.
For $b_2^+ > 1$ $Z_\nu$ is a linear combination of SW invariants with coefficients in a ring of modular forms for $\tau_{uv}$ and obeys the "proper" S-duality covariance.

Today I will skip much of the physics background – See previous talks.
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6 LEET Near Cusps & Explicit Results
``Equations Of Motion''

\[
A \in \mathcal{A}(P) \quad M \in \Gamma(W^+ \otimes \text{ad}P \otimes \mathbb{C})
\]

\[W^+ \to X: \text{ Positive chirality rank two bundle associated to uv spin-c structure } \varsigma\]

\[Q - \text{fixed point equations (need Riemannian metric)}\]

\[F^+ + [M, \bar{M}] = 0 \quad DM = 0\]

``Nonabelian monopole/SW equations''

[Labastida-Marino; Losev-Shatashvili-Nekrasov]

When \(\varsigma\) is associated to an ACS these are equivalent to the Vafa-Witten equations.
Index Computations

\[ v \dim \mathcal{M}_{Q,k} = \dim G \frac{c_{uv}^2 - (2\chi + 3\sigma)}{4} \]

N.B. Independent of instanton number \( k \)!

\[ \dim \mathcal{M}_k = 8k - \frac{3}{2}(\chi + \sigma) \]

**Index** \( D = -8k + \frac{3}{8}(c_{uv}^2 - \sigma) \)

\[ \Rightarrow \text{Correlation functions on } H_*(X) \text{ infinite } q_{uv} - \text{series} \]
Operators In The TQFT

\[ \mathcal{O}: H_\ast(X, \mathbb{Z}) \to Q - coho \]

\[ p \in H_0(X; \mathbb{Z}) \quad \mathcal{O}(p) = \left[ Tr \phi^2(p) \right] \]

\[ S \in H_2(X; \mathbb{Z}) \quad \mathcal{O}(S) = \left[ \int_S Tr(\phi F + \psi^2) \right] \]

What do these mean mathematically?
$U(1)_b$ Symmetry

$F^+ + [M, \overline{M}] = 0 \quad DM = 0$

$U(1)_b : M \to e^{i\theta} M$

$U(1)_b$ acts on the moduli space $\mathcal{M}_{Q,k}$ of these eqs.

$Q$-coho $\cong H^*_{U(1)_b}(\mathcal{M}_{Q,k})$

$t = \frac{m}{\Lambda} : U(1)_b$ equivariant parameter

[Labastida-Marino; Losev-Shatashvili-Nekrasov]

$O(x) \leftrightarrow \mu(x)$
Generating Function Of Correlators

\[ Z_v(x; \tau_{uv}, c_{uv}, t) := \langle e^{\mathcal{O}(x)} \rangle_{\mathcal{N}=2^*} \]

\( Q\)-symmetry: Path integral \( \rightarrow \int_{\mathcal{M}_{Q,k}} \ldots \)

\[ \langle e^{\mathcal{O}(x)} \rangle_{\mathcal{N}=2^*} = \sum_{k \geq 0} q_{uv}^k \int_{\mathcal{M}_{Q,k}} e^{\mu(x)} \text{Eul}(\mathcal{E}_z; t) \]

\( \mathcal{E}_z \) : Obstruction bundle for elliptic complex
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S-Duality

In the $SU(2)$ theory $Z_\nu$ is the partition function in the presence of ‘t Hooft flux

$\nu \in H^2(X; \mathbb{Z}_2)$

The $Z_\nu$ span a vector space $\mathcal{V}$

But arbitrary linear combinations aren’t physically meaningful
Three Distinct Theories

\[ \mathcal{T}(SU(2)) \]

\[ \mathcal{T}(SO(3)_-) \quad \mathcal{T}(SO(3)_+) \]

Gaiotto, Moore, Neitzke 2009;
Aharony, Seiberg, Tachikawa 2013
Partition Functions For The $SO(3)_\pm$ Theories

\[ Z^\text{SO(3)}_+ = \sum_\rho e^{i \pi \nu \cdot \rho} Z_\rho \]

\[ Z^\text{SO(3)}_- = \sum_\rho e^{i \frac{\pi}{2} \rho^2 - i \pi \nu \cdot \rho} Z_\rho \]

\[ \Delta S = \frac{i \pi}{2} \int P_2(w_2(P)) \]

Aharony, Seiberg, Tachikawa 2013
S-Duality Transformations

\[ T: Z_\nu \rightarrow \xi_\nu Z_\nu \]

\[ S: Z_\nu \rightarrow (-i \tau_0)^w \sum_{\rho} e^{i \pi \nu \cdot \rho} Z_\rho \]

\[ w = -\frac{\chi}{2} - 4\ell \quad \ell = \frac{c(\varsigma)^2 - 2\chi - 3\sigma}{8} \]

\[ \xi_\nu = \omega_{12}^{-\chi - 2\ell} \omega_4^{-\nu^2} \]

Derivation from 6d ?
The $Z_\nu$ span a vector space $\mathcal{V}$

The physical partition functions of the theories form an orbit in that vector space.

It is a finite covering of the triangle of theories.
For simplicity, work in $\mathbb{P}_0$.

\[
\begin{align*}
&\left[Z_{\nu}^{SU(2)}\right] \\
&\left[Z_{\nu}^{SO(3)_+}\right] \\
&\left[Z_{\nu}^{SO(3)_-}\right] \\
&\left[Z_{\nu+w_2(X)}^{SO(3)_+}\right] \\
&\left[Z_{\nu+w_2(X)}^{SO(3)_-}\right] \\
&\left[Z_{\nu}^{SU(2)}\right]
\end{align*}
\]
Full Modular Transformation Law

\[ x = (p, S) \in H_0(X) \oplus H_2(X) \quad \tau := \tau_{\nu \nu} \]

\[
Z_{\nu} \left( \tilde{p}, \tilde{S}, \frac{a \tau + b}{c \tau + d} \right) = (c \tau + d)^{\nu} \sum_{\mu} B_{\mu, \nu}(\gamma)Z_{\mu}(p, S, \tau)
\]

\[
\tilde{S} = \frac{S}{(c \tau + d)^2}
\]

\[
\tilde{p} = \frac{1}{(c \tau + d)^2} \left( p - 2\pi i \; c \; (c \tau + d)S^2 \right)
\]
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Coulomb Branch Integral

In principle defined for general class S theory.

\[ Z^C_B = \int_{\mathcal{B}} du \, d\bar{u} \, \mathcal{H} \, \Psi \]

\( \mathcal{H} \) is \textcolor{red}{holomorphic} and \textcolor{red}{metric-independent}

\( \Psi: \textbf{NOT holomorphic} \) and \textcolor{red}{metric- DEPENDENT}

``indefinite theta function``

Today: \( u \in \mathbb{C} \cong \mathcal{B} \)
Coulomb Branch Integral: Measure & Evaluation

5a. Seiberg-Witten Review

5b. Formulating The Measure And Integral

5c. Evaluation Using Mock This & That
\[ E_u \quad y^2 = \prod_{i=1}^{3} (x - \alpha_i) \quad \alpha_i = u e_i(\tau_{uv}) + m^2 e_i(\tau_{uv})^2 \]

\[ e_i(\tau_{uv}) \quad \text{half-periods of} \quad E_{\tau_{uv}} = \mathbb{C}/(\mathbb{Z} + \tau_{uv}\mathbb{Z}) \]

Discriminant \sim \eta^{24}(\tau_{uv}) \prod_{i=1}^{3} (u - m^2 e_i(\tau_{uv}))^2
\[ u_j = m^2 e_j(\tau_0) \]
Special Geometry

$H_1(E_u; \mathbb{Z})$: Fibers of a local system over $\mathcal{B}^*$

**Definition:** A ``duality frame'' is a local choice of $A, B$ --cycles

Periods of $\lambda$ define homomorphism $Z_u: H_1(E_u; \mathbb{Z}) \to \mathbb{C}$

$$a(u) := \oint_A \lambda \quad \quad a_D(u) := \oint_B \lambda$$

**Fact:** There is a locally holomorphic function $\mathcal{F}(\alpha)$

$$a_D = \frac{d\mathcal{F}}{d\alpha}$$
\[
\frac{da}{du} = \int_A \frac{dx}{y} \quad \frac{da_D}{du} = \int_B \frac{dx}{y} \quad \tau = \frac{da_D}{da} = \frac{d^2\mathcal{F}}{da^2}
\]

N.B.
\[
\tau(u, m, \tau_{uv}) \text{ should not be confused with } \tau_{uv}
\]

\[
\lim_{m \to 0} \tau(u, m, \tau_{uv}) = \tau_{uv} \quad \lim_{u \to \infty} \tau(u, m, \tau_{uv}) = \tau_{uv}
\]
Weak Coupling Prepotential

\( u \to \infty: \exists \text{ Canonical duality frame (``weak coupling'')} : \)

\[
F(a, m) = \frac{1}{2} \tau_{uv} a^2 + \\
+ m^2 \left( \log \left( \frac{2a}{m} \right) - \frac{3}{4} + \frac{3}{2} \log \left( \frac{m}{\Lambda} \right) \right) \\
+ a^2 \sum_{n=2}^{\infty} f_n(\tau_{uv}) \left( \frac{m}{a} \right)^{2n}
\]

\( f_n(\tau_{uv}): \text{ polynomials:} \)

\( E_2, E_4, E_6 \quad \text{wt} = 2n - 2 \)

[Minhahan, Nemeschansky, Warner; Dhoker, Phong]

Nekrasov: Instanton partition function \( \Rightarrow \)

\( \Lambda, m \text{ dependence (also A,B couplings):} \)

[Manschot, Moore, Xinyu Zhang 2019]
Remarkably: One can invert these equations and express periods as bimodular forms in $\tau, \tau_{uv}$

$$m^2 \left( \frac{da}{du} \right)^2 = \frac{\vartheta_4^4(\tau) \vartheta_3^4(\tau_{uv}) - \vartheta_3^4(\tau) \vartheta_4^4(\tau_{uv})}{\eta^6(\tau_{uv})}$$

$$m^{-2} u(\tau, \tau_{uv}) = \frac{e_1^2(\tau_{uv}) e_{23}(\tau) + cycl.}{e_1(\tau_{uv}) e_{23}(\tau) + cycl}$$

$$\mathcal{B} \cong \mathcal{H} / \Gamma(2) \cong \mathcal{F}(\Gamma(2))$$
\( \tau = i \infty \leftrightarrow u = u_1 \)

\( \tau = 0 \leftrightarrow u = u_2 \)

\( \tau = 1 \leftrightarrow u = u_3 \)
Coulomb Branch Integral: Measure & Evaluation

Seiberg-Witten Review

Formulating The Measure And Integral

Evaluation Using Mock This & That
Coulomb Branch Measure

\[ Z^\text{CB}_\nu = \int_{\mathcal{F}(\Gamma(2))} \Omega \]

\[ \Omega = d\tau \wedge d\bar{\tau} \ \mathcal{H} \ \Psi^J_\nu \]

Begin with Maxwell partition function \( \Psi^J_\nu \)

\[ \Psi \sim \sum_{\text{fluxes}} e^{-S_{\text{classical}}} \]

Frame dependent.

Not holomorphic.

Metric dependent.
The "Period Point" $J$

\[ b_2^+ > 1 \Rightarrow Z_{v}^{CB} = 0 \]

\[ b_2^+ = 1 \quad Z_{v}^{CB} \neq 0 \]

$H^2(X; \mathbb{R})$

\[ *J = J \]

\[ J^2 = 1 \]

$J \in$ Forward Light Cone
Maxwell Partition Function

\[ \Psi_{\nu} \sim \sum_{\text{fluxes}} e^{-\int \bar{\tau}(u)f_+^2 + \tau(u)f_-^2} \]

Sum over the first Chern class
\( \lambda \in 2L + \bar{\nu} \), \( L = H^2(X; \mathbb{Z}) \)

\[ \Psi_{\nu}^J = \sum_{\lambda \in 2L+\bar{\nu}} \partial_{\bar{\tau}} E^J_{\lambda} q^{-\frac{1}{4}\lambda^2} e^{\pi i \lambda \cdot z} \]

\[ z = c_{uv} \nu(\tau, \tau_{uv}) + S \frac{du}{da} \]
Maxwell Partition Function

\[ \Psi_{\nu}^J = \sum_{\lambda \in 2L+\nu} \partial_{\bar{\tau}} E_{\lambda}^J \ q^{-\frac{1}{4}\lambda^2} \ e^{\pi i \lambda \cdot z} \]

\[ z = c_{uv} \ \nu(\tau, \tau_{uv}) + S \ \frac{du}{da} \]

\[ E_{\lambda}^J = Erf(x_{\lambda}) \quad Erf(x) := \int_0^x e^{-\pi t^2} \ dt \]

\[ x_{\lambda} = \sqrt{Im \tau}(\lambda + \frac{Im \ z}{Im \ \tau}) \cdot J \]
Maxwell Coupling To $\mathcal{S}_{\nu\nu}$

\[ \sim \exp\left( \int_X \nu \ F^+_b f^+ + \bar{\nu} F^-_b f^- \right) \]

\[ \nu := \frac{d^2 \mathcal{F}}{da \, dm} = \frac{(a_D - a\tau)}{m} \]

Determines bimodular $\nu(\tau, \tau_{\nu\nu})$

\[ \frac{\vartheta_2(\nu, 2\tau)}{\vartheta_3(\nu, 2\tau)} = \frac{\vartheta_2(0, 2\tau_{\nu\nu})}{\vartheta_3(0, 2\tau_{\nu\nu})} \]
Holomorphic Part Of Measure

\[ \mathcal{H}_{\text{bare}} = A_1^\sigma A_2^\chi A_3^{c_{uv}^2} \]

