Algebraic structure of the IR limit of massive d=2 N=(2,2) theories

IAS, October 13, 2014

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collaboration with
Davide Gaiotto & Edward Witten

draft is ``nearly finished"…

So, why isn't it on the arXiv ?

The draft seems to have stabilized for a while at around 350 pp ….. So, why isn't it on the arXiv?

In our universe we are all familiar with the fact that

$$e^{i\pi} - 1 = -2$$

In that part of the multiverse in which we have the *refined* identity

$$e = i = \pi = -1 = -2$$

our paper has definitely been published!

Much ``written” material is available:

Several talks on my homepage.

Davide Gaiotto: Seminar at Perimeter, Fall 2013:
``Algebraic structures in massive (2,2) theories

In the Perimeter online archive of talks.

Davide Gaiotto: ``BPS webs and
Landau-Ginzburg theories,”
Talk at String-Math 2014. On the web.

Three Motivations

1. IR sector of ***massive*** 1+1 QFT with N =(2,2) SUSY

2. Knot homology.

3. Spectral networks & categorification of 2d/4d wall-crossing formula [Gaiotto-Moore-Neitzke].

(A unification of the Cecotti-Vafa and Kontsevich-Soibelman formulae.)

Outline

- Introduction & Motivations
- Overview of Results
- Some Review of LG Theory
- LG Theory as SQM
- Boosted Solitons & Soliton Fans
- Motivation from knot homology & spectral networks
- Conclusion

d=2, N=(2,2) SUