

Quantum Field Theory And Invariants Of Smooth Four-Dimensional Manifolds

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Joint Mathematics Meeting, Jan. 18, 2020, Denver

1 Physical Mathematics

2 Math & Physics Problems

3 Instantons & Donaldson Invariants

4 Topological Field Theory

5 Low Energy Effective Field Theory

6 New Results On Z_u

7 Directions For Future Research



Phys-i-cal Math-e-ma-tics, n.

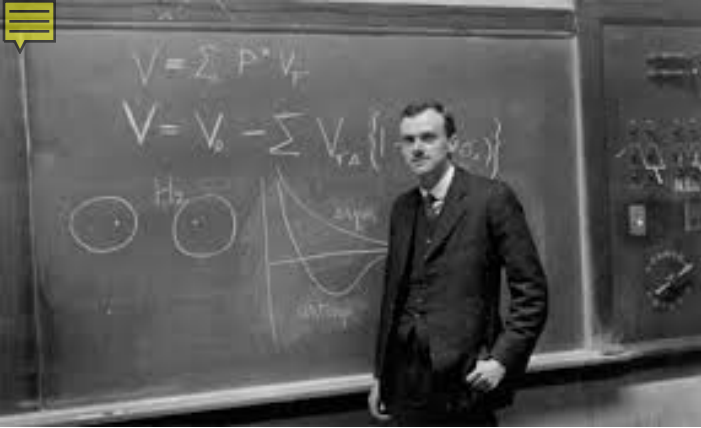
Pronunciation: Brit. /'fɪzɪkl ˌmæθ(ə)'mæɪtɪks / , U.S. /'fɪzək(ə)l ˌmæθ(ə)'mædɪks/

Physical mathematics is a fusion of mathematical and physical ideas, motivated by the dual, but equally central, goals of

1. Elucidating the laws of nature at their most fundamental level,

together with

2. Discovering deep mathematical truths.



1931: Dirac's Paper on Monopoles

Quantised Singularities in the Electromagnetic Field

P.A.M. Dirac

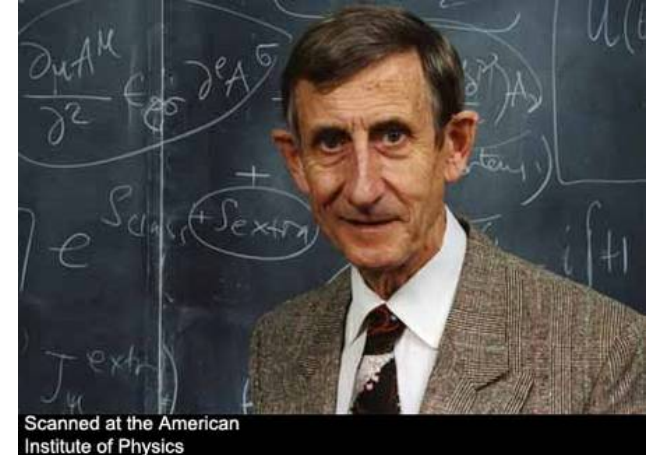
Received May 29, 1931

§ 1. *Introduction*

The steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced. This is only natural and to be expected. What, however, was not expected by the scientific workers

for the description of general facts of the physical world. It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalisation of the axioms at the base of the mathematics rather than with a

1972: Dyson's Announcement



MISSED OPPORTUNITIES¹

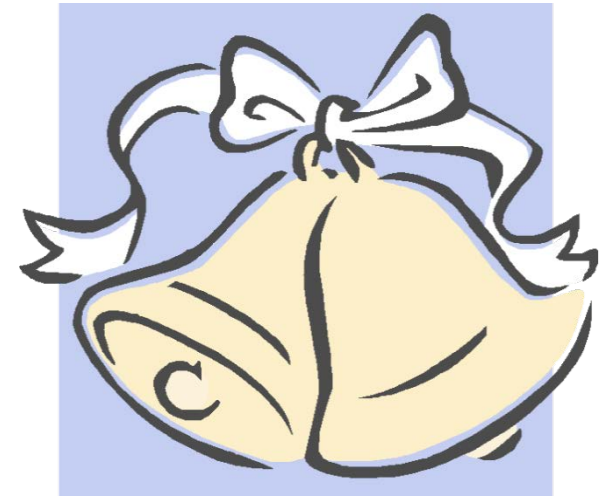
BY FREEMAN J. DYSON

It is important for him who wants to discover not to confine himself to one chapter of science, but to keep in touch with various others.

JACQUES HADAMARD

1. **Introduction.** The purpose of the Gibbs lectures is officially defined as “to enable the public and the academic community to become aware of the contribution that mathematics is making to present-day thinking and to modern civilization.” This puts me in a difficult position. I happen to be a physicist who started life as a mathematician. **As a working physicist, I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce.** Discussing this divorce, the

Well, I am happy to report that
Mathematics and Physics have
remarried!



Change began in the 1970's

Some great mathematicians
got interested in aspects of
fundamental physics

While some great physicists started
producing results requiring ever
increasing mathematical
sophistication,



Physical Mathematics

In the past few decades a new field has emerged with its own distinctive character, its own aims and values, its own standards of proof.

One of the guiding principles is certainly the discovery of the ultimate foundations of physics.

This quest has led to ever more sophisticated mathematics...

A second guiding principle is that physical insights can lead to surprising and new results in mathematics

Such insights are a great success - just as profound and notable as an experimental confirmation of a theoretical prediction.



Today:

I will explain just one emblematic example of a remarkable convergence of physical and mathematical ideas.

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Four Dimensional Differentiable Topology

X : Four-dimensional, compact, oriented, simply connected, smooth manifold without boundary.

We do not know anything even close to a complete topological invariant.

One thing we know for sure is that the world of such four-dimensional manifolds is really wild.

Will We Ever Classify Simply-Connected Smooth 4-manifolds?

Ronald J. Stern

ABSTRACT. These notes are adapted from two talks given at the 2004 Clay Institute Summer School on *Floer homology, gauge theory, and low dimensional topology* at the Alfred Rényi Institute. We will quickly review what we do and do not know about the existence and uniqueness of smooth and symplectic structures on closed, simply-connected 4-manifolds. We will then list the techniques used to date and capture the key features common to all these techniques. We finish with some approachable questions that further explore the relationship between these techniques and whose answers may assist in future advances towards a classification scheme.

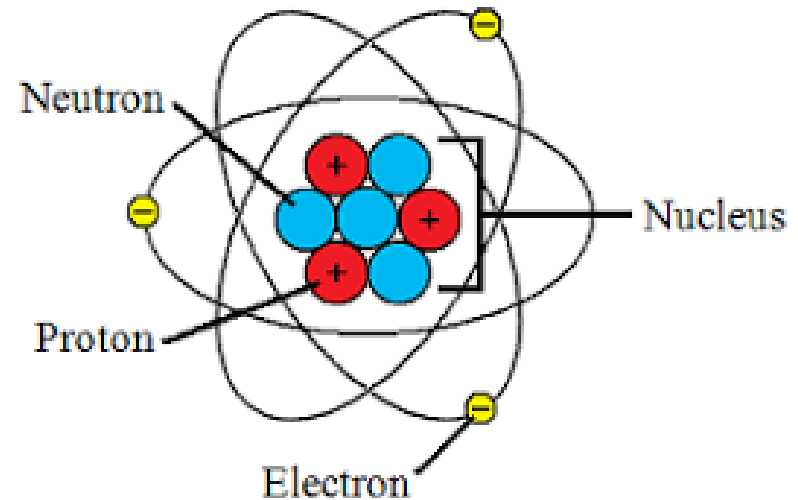
1. Introduction

Nuclear Force

Protons have positive electric charge

It's cozy in there:

$$r = 10^{-15} m.$$



Electrical force between two protons at this distance produces an acceleration ... $\sim 10^{28} g$