Include observables:

\[ \mathcal{H} = \mathcal{H}_{\text{bare}} A_4^p A_5^{c_{uv}\cdot S} A_6^{S^2} \]

Depend on duality frame – but the local system has nontrivial monodromy.
Local Topological Interactions

\[ A_1 = \prod_i (u - u_i)^{\frac{1}{8}} = \] 
\[ (2m)^6 \frac{\eta(\tau_{uv})^{24} \eta(\tau)^{12}}{(\vartheta_4(\tau)^4 \vartheta_3(\tau_{uv})^4 - \vartheta_3(\tau)^4 \vartheta_4(\tau_{uv})^4)^3} \]

\[ A_2 = \left( \frac{da}{du} \right)^{-\frac{1}{2}} \]

\[ A_3 := \exp \left( -2 \pi i \frac{d^2 F}{dm^2} \right) = \left( \frac{\Lambda}{m} \right)^{\frac{3}{2}} \frac{\vartheta_1(2\tau, 2\nu)}{\vartheta_2^2(\tau_{uv})\vartheta_4(2\tau)} \]
With all these ingredients we can now check that the CB measure is indeed monodromy invariant and hence well-defined. (Nontrivial!)

What about defining the integral of the measure?
Do the phase integral first.
(as in string theory)
Coulomb Branch Integral: Measure & Evaluation

Seiberg-Witten Review

Formulating The Measure And Integral

Evaluation Using Mock This & That
$Z_{v}^{CB}$ : A sum of integrals of the form

$$I_f = \int_{F_{\infty}} d\tau d\bar{\tau} \ (Im \ \tau)^{-s} \ f (\tau, \bar{\tau})$$

Support of $c$ is bounded below

$$f(\tau, \bar{\tau}) = \sum_{m-n \in \mathbb{Z}} c(m, n) q^m \bar{q}^n$$

Strategy: Find $\hat{h}(\tau, \bar{\tau})$ such that

$$\partial_{\tau} \hat{h} = (Im \ \tau)^{-s} \ f (\tau, \bar{\tau})$$

$\hat{h} (\tau, \bar{\tau})$ is modular of weight (2,0)
We choose an explicit solution

\[ \partial_{\bar{\tau}} R = (Im\tau)^{-s} f(\tau, \bar{\tau}) \]

vanishing exponentially fast at \( Im\tau \to \infty \)

\( R \) is not modular, but it’s failure to be modular must be holomorphic.

\[ \hat{h}(\tau, \bar{\tau}) = h(\tau) + R \]

\( h(\tau) \) : mock modular form

\[ h(\tau) = \sum_{m \in \mathbb{Z}} d(m)q^m \quad q = e^{2\pi i \tau} \]
Doing The Integral

\[ I_f = \int_{\mathcal{F}_\infty} d\tau d\bar{\tau} y^{-s} f(\tau, \bar{\tau}) \]

\[ \partial_{\bar{\tau}} \hat{h} = y^{-s} f(\tau, \bar{\tau}) \]

\[ I_f = d(0) \]

\[ h(\tau) = \sum_{m \in \mathbb{Z}} d(m) q^m \]

Note: \( d(0) \) undetermined by diffeq but fixed by the modular properties: Subtle!
Evaluation Of CB Integral

\[ Z_{v}^{CB} = \int_{\mathcal{F}(\Gamma(2))} \Omega \]
\[ \Omega = d\tau \wedge d\bar{\tau} \quad \mathcal{H} \quad \Psi_{v}^{J} \]

\[ \Psi_{v}^{J} = \sum_{\lambda \in 2L+v} \partial_{\bar{\tau}} E_{\lambda}^{J} \quad q^{\frac{1}{4}\lambda^2} \quad e^{-2\pi i \lambda \cdot z} \]

\[ z = c_{uv} \nu(\tau, \tau_{uv}) + S \frac{du}{da} \]

\[ \Omega = d \Lambda \quad \Lambda = d\tau \mathcal{H} \quad \hat{G} \quad \Psi_{v}^{J} = \partial_{\bar{\tau}} \hat{G} \]
Evaluation Of CB Integral?

\[ \Psi^J_\nu = \sum_{\lambda \in 2L+\nu} \partial_{\bar{\tau}} E^J_\lambda q^{\frac{1}{4}\lambda^2} e^{-2\pi i \lambda \cdot z} \]

\[ \Psi^J_\nu = \partial_{\bar{\tau}} \hat{G} \]

\[ \hat{G} = \sum_{\lambda \in 2L+\nu} E^J_\lambda q^{\frac{1}{4}\lambda^2} e^{-2\pi i \lambda \cdot z} \]

?? NO!!!