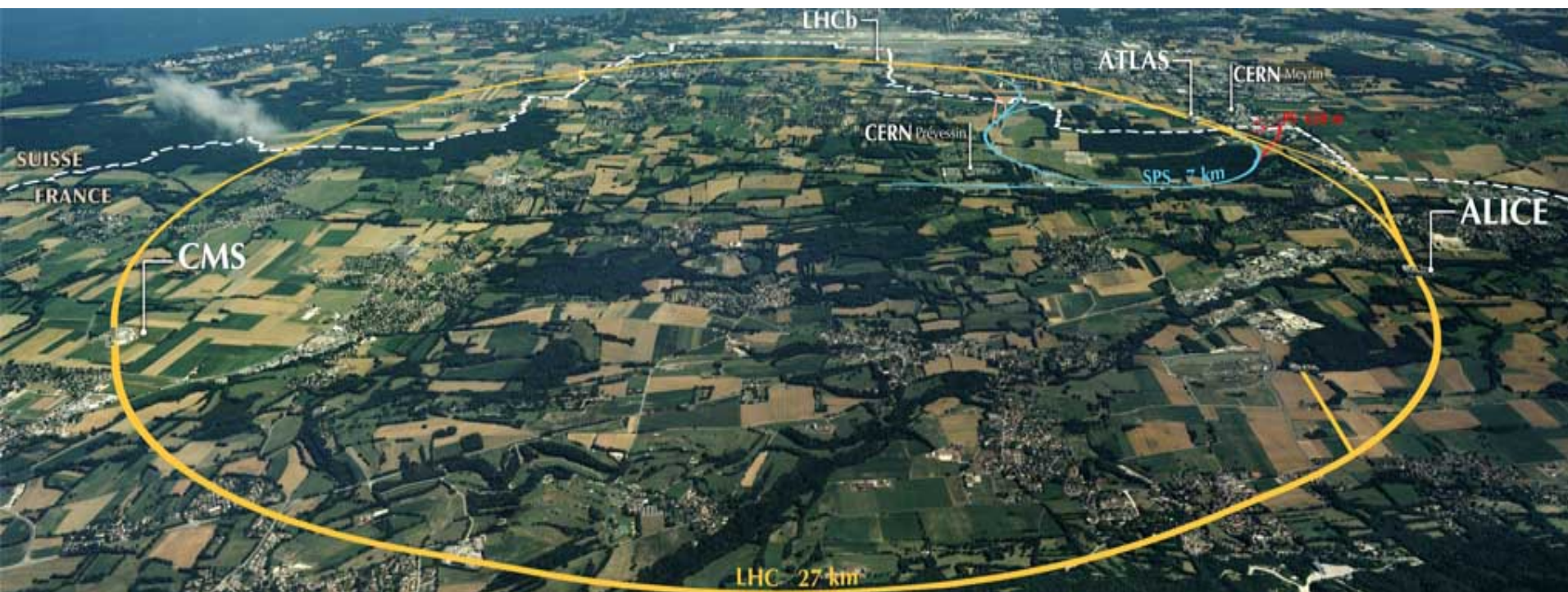
Fastest roller coaster $\sim 6 g$



Fighter pilots $\sim 9 g$



The strong force is very subtle – it has been studied with particle accelerators for decades - up to the present day...



Large Hadron Collider at CERN

Mathematical Formulation Of The Strong Force: Yang-Mills Theory

G : Compact finite dimensional Lie group

Lie algebra \mathfrak{g} equipped with invariant bilinear form tr

\mathcal{A} : Space of connections on principal G -bundles over X

\mathcal{A}/G : Connections up to isomorphism (gauge equivalence classes)

Given ∇_P : connection on $P \rightarrow X$ AND a Riemannian metric on X

Yang-Mills action $A_{YM} = \int_X tr(F \wedge * F)$

$*$: Hodge star: Depends on the metric $g_{\mu\nu}$ on X

A_{YM} descends to a function on \mathcal{A}/G

Formally, \exists natural translation invariant measure on \mathcal{A} pushes forward to a measure $d\mu$ on \mathcal{A}/\mathcal{G}

Physicists want: $d\mu e^{-A_{YM}}$ as a probability measure on \mathcal{A}/\mathcal{G}

Expectation values: “path integral”

Overwhelming evidence suggests it makes mathematical sense

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Most important connections:

The minima of A_{YM}

$$2 \int_X \text{tr}(F * F) = \int_X \text{tr}(F + * F)^2 - 2 \int_X \text{tr} F^2$$

With suitable definition of trace:

$$k := \int_X \text{tr} F^2 \in \mathbb{Z}$$

is a Chern-Weil
representation of a
characteristic class:

$$\mathcal{A} = \coprod_{k \in \mathbb{Z}} \mathcal{A}_k$$

Instantons -1.1

$$2 \int_X \text{tr}(F * F) = \int_X \text{tr}(F + * F)^2 - 2 \int_X \text{tr} F^2$$

Connections minimizing the action for $k > 0$
solve anti-self-dual YM equation:

$$F_+ := \frac{1}{2} (F + * F) = 0$$

Reminder On Self-Duality

$*^2 = 1$. For any 2-form ω define (anti-)self-dual projections:

$$\omega_{\pm} := \frac{1 \pm *}{2} \omega = \frac{1}{2} (\omega \pm * \omega)$$

$$H^2(X) = \{ \text{closed/exact 2-forms} \}$$

$H^2(X)$ splits into orthogonal sum of (anti-)self-dual forms dimensions b_2^{\pm}

$$b_2^+ + b_2^- = \dim H^2(X)$$



Instantons – 1.2

Solutions to $F_+ = 0$ are called instantons.

Ever since their discovery in 1975 by Belavin, Polyakov, Schwartz, and Tyupkin they have been intensively studied by physicists interested in Yang-Mills theory.

Meanwhile, back at the ranch



Starting in the late 1970's mathematicians such as Atiyah, Bott, Drinfeld, Hitchin, Manin, Singer, Uhlenbeck.... were producing remarkable mathematical results about these instantons.

There are continuous families of instanton solutions:

$$\mathcal{M} \subset \frac{\mathcal{A}}{\mathcal{G}} : \text{Moduli space of instantons} \quad \mathcal{M} = \coprod_{k \in \mathbb{Z}} \mathcal{M}_k$$

G : Simple Lie group:

$$\dim_{\mathbb{R}} \mathcal{M}_k = 4hk - \dim G \frac{(\chi + \sigma)}{2}$$

$$\dim_{\mathbb{R}} \mathcal{M}_k = 8k - \frac{3}{2}(\chi + \sigma) \quad G = SU(2), SO(3)$$

Donaldson Invariants

$S \subset X$: smooth surface.

$\mathcal{M}(S)$: subspace where the Dirac equation on S coupled to ∇_P has a solution

Poincare dual to $\mathcal{M}(S)$ defines $\mu(S) \in H^2(\mathcal{M}; \mathbb{Z})$

$$Z_D(S) := \int_{\mathcal{M}} e^{\mu(S)} = \sum_k \int_{\mathcal{M}_k} \frac{\mu(S)^r}{r!}$$

Formally a function on $H_2(X; \mathbb{C})$

Metric independent \Rightarrow Topological Invariant of X !

The discovery of these and related topological invariants led to major advances in the differential topology of four-manifolds







Donaldson invariants are extremely
hard to compute

It took a lot of effort to compute
a few special examples....

Mathematicians hit a wall...