\[ \lim_{\lambda^2 \to +\infty} E^J_\lambda = \pm 1 \]
Evaluating Difference Of CB Integrals

\[ \Psi^J_1 - \Psi^J_2 = \partial_{\tau} \hat{G}^J_{1,2} \]

\[ \hat{G}^J_{1,2} = \sum_{\lambda \in 2L + \nu} E^J_{1,2} \ e^{-\frac{1}{4} \lambda^2} e^{-2\pi i \lambda \cdot z} \]

\[ E^J_{1,2} = \text{Erf}(x^J_1) - \text{Erf}(x^J_2) \]

Converges nicely!

\[ \Rightarrow \text{Can use this to evaluate the difference } Z^{CB,J_1}_v - Z^{CB,J_2}_v \text{ by a sum of residues.} \]
Metric Dependence

Discontinuous jumps across walls:
Involves modular functions

For the boundary at $u \to \infty$ the modular parameter $	au \to \tau_{uv}$. This leads to continuous metric dependence.

Closely related: Nonholomomorphic in $\tau_{uv}$

($\mathbb{CP}^2$ is a degenerate case.)
The Coulomb Branch Integral As Harmonic Maass Form

\[ Z^C_B(\tau_{uv}) = \int_{F(\Gamma(2))} \Omega(\tau, \tau_{uv}) \]

\[ Z^C_B \] transforms under \( SL(2, \mathbb{Z}) \) as above

\[ \frac{\partial}{\partial \bar{\tau}_{uv}} Z^C_B = y^{-\frac{3}{2}} \eta^{-2} \chi \sum_{\lambda} K[\lambda_+, \lambda_-] \bar{q}^{\lambda^2_+} q^{-\lambda^2_-} \]
The Special Period Point

For any manifold with $b_2^+ = 1$

$\exists$ special $J_0$ such that $\Psi^J_0$ factorizes:

$$\Psi^J_0 = f_\nu \Theta_{L-}(\tau, z)$$

$$f_\nu = \sum_{\lambda \in 2\mathbb{Z}-\nu} \partial_\tau E^J_\lambda \ q^{-\frac{1}{4} \lambda^2} \ e^{-2\pi i \lambda \cdot z}$$
Measure As A Total Derivative

\[ \Omega = d \Lambda \quad \Lambda = d\tau \mathcal{H} \quad \hat{G} \]

Where we can write \( \hat{G} \) explicitly so that \( \Lambda \) is:

1. Well-defined
2. Nonsingular away from \( \tau \in \{ 0, 1, i \infty, \tau_{uv} \} \)
3. Good \( q_i \) expansion near cusps
Harmonic Jacobi-Maass Forms

These conditions determine $\hat{G}$ uniquely.

Modular completion of an Appel-Lerche sum

$$F(\tau, z) \sim \frac{e^{-2\pi i z}}{\vartheta_4(2\tau)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{n^2 - \frac{1}{4}}}{1 + e^{4\pi i z q^{2n-1}}}$$

$$z = c_{uv} \nu(\tau, \tau_{uv}) + S \frac{du}{da}(\tau, \tau_{uv})$$
The Integral Is a Mock Modular Form

For $\varsigma = \varsigma(J)$ we find:

$$Z^C_B = \hat{g}_v(\tau_{uv}, \bar{\tau}_{uv}) \Theta_{L-}(\tau_{uv})/\eta^{2\chi}(\tau_{uv})$$

$$g_v = 3 \sum_{n \geq 0} H(4n - 2\mu)q_{uv}^{n-\nu/2}$$
... but other $\mathcal{S}$ generalize ...

For $\mathbb{CP}^2$ & $c_{uv} = 1$ (acs $\Rightarrow c_{uv} = 3$)

$$\frac{\partial}{\partial \bar{\tau}_{uv}} Z_v = y_{uv}^{-\frac{3}{2}} \eta^{-2} \tilde{E}_2 \Theta_v(-\bar{\tau}_{uv})$$
Including Observables

<table>
<thead>
<tr>
<th>n</th>
<th>Hol. part of $\eta(\tau_{uv})^6 \Phi_{1/2}^{P^2} [u_D^n]$</th>
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<tr>
<td>0</td>
<td>$\frac{3}{4} q_{uv} + 3 q_{uv}^{7/4} + 3 q_{uv}^{11/4} + 6 q_{uv}^{15/4} + \ldots$</td>
</tr>
<tr>
<td>1</td>
<td>$-m^2 \left( \frac{3}{2} q_{uv}^{7/4} + 12 q_{uv}^{11/4} + 35 q_{uv}^{15/4} + \ldots \right)$</td>
</tr>
<tr>
<td>2</td>
<td>$m^4 \left( \frac{19}{16} q_{uv}^{7/4} + \frac{31}{2} q_{uv}^{11/4} + 89 q_{uv}^{15/4} + \ldots \right)$</td>
</tr>
<tr>
<td>3</td>
<td>$-m^6 \left( \frac{15}{4} q_{uv}^{11/4} + \frac{971}{16} q_{uv}^{15/4} + \ldots \right)$</td>
</tr>
<tr>
<td>4</td>
<td>$m^8 \left( \frac{85}{32} q_{uv}^{11} + \frac{15151}{256} q_{uv}^{15/4} + \ldots \right)$</td>
</tr>
</tbody>
</table>
Contributions Of The Cusps $u_j$

Physics $\Rightarrow$ Near each cusp $u_j, j = 1,2,3$
the description of the vacuum changes:
We have a U(1) VM coupled to a charge 1 HM.
(In the appropriate duality frame) [Seiberg-Witten 94]

There is a separate contribution to the path integral
coming from the path integral of these three LEET.

We add the contributions, because we sum over vacua:

$$Z_v = Z_v^{CB} + \sum_{j=1}^{3} Z_v^{SW}$$
When $b_2^+ > 1$ $Z^{CB}_v$ vanishes – we get true topological invariants:

$$Z_v = \sum_{j=1}^{3} Z_{v,j}^{SW}$$

So it is quite interesting to determine The three effective actions
\[ u_j = m^2 e_j(\tau_0) \]
General Form Of Effective Action Near $u_j$

$a$: Local special coordinate vanishing at $u_j$

$$S_j^{SW} = \int \sum_n \kappa_n(a; \tau_{uv}; t) \delta_n + Q(*)$$

$\delta_n$: Possible *local* topological couplings

$$e^{-S_j^{SW}} \bigg|_{\text{localize}} = \prod_n F_{n,j}(\tau_{uv}, t)^{\Delta_n}$$
Possible Topological Couplings $\Delta_n$

\[
X \Rightarrow \chi \quad \sigma
\]

\[
c_{uv}, \nu \Rightarrow c_{uv}^2 \quad c_{uv} \cdot \nu \quad \nu^2
\]

\[
c_{ir} \Rightarrow c_{ir}^2 \quad c_{ir} \cdot c_{uv} \quad c_{ir} \cdot \nu
\]

\[
S \Rightarrow S^2 \quad S \cdot c_{ir} \quad S \cdot c_{uv}
\]
MW97: The couplings $\kappa_n$ at $u_j$ can be determined from the wall-crossing behavior of $Z_v^{CB}$ from $u_j$. Explicit formulae!

$$Z_v^{SW} = \sum_{c_{ir}} SW(c_{ir}) \prod_{n=1}^{12} F_{n,j}(\tau_{uv}; t)^{\Delta_n}$$
Comparison: Witten Conjecture

\[ Z^K_{\nu}^{MW} (p, S) = 2^{1+\kappa-\chi_h} e^{-\frac{i\pi}{2} \nu \cdot c_{uv}} [Z_{\nu,2}(p, S) + Z_{\nu,3}(p, S)] \]

\[ \chi_h := \frac{\chi + \sigma}{4} \quad \kappa = 2\chi + 3\sigma \]

\[ Z_{\nu,2}(p, S) = e^{\frac{1}{2}S^2 + p} \sum_{c_{ir}} SW(c_{ir}) e^{c_{ir} \cdot S} e^{\frac{i\pi}{2} \nu \cdot c_{ir}} \]

\[ Z_{\nu,3}(p, S) = e^{-\frac{1}{2}S^2 - p} \sum_{c_{ir}} SW(c_{ir}) e^{-ic_{ir} \cdot S} e^{\frac{i\pi}{2} \nu \cdot c_{ir}} \]
\[ Z_v = \sum_{j=1}^{3} Z_{v,j} \]

\[ Z_{v,j}^{SW} = \sum_{c_{ir}} SW(c_{ir}) \prod_{n=1}^{12} F_{n,j}(\tau_{uv}; t)^{\Delta n} \]
$$Z_{v,2}^{SW} = F_1^\ell F_2^h F_3^\kappa \sum_{c_{ir}} SW(c_{ir}) F_4 \left(\frac{c_{ir} + c_{uv}}{2}\right)^2$$

$$F_1 = t^3 \left(\eta^4(\tau_{uv}) \vartheta_3(\tau_{uv}/2)\right)^{-1}$$

$$F_2 = \left(2 \eta^{12}(\tau_{uv}/2)\right)^{-1}$$

$$F_4 = \vartheta_3(\tau_{uv}/2)/\vartheta_4(\tau_{uv}/2)$$
\[
F_5^p \quad F_6^S \quad F_7^{S \cdot c_{uv}} \quad F_8^{S \cdot c_{ir}}
\]

\[
F_5^p = \exp \left( -\frac{t^2}{12} (\vartheta_2^4 + \vartheta_3^4) p \right)
\]

\[
F_8^{S \cdot c_{ir}} = \exp \left( -\frac{it}{4} (\vartheta_2 \vartheta_3)^2 S \cdot c_{ir} \right)
\]
There are similar expressions for the other two cusps.