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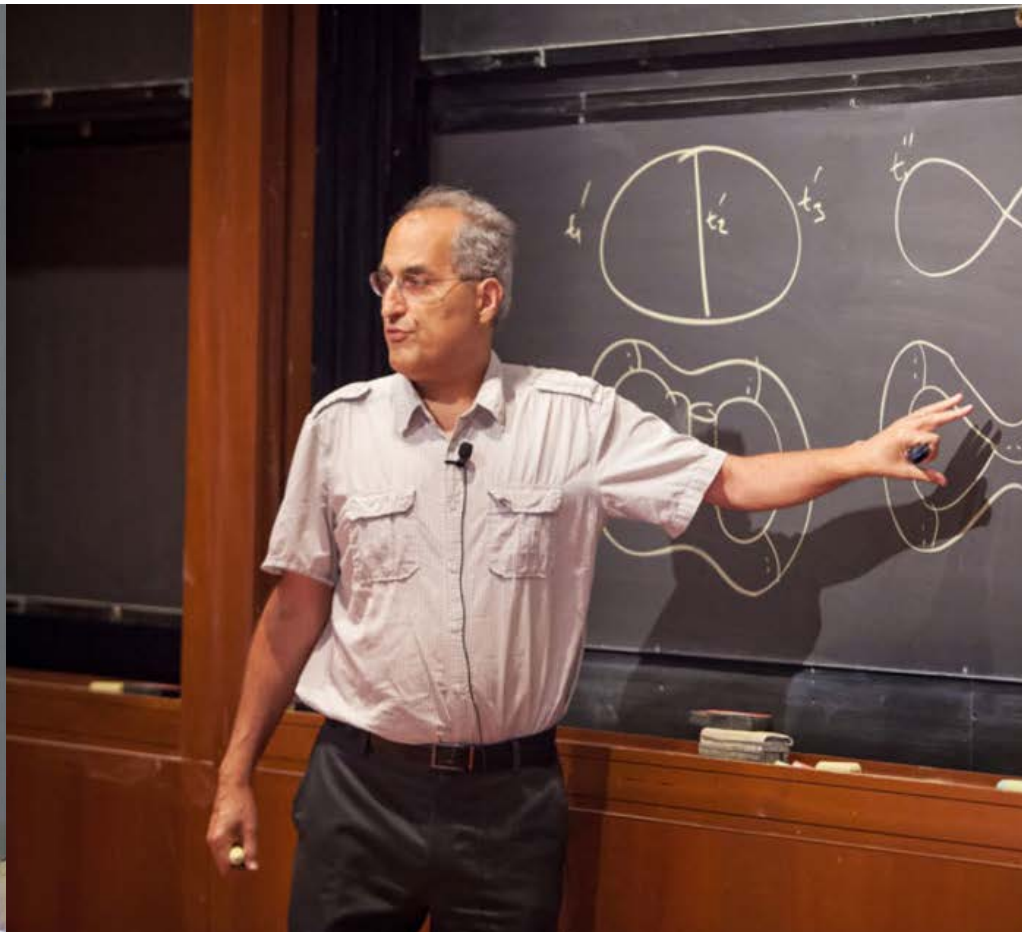
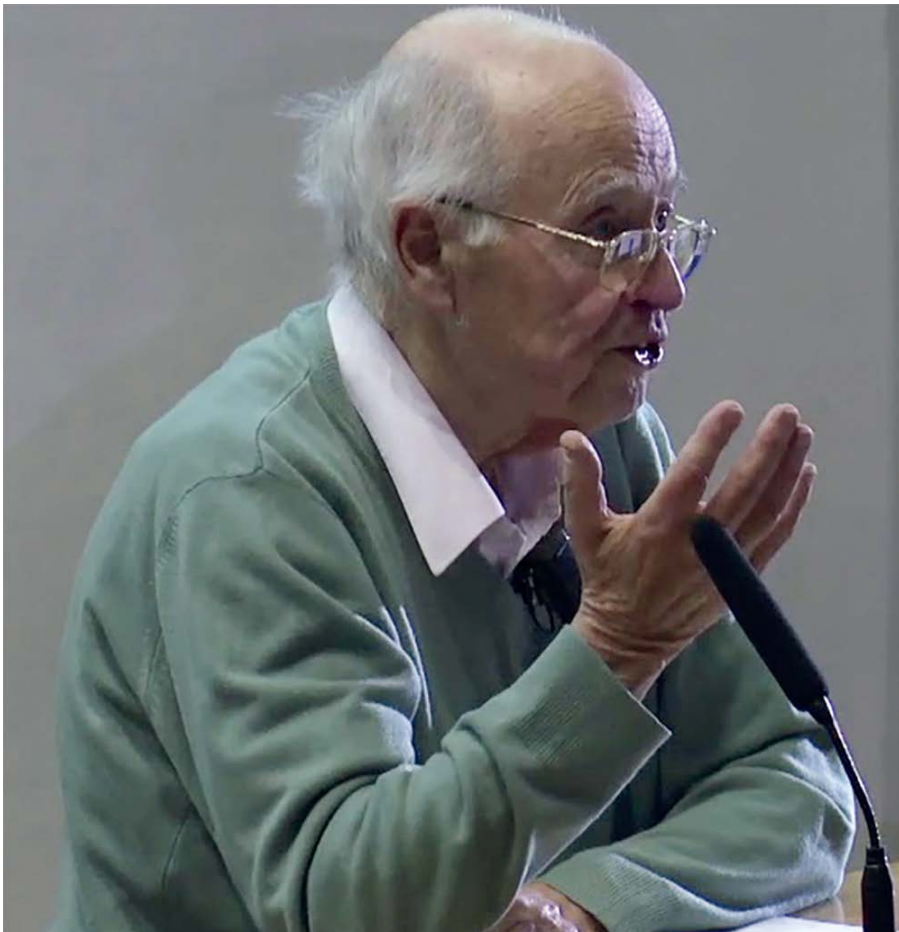
4 Topological Field Theory

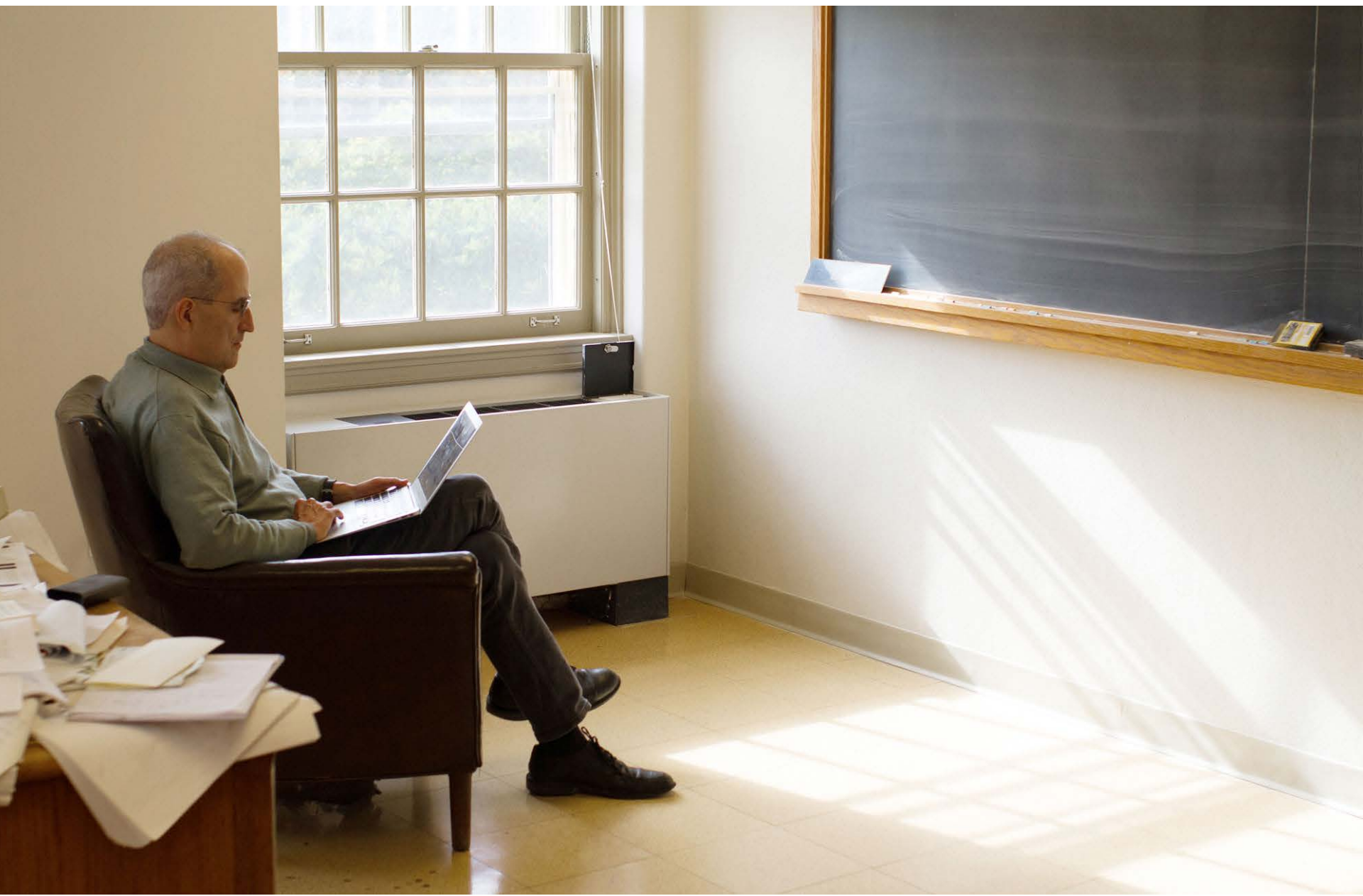
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A Turning Point: Atiyah's Question: What is *the physical interpretation* of the Donaldson invariants?





Witten's Answer

Donaldson invariants can be computed within the framework of a Yang-Mills field theory (with gauge group $SU(2)$)

Topological Quantum Field Theory

Edward Witten[★]

School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

Abstract. A twisted version of four dimensional supersymmetric gauge theory is formulated. The model, which refines a nonrelativistic treatment by Atiyah, appears to underlie many recent developments in topology of low dimensional manifolds; the Donaldson polynomial invariants of four manifolds and the Floer groups of three manifolds appear naturally. The model may also be interesting from a physical viewpoint; it is in a sense a generally covariant quantum field theory, albeit one in which general covariance is unbroken, there are no gravitons, and the only excitations are topological.

N=2 Super-Yang-Mills Theory

Fields: $\nabla_P \in \mathcal{A}$ $\phi \in \Gamma(adP \otimes \mathbb{C})$

Fermions/anticommuting: $\psi \in \Gamma(S^+ \otimes E)$

S^+ : Chiral spin bundle over X

$E \rightarrow X$: Rank 2 complex vector bundle with connection ∇_E and structure group $SU(2)_R$

N.B. If X is not spin we can still define ψ making a proper choice of E

Topological Twisting

In SYM we replace $\frac{\mathcal{A}}{\mathcal{G}}$ by an infinite-dimensional super-manifold \mathcal{F} fibered over \mathcal{A}/\mathcal{G}

There is still a formal probability measure

$$d\mu e^{-A_{SYM}} \text{ on } \mathcal{F} \text{ with } A_{SYM}:$$

$$\int_X \frac{1}{g^2} \text{tr} (F * F + D\phi * D\phi^* + [\phi, \phi^*]^2 + \bar{\psi} D_E \psi + \dots)$$

For a special choice of E, ∇_E the variation of the measure wrt metric $g_{\mu\nu}$ on X is a total derivative



Observables In Witten's Theory

$S \subset X$: smooth surface.

$$\mathcal{O}(S) := \int_S \text{tr}(\phi F + \psi^2)$$

$$Z_W(S) := \langle e^{\mathcal{O}(S)} \rangle$$

$Z_W(S)$ is a function on $H_2(X; \mathbb{C})$

With Top. Twist, i.e., special E, ∇_E ,
 $Z_W(S)$ independent of metric

Localization

M. Atiyah & L. Jeffrey: The measure can be interpreted as a representative of a Thom class for the normal bundle of $\mathcal{M} \subset \mathcal{A}/\mathcal{G}$

\Rightarrow Path integral over \mathcal{F} (assumed to exist) localizes to an integral over \mathcal{M}

L. Baulieu & I. Singer: Under localization $\mathcal{O}(S)$ pulls back to $\mu(S)$

Claim: $Z_D(S) = Z_W(S)$





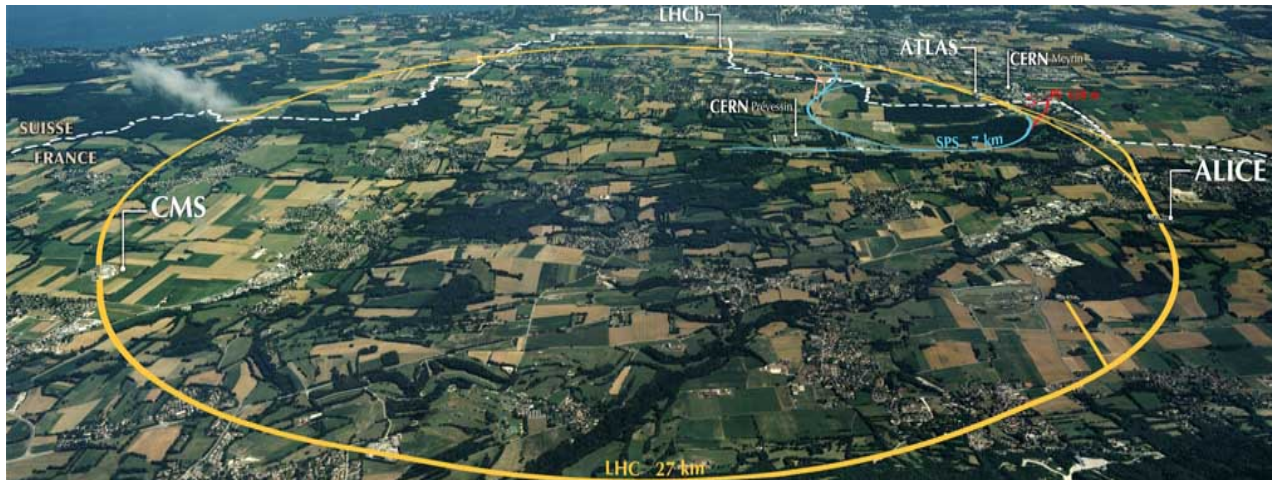
Quantum Field Theory

Computing Donaldson invariants
a la Witten requires computing
expectation values in a
Yang-Mills theory

How hard can that be?

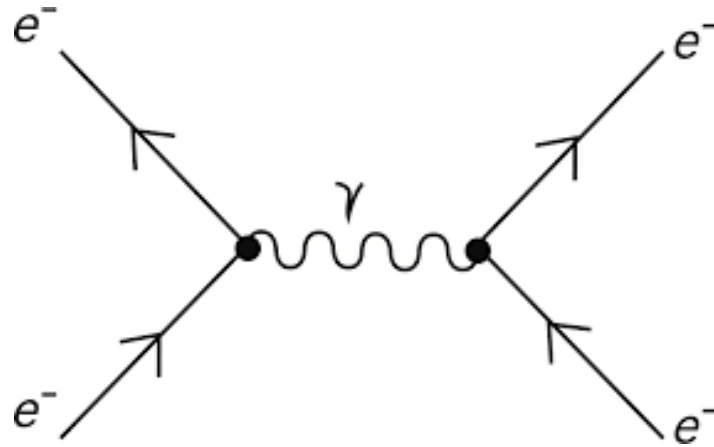


Abelian field theory



Nonabelian field theory

Abelian: Maxwell's theory: Hard, but solvable -
Feynman, Schwinger, Tomonaga (1946-1949)



Nonabelian: *MUCH*ⁿ harder:
Not solved yet

So, computing correlation functions of
operators in *nonabelian*
Yang-Mills theory is extremely difficult

So we seem to have exchanged
one hard problem for another.....