\[ Z_{SW,1,\mu}(\tau_{uv}) = (-2 \eta(2\tau_{uv})^{12})^{-\chi_h} \left( \frac{(\Lambda/m)^3}{4 \eta(\tau_{uv})^4 \vartheta_3(2\tau_{uv})^4} \right)^\ell \left( \frac{\eta(\tau_{uv})^2}{\vartheta_3(2\tau_{uv})} \right)^\lambda \]

\[ \times \sum_{x=2\mu \mod 2L} \text{SW}(c_{ir}) \left( \frac{\vartheta_3(2\tau_{uv})}{\vartheta_2(2\tau_{uv})} \right)^x \]

\[ Z_{SW,3,\mu}(\tau_{uv}) = 2 e^{2\pi i \mu^2} \left( \frac{-(\Lambda/m)^3}{\eta(\tau_{uv})^4 \vartheta_3((\tau_{uv} + 1)/2)^4} \right)^\ell \]

\[ \times (2 \eta((\tau_{uv} + 1)/2)^{12})^{-\chi_h} \left( \frac{2 \eta(\tau_{uv})^2}{\vartheta_3((\tau_{uv} + 1)/2)} \right)^\lambda \]

\[ \times \sum_{x \in L} \text{SW}(c_{ir}) (-1)^{2B(x,\mu)} \left( \frac{\vartheta_3((\tau_{uv} + 1)/2)}{\vartheta_4((\tau_{uv} + 1)/2)} \right)^x \].
Relation To Previous Results

For $\varsigma(I)\text{ and } m \to 0$ we recover and generalize formulae of [VW;DPS] for VW invariants.

For $c_{uv} = 0$ we recover formulae of Labastida-Lozano

For $m \to \infty, q_{uv} \to 0$ after suitable renormalization we recover the ``Witten conjecture” for the Donaldson invariants in terms of the Seiberg-Witten invariants.

Recover and generalize explicit evaluation of u-plane integral for $\mathbb{CP}^2, S^2 \times S^2$ of Moore-Witten, Malmendier-Ono

A generalization and unification of the 1990’s formulae:
VIRTUAL REFINEMENTS OF THE VAFA-WITTEN FORMULA

LOTHAR GÖTTSCHE AND MARTIJN KOOL

with an appendix by Lothar Göttsche and Hiraku Nakajima

VERLINDE FORMULAE ON COMPLEX SURFACES I: K-THEORETIC INVARIANTS

L. GÖTTSCHE, M. KOOL, AND R. A. WILLIAMS

REFINED SU(3) VAFA-WITTEN INVARIANTS AND MODULARITY

LOTHAR GÖTTSCHE AND MARTIJN KOOL

VIRTUAL SEGRE AND VERLINDE NUMBERS OF PROJECTIVE SURFACES

L. GÖTTSCHE AND M. KOOL

SHEAVES ON SURFACES AND VIRTUAL INVARIANTS

L. GÖTTSCHE AND M. KOOL
<table>
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<tr>
<th>Concept</th>
<th>This paper</th>
<th>GKNW</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometry</td>
<td>Smooth, compact four-manifold $X$ with $b_1 = 0$ of SW simple type</td>
<td>Projective complex surface $S$ with $b_2^+ &gt; 1$, $b_1 = 0$ of SW simple type</td>
</tr>
<tr>
<td>Mass/Scale</td>
<td>$m/\Lambda = t$</td>
<td>$t$</td>
</tr>
<tr>
<td>Modular parameter</td>
<td>$q_{uv}$</td>
<td>$x^4$</td>
</tr>
<tr>
<td>UV Spin-c structure</td>
<td>$c_{uv} \in \bar{w}_2(X) + H_2(X, 2\mathbb{Z})$</td>
<td>Canonical class $K_S$</td>
</tr>
<tr>
<td>IR Spin-c structure</td>
<td>$c_{ir} \in \bar{w}_2(X) + H_2(X, 2\mathbb{Z})$</td>
<td>SW basic class $K_S - 2a_i$</td>
</tr>
<tr>
<td>'t Hooft flux</td>
<td>$2\mu \in H^2(X, \mathbb{Z})$</td>
<td>first Chern class $c_1$</td>
</tr>
<tr>
<td>0-observable</td>
<td>$p$</td>
<td>$-u$</td>
</tr>
<tr>
<td>2-observable</td>
<td>$S$</td>
<td>$i\alpha z$</td>
</tr>
</tbody>
</table>