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Low Energy Effective Theory – 1.1

$Z_W(S)$ independent of $g_{\mu\nu}$ on X

Consider metric $L^2 g_{\mu\nu}$ in the limit $L \rightarrow \infty$

Quantum theory: Length $\sim 1/\text{Energy}$

Minima of energy (or action): “Vacua”

Small deviations from a vacuum can be described by a DIFFERENT QFT: The LEET

Low Energy Effective Theory – 1.2

We ask a question Q of a QFT, with action A ,

We answer using a *different* QFT

with action A_{LEET}

(perhaps with a totally new set of fields)

The answer to Q is the SAME as the answer to an analogous question Q_{LEET} .

Moduli Spaces Of Vacua

$$A_{SYM} = \int_{\mathbb{R}^4} \text{tr}(F * F + |D\phi|^2 + [\phi, \phi^*]^2)$$

Vacua: $\phi(x)$ is constant: $\phi(x) = \phi_{vac} \in \mathfrak{g}$

$[\phi, \phi^*] = 0 \Rightarrow \phi_{vac}$ is semisimple.

Any constant will do! So there is a moduli space of vacua:

$$SU(2): u := \langle \text{tr}(\phi^2) \rangle_u \sim \text{tr}(\phi_{vac}^2) \in \mathbb{C}$$

There is not one probability measure on \mathcal{F} but (for $G = SU(2)$) a continuous family labeled by "vacua" $u \in \mathbb{C}$

Spontaneous Symmetry Breaking

For $G = SU(2)$: gauge ϕ_{vac} to the form

$$\phi_{vac} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$

Automorphisms of the vacuum are
U(1) gauge transformations.

\Rightarrow The LEET of small field deviations around
a vacuum u is an ABELIAN GAUGE THEORY!



Seiberg-Witten Paper

Seiberg & Witten (1994)



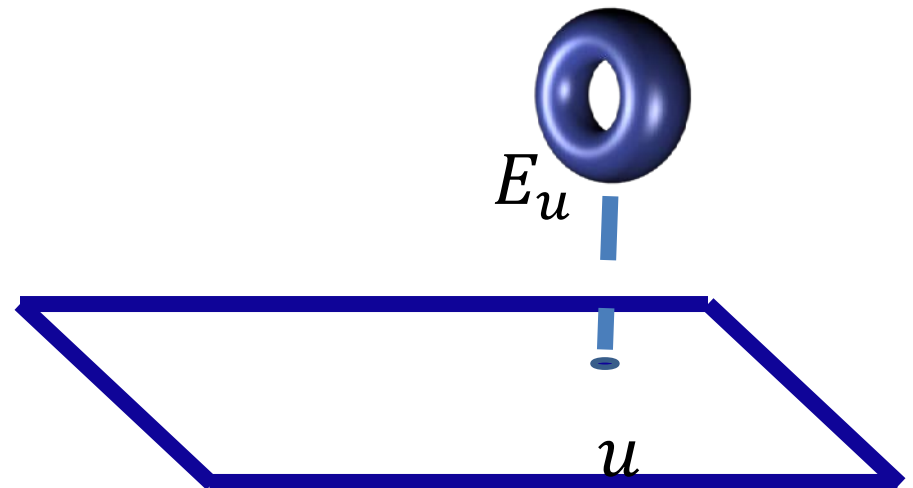
The LEET has action A_{LEET}^u that depends on u

Seiberg-Witten: A_{LEET}^u can be computed from the periods of a meromorphic 1-form λ_u on a family of Riemann surfaces Σ_u

$$G = SU(2):$$

$$E_u: y^2 = x^2(x - u) + \frac{1}{4}x$$

$$\lambda = \frac{dx}{y}(x - u)$$



Local System Of Charges

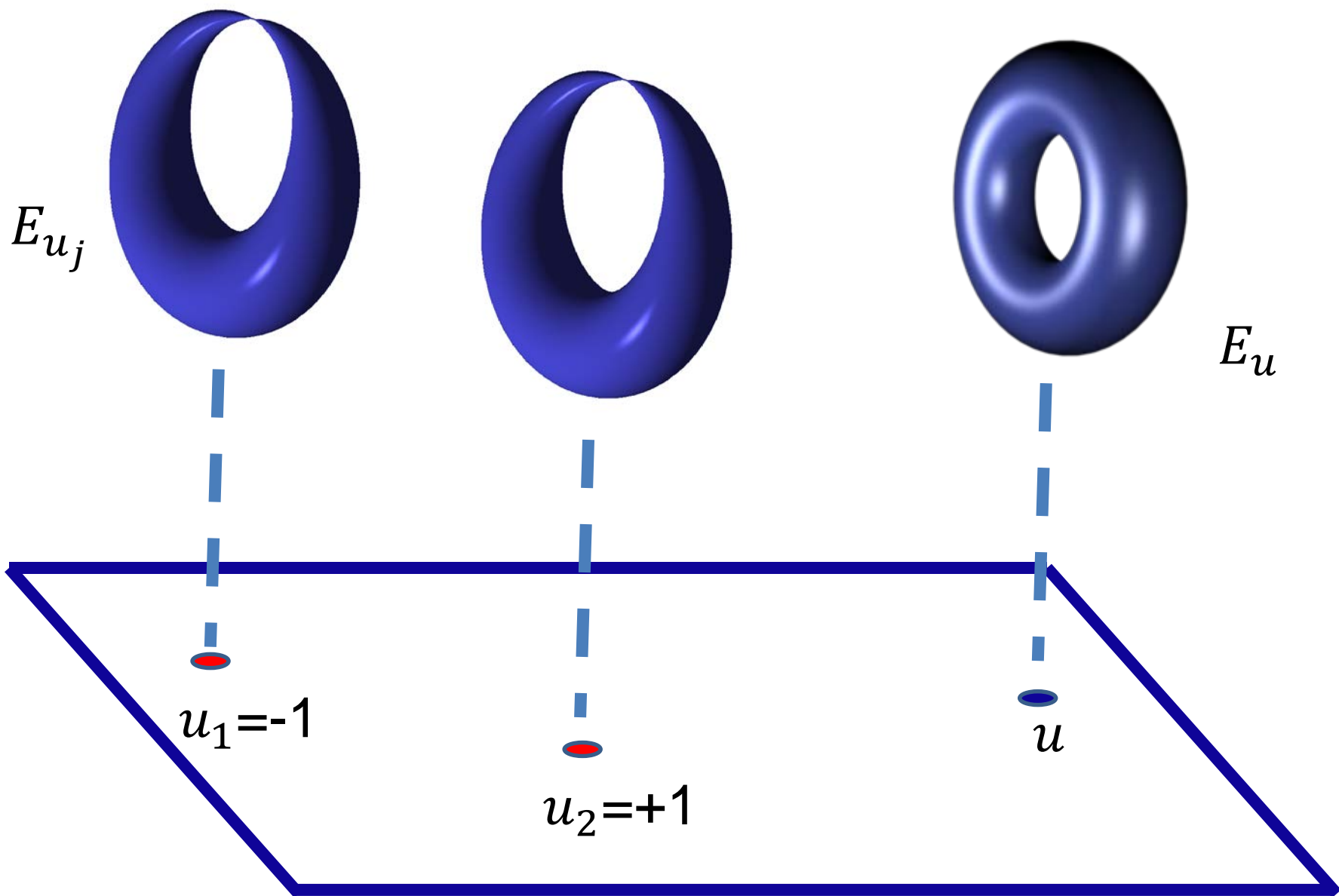
Electro-mag. charge lattice: $\Gamma_u = H_1(E_u; \mathbb{Z})$