**Table 9:** Dictionary between some of the concepts in this paper and in [13, 14, 85]
\[ U(1)_b \text{ Localization} \]

\[ F^+ + [M, \bar{M}] = 0 \quad DM = 0 \]

Fixed point set for \( M \to e^{i\theta} M \) has TWO branches

Branch 1: \( M_{asd} \): \( M = 0 \) \& \( F^+ = 0 \)

Branch 2: \( M_{ab} \): \( M \sim \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \)

\[ U(1)_b: \quad \int_{M_{Q,k}} \cdots \to \int_{M_{asd,k}} \cdots + \int_{M_{ab}} \cdots \]
\[ \int_{\mathcal{M}_{asd}} \cdots \sim Z_{\nu,2}^{SW} + Z_{\nu,3}^{SW} \]

\[ \int_{\mathcal{M}_{ab}} \cdots \sim Z_{\nu,1}^{SW} \]
Concluding Remarks

Twisted $N = 2^*$ on four-manifolds with a spin-c structure unifies and generalizes previous expressions for invariants of 4-manifolds derived from SYM.

Paper on the arXiv should appear ``soon."

Hamiltonian formulation (Floer theory)?

Derivation from 6d (2,0) theory?

Generalization of these techniques to class S $X$ complex: Compute Refined Versions From Physics
REMARKS ON CLASS S: SLIDES FROM MY STRING MATH 2018 TALK IN SENDAI, JAPAN
Class S: General Remarks

\[ \mathcal{H} = \alpha^\chi \beta^\sigma \det \left( \frac{d\alpha^i}{d\nu_j} \right)^{1-\frac{\chi}{2}} \frac{\sigma}{\Delta^8_{phys}} \]

\(\Delta_{phys}\) a holomorphic function on \(\mathcal{B}\) with first-order zeros at the loci of massless BPS hypers

\(\alpha, \beta\) will be automorphic forms on Teichmuller space of the UV curve \(C\)

\(\alpha, \beta\) are related to correlation functions for fields in the (0,2) QFT gotten from reducing 6d (0,2)
Class S: General Remarks

\[ \Psi \sim \sum_{\lambda} e^{i \pi \lambda \cdot \xi} e^{-i \pi \bar{\tau}(\lambda_+, \lambda_+)}-i \pi \tau(\lambda_- \cdot \lambda_-) + \cdots \]

\( \lambda \in \lambda_0 + \Gamma \otimes H^2(X; \mathbb{Z}) \)

\( \xi \in \Gamma \otimes H^2(X; \mathbb{R}) \)

\( \Gamma \subset H^1(\Sigma; \mathbb{Z}) \)

Lagrangian sublattice

If \( \xi = \rho \otimes w_2(X) \mod 2 \) then WC from interior of \( \mathcal{B} \) will be cancelled by SW invariants

\[ \Rightarrow \text{No new four-manifold invariants...} \]
\( \Psi \) comes from a ``partition function'' of a level 1 SD 3-form on \( M_6 = \Sigma \times X \)

Quantization: Choose a QRIF \( \Omega \) on \( H^3(M_6; \mathbb{Z}) \)

Natural choice: [Witten 96,99; Belov-Moore 2004]

\[
\Omega(x) = \exp \left( i \pi \ WCS(\theta \cup x; \ S^1 \times M_6) \right)
\]

Choice of weak-coupling duality frame + natural choice of \( spin^c \) structure gives

\[
\xi = \rho \otimes w_2(X)
\]
HOWEVER!
Need For U(1)-valued QRIF

\[ e^{i \pi \lambda \cdot \xi} \text{ is a 6d generalization of the famous Witten phase: } (-1)^{w_2(X) \cdot \lambda} \]

\[ e \int \bar{\nu} F_b^+ F_{dyn}^+ + \nu F_b^- F_{dyn}^- \rightarrow e^{i \pi \int w_2(X) \frac{F_{dyn}}{2\pi}} \]

So the \( \mathbb{Z}_2 \)-phase generalizes to a U(1)-valued phase.

Important implications for the generalization of CB integral to class S theories: We do not want a \( \mathbb{Z}_2 \) –valued QRIF.
\[ \mathcal{N} = 2^* SU(2) \]

\( SL(2) \) Hitchin system on \( E_{uv} = \mathbb{C}/(\mathbb{Z} + \tau_{uv}\mathbb{Z}) \)

Regular singularity at \( z = 0 \)

Monodromy \( \sim \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} \)

\( \lambda: \) Liouville form pulled back to \( \Sigma \subset T^* E_{uv} \)