Electromagnetic duality: To write a LEET
we must choose a splitting:

$$\Gamma_u \cong \mathbb{Z}\gamma_e \oplus \mathbb{Z}\gamma_m \Rightarrow \tau(u)$$

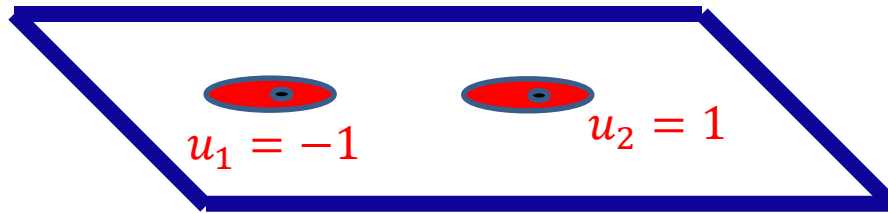
$$A_{LEET}^u \sim \int_X \overline{\tau(u)} F_+^2 + \tau(u) F_-^2 + \dots$$

Γ_u has nontrivial monodromy around
discriminant locus: $\Delta(u_j) = 0$



Seiberg-Witten Theory - II

LEET breaks down near discriminant locus:



BUT!

Near u_j we can use a different LEET – there are new fields (“monopole fields”) which need to be included.

It is also an Abelian gauge theory and

has an action $A_{LEET}^{u_j}$

SUMMING OVER VACUA

Quantum effects \Rightarrow the path integral for compact X will be given by a sum over ALL the vacua on \mathbb{R}^4 .

Two kinds of vacua

Continuous moduli space of vacua
parametrized by $u := \langle \text{tr} \phi^2 \rangle_u \in \mathbb{C}$

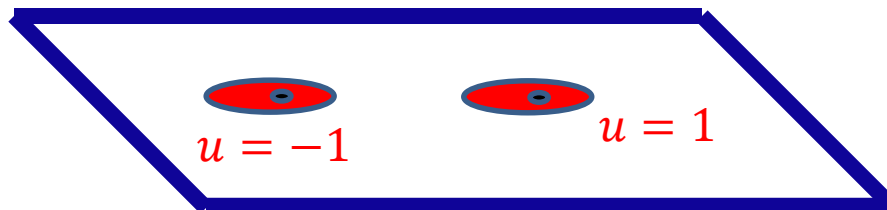
ALSO: The LEET $A_{LEET}^{u_j}$ near the discriminant loci
 $u_j \in \{ -1, +1 \}$ each has its own set of vacua.

Vacua of $A_{LEET}^{u_j}$ are enumerated by the solutions to
the renowned Seiberg-Witten equations

Evaluate $Z_{DW}(S)$ Using LEET

$$Z_D(S) = \int_{\mathcal{M}} e^{\mu(S)} = Z_W(S) = \langle e^{\theta(S)} \rangle$$

$$Z_W = Z_u + \sum_{u_j} Z_j^{SW}$$



Preliminary Comments On Z_u

Z_u an explicit integral over complex u –plane computed using QFT techniques

We will return to it and
discuss it in detail.

But first let's finish writing down the
full answer for the path integral.

Contributions From u_j

Path integral for the LEET with action $A_{LEET}^{u_j}$ takes the general form of a sum over vacua (soln's to SWE):

spinc structure:

$$w_2(TX) = \lambda \pmod{2}$$

$$\sum_{\lambda \in \mathfrak{Spinc}} SW(\lambda) R_\lambda(S)$$

$$R_\lambda(S) := \text{Res}_{u_j} \left[\left(\frac{du}{u} \right) e^{s^2 T(u) + i \frac{S \cdot \lambda}{\omega(u)}} C(u)^{\lambda^2} P(u)^\sigma E(u)^\chi \right]$$

$\omega(u)$: Nonvanishing period in the neighborhood of u_j

$$T(u) \sim \omega(u)^{-2} E_2(\tau(u))$$

Everything is already known **EXCEPT:**

$$C(\mathbf{u})^{\lambda^2} P(u)^{\sigma} E(u)^{\chi}$$

Functions C, P, E : independent of X
but difficult to derive from first principles.

Using a phenomenon known as
“wall crossing” one can derive C, P, E ,
directly from Z_u

$\Rightarrow Z_u$ is of fundamental importance.

Witten Conjecture

$Z_u = 0$ when $b_2^+ > 1 \Rightarrow$

If X has $b_2^+ > 1$ then:

$$Z_{DW}(S) = 2^{c^2 - \chi_h} \left(e^{\frac{1}{2}S^2} \sum_{\lambda} SW(\lambda) e^{S \cdot \lambda} + i\chi_h e^{-\frac{1}{2}S^2} \sum_{\lambda} SW(\lambda) e^{-iS \cdot \lambda} \right)$$

$$\chi_h = \frac{\chi + \sigma}{4}$$

$$c^2 = 2\chi + 3\sigma$$

Example: $X = K3$:

$$Z_{DW}(S) = \sinh\left(\frac{S^2}{2}\right)$$



The Donaldson invariants can be written in terms of the Seiberg-Witten invariants:
They carry the same information about the four-dimensional space



Much easier
to compute!





Mad Dash After A Breakthrough

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY

In the last three months of 1994 a remarkable thing happened: this research area was turned on its head by the introduction of a new kind of differential-geometric equation by Seiberg and Witten: in the space of a few weeks long-standing problems were solved, new and unexpected results were found, along with simpler new proofs of existing ones, and new vistas for research opened up.

opments, which are due to various mathematicians, notably Kronheimer, Mrowka, Morgan, Stern and Taubes, building on the seminal work of Seiberg [S] and Seiberg and Witten [SW]. It is written as an attempt to take stock of the progress stemming



Comments On The Witten Conjecture

Agrees with structure theorem of P. Kronheimer and T. Mrowka – whose work provided important background for Witten's conjecture.

Physical derivation sketched above was given by G. Moore and E. Witten in 1997

P. Feehan and T. Leness have outlined a rigorous proof of this conjecture using standard techniques of differential geometry.

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u -Plane Integral Z_u

Can be computed explicitly from QFT of LEET

Vanishes if $b_2^+ > 1$. When $b_2^+ = 1$:

$$Z_u = \int du d\bar{u} \mathcal{H} \Psi$$

\mathcal{H} is holomorphic and metric-independent

Ψ : Sum over principal $U(1)$ bundles for gauge field of the Abelian LEET:

$$\Psi \sim \sum_{v=c_1(L)} e^{-i \pi \bar{\tau}(\bar{u}) v_+^2 - i \pi \tau(u) v_-^2}$$

NOT holomorphic and metric-DEPENDENT

u-Plane: Recent Progress 1.1

The u-plane integral turns out to be closely related to the theory of mock modular (and Jacobi) forms

Such a connection was noted for special cases $X = \mathbb{C}\mathbb{P}^2$ & $S^2 \times S^2$ by Moore-Witten (1997) and Malmendier-Ono (2008)

New point: The relation holds for ALL 4-manifolds with $b_2^+ = 1$

G. Korpas, J. Manschot, G. Moore, I. Nidaiev (2019)

u-Plane: Recent Progress 1.2

$$E_u: y^2 = x^2(x - u) + \frac{1}{4}x$$

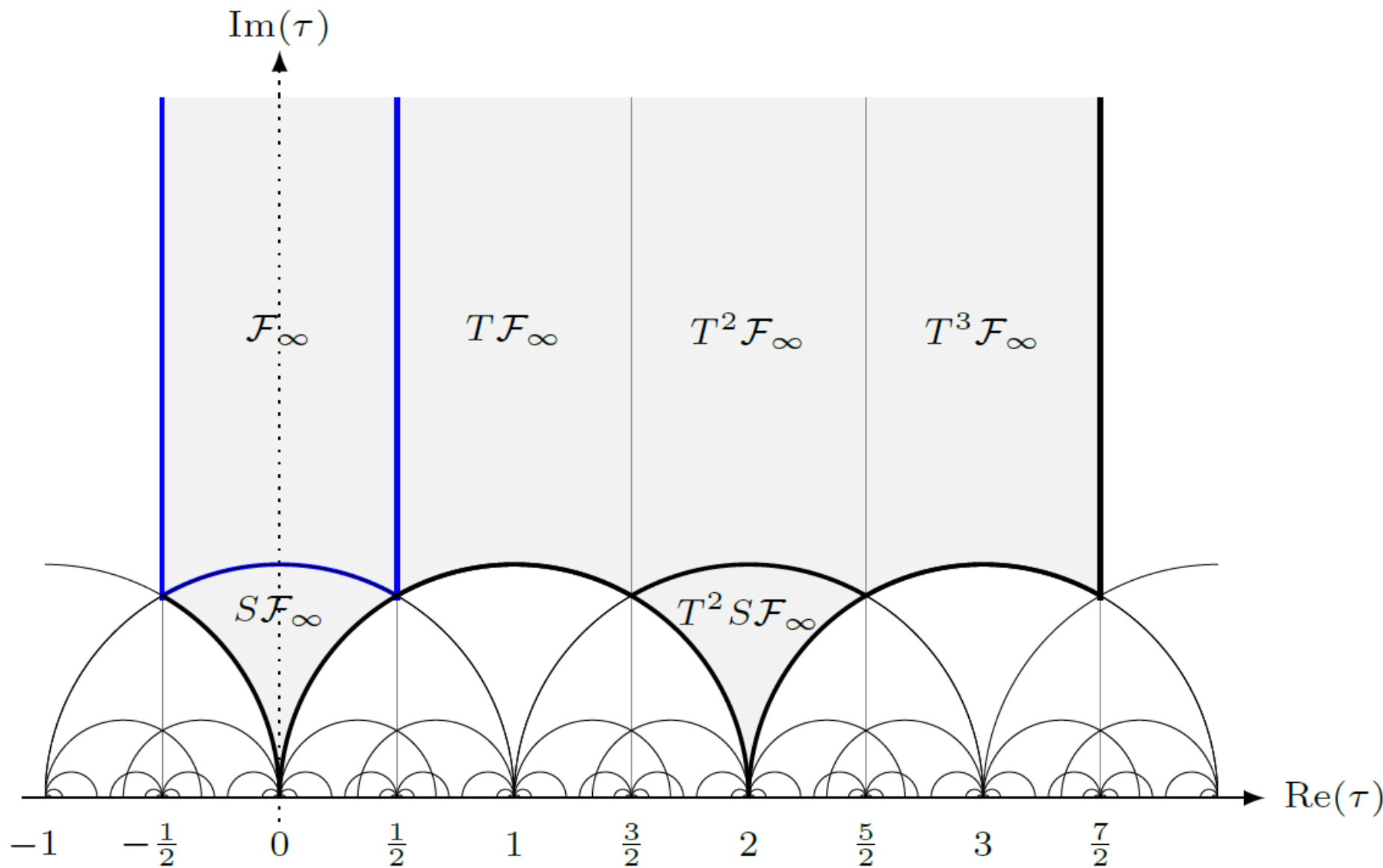
There is a unique A-cycle in $H_1(E_u; \mathbb{Z})$ invariant under monodromy $u \rightarrow e^{2\pi i} u$ (for $|u| > 1$)

$$\omega := \oint_A \frac{dx}{y}$$

Choose B-cycle: \Rightarrow

$$\tau(u) = \oint_B \frac{dx}{y} / \oint_A \frac{dx}{y}$$

$$u = \frac{\vartheta_2^4 + \vartheta_4^4}{2 \vartheta_2^2 \vartheta_4^2} = \frac{1}{8} q^{-\frac{1}{4}} + \frac{5}{2} q^{\frac{1}{4}} + \dots \quad q = e^{2\pi i \tau}$$



$$\Gamma^0(4) = \left\{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \text{ mod } 4 \right\} \subset SL(2, \mathbb{Z})$$

Relation To Mock Modular Forms – 1.1

Z_u : A sum of integrals of the form :

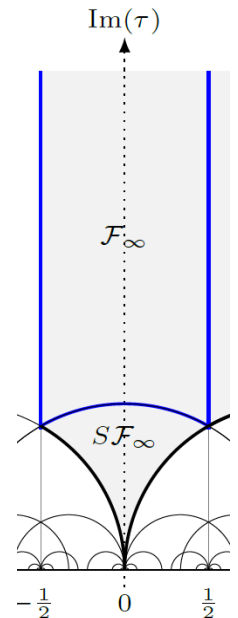
$$I_f = \int_{\mathcal{F}_\infty} d\tau d\bar{\tau} (\text{Im } \tau)^{-s} f(\tau, \bar{\tau})$$

Support of c is bounded below $f(\tau, \bar{\tau}) = \sum_{m-n \in \mathbb{Z}} c(m, n) q^m \bar{q}^n$

Strategy: Find $\hat{h}(\tau, \bar{\tau})$ such that

$$\partial_{\bar{\tau}} \hat{h} = (\text{Im } \tau)^{-s} f(\tau, \bar{\tau})$$

$\hat{h}(\tau, \bar{\tau})$ is modular of weight (2,0)



Relation To Mock Modular Forms – 1.2

$$\hat{h}(\tau, \bar{\tau}) = h(\tau) + R$$

We choose an explicit solution

$$\partial_{\bar{\tau}} R = (Im\tau)^{-s} f(\tau, \bar{\tau})$$

vanishing exponentially fast at $Im\tau \rightarrow \infty$

$h(\tau)$: mock modular form

$$h(\tau) = \sum_{m \in \mathbb{Z}} d(m) q^m \quad q = e^{2\pi i \tau}$$

$$h\left(-\frac{1}{\tau}\right) = \tau^2 h(\tau) + \tau^2 \int_{-i\infty}^0 \frac{f(\tau, \bar{v})}{(\bar{v} - \tau)^s} d\bar{v}$$

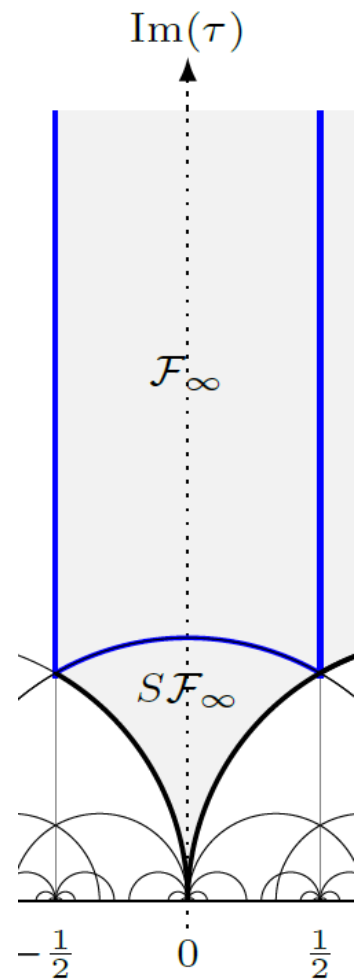
Doing The Integral

$$I_f = \int_{\mathcal{F}_\infty} d\tau d\bar{\tau} y^{-s} f(\tau, \bar{\tau})$$

$$\partial_{\bar{\tau}} \hat{h} = y^{-s} f(\tau, \bar{\tau})$$

$$h(\tau) = \sum_{m \in \mathbb{Z}} d(m) q^m$$

$$I_f = d(0)$$



Note: $d(0)$ undetermined by diffeq but fixed by the modular properties: Subtle!

\exists Long history of the definition & evaluation of such integrals with singular modular forms – refs at the end.

Examples 1.1

$$X = \mathbb{CP}^2: \quad b_2 = b_2^+ = 1$$

$$Z_u = \int_{\mathcal{F}_\infty} d\tau d\bar{\tau} \mathcal{H} \Psi$$

$$\mathcal{H} = \frac{\vartheta_4^{12}}{\eta^9} \exp[S^2 T(\tau)]$$

$$\Psi = e^{-2\pi y b^2} \sum_{k \in \mathbb{Z} + \frac{1}{2}} \partial_{\bar{\tau}}(\sqrt{y} (k + b)) (-1)^k \bar{q}^{k^2} e^{-2\pi i \bar{z} k}$$

$$y = \text{Im}(\tau) \quad z = \frac{S}{\omega} \quad b = \frac{\text{Im}(z)}{y}$$

Examples 1.2

$$h(\tau, z) = \frac{r}{\vartheta_4(\tau)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{\frac{n^2}{2} - \frac{1}{8}}}{1 - r^2 q^{n - \frac{1}{2}}}$$

$$r = e^{i\pi z} \quad z = \frac{S}{\omega} \quad \omega = \vartheta_2(\tau)\vartheta_3(\tau)$$

$$Z_u = Z_{DW}(S) = [\mathcal{H} h(\tau, z)]_q^0$$

$$Z_{DW}(S) = -\frac{3}{2}S + S^5 + 3S^9 + 54S^{13} + 2540S^{17} + \dots$$

Examples 1.3

$$b_2 = 2 \quad \& \quad b_2^+ = 1$$

$s_1 + s_2$	$P_{\mu}(x_1, x_2)$ for $\mu = (0, 0)$
1	$-2x_1 - 2x_2$
5	$x_1^5 - x_1^4 x_2 + x_1^3 x_2^2 + x_1^2 x_2^3 - x_1 x_2^4 + x_2^5$
9	$-40x_1^9 + 24x_1^8 x_2 - 12x_1^7 x_2^2 + 4x_1^6 x_2^3 + 4x_1^3 x_2^6 - 12x_1^7 x_2^2 + 24x_1 x_2^8 - 40x_2^9$

1 Physical Mathematics

2 Math & Physics Problems

3 Instantons & Donaldson Invariants

4 Topological Field Theory

5 Low Energy Effective Field Theory

6 New Results On Z_u

7 Directions For Future Research



FUTURE DIRECTIONS

1. Other N=2 Theories

2. Family Invariants

3. $b_2^+ = 0$

4. Manifolds with boundary and Floer theory

5. Physical Interpretation of Bauer-Furuta Invariant?

6. Vafa-Witten theory (twisted N=4 SYM)

What About Other $\mathcal{N}=2$ Theories?

There are infinitely many other four-dimensional $\mathcal{N}=2$ supersymmetric quantum field theories.

Topological twisting makes sense
for any $\mathcal{N} = 2$ theory \mathcal{T} .

Natural Question: Given the successful application of $\mathcal{N} = 2$ SYM for $G = SU(2)$ to the theory of 4-manifold invariants, are there interesting applications of OTHER $\mathcal{N} = 2$ field theories?

How Lagrangian Theories Generalize Donaldson Invariants

$$Z(S) = \langle e^{\theta(S)} \rangle_{\mathcal{T}} = \int_{\mathcal{M}} e^{\mu(S)} \varepsilon(\mathcal{V})$$

But now \mathcal{M} : is the moduli space of:

$$F^+ = \mathcal{D}(M, \bar{M}) \quad \gamma \cdot D M = 0$$


$$M \in \Gamma(W^+ \otimes V)$$

W^+ : Rank 2 bundle associated to a spin^c structure

“Generalized monopole equations”

[Labastida-Marino; Losev-Shatashvili-Nekrasov]

U(1) case: Seiberg-Witten equations.



Example: $SU(2) \mathcal{N} = 2^*$

$$M \in \Gamma(W^+ \otimes ad P \otimes \mathbb{C})$$

It has been a long-standing problem to derive a formula for Z_u for this theory when X is not spin.

The generalization is nontrivial and involves unknown interactions in the LEET that depend on the *spin*c structure.

Recent work with Jan Manschot has derived these interactions – they violate some folk wisdom.

Puzzle Of Holomorphy vs. S-duality

The definition of the $\mathcal{N} = 2^*$ theory depends on a choice of τ_0 :

$$\tau_0 = \frac{\theta}{2\pi} + i \frac{4\pi}{g^2}$$

S-duality: Physical quantities only depend on the isomorphism class of $E_{\tau_0} = \mathbb{C}/(\mathbb{Z} + \tau_0\mathbb{Z})$

Formal arguments: The path integral $Z_{2^*}(S; \tau_0)$ is both a modular form in τ_0 and holomorphic in τ_0 .

Problem: For $X = \mathbb{CP}^2$ There is no suitable modular form of the required weight and multiplier system!



Z_u To The Rescue

Resolution: The path integral is modular in τ_0 , but not holomorphic.

It is a nonholomorphic completion of a mock modular form.

Now Z_u , not just its integrand, is a mock modular form (of τ_0)

Z_u depends on the *spin*c structure.

Example

$$c := c_1(\det W^+)$$

For $c^2 = 9$ and $w_2(adP) = 0$

$$Z(0, \tau_0) = h(\tau_0) + \int_{-\bar{\tau}_0}^{i\infty} \frac{\vartheta_3(2u)}{(\tau_0 + u)^{\frac{3}{2}}} du$$

Fourier expansion of $h(\tau_0)$ is related to class numbers

This case should be compared with result of Vafa-Witten for $N=4$ SYM

Class S: Conjecture A

Another class of theories is derived from *six-dimensional* QFT by compactification on a Riemann surface C . [Lerche et. al.; Witten; Gaiotto; Gaiotto, Moore, Neitzke]

Not always described by an action principle – somewhat more mysterious.

Conjecture A: For all Lagrangian and all class S theories the four-manifold invariants can be written in terms of Seiberg-Witten invariants, using formulae generalizing the Witten conjecture.

Nontrivial checks: Pure $SU(N)$ [M. Marino & G. Moore]
& simplest non-Lagrangian [Moore-Nidaiev]

Family Donaldson Invariants – 1.1

There is an interesting generalization to invariants for families of four-manifolds.

Mentioned by Donaldson long ago.
Little work has been done.

$$Z_D(S) \in H^*(BDiff(X))$$

Family Donaldson Invariants – 1.2

Couple to “topologically twisted supergravity”

$$Z[g_{\mu\nu}, \Psi_{\mu\nu}] = \langle \exp\left(\int_X \Psi^{\mu\nu} \Lambda_{\mu\nu}\right) \rangle$$

Closed & horizontal $\text{diff}(X)$ -equivariant
differential form on $\text{Met}(X)$

Descends to cohomology class $\in H^*\left(\frac{\text{Met}(X)}{\text{Diff}(X)}\right)$

**Conjecture B: These are the
family Donaldson invariants**

Under discussion with J. Cushing, M. Rocek, and V. Saxena

NOT

That's all Folks!

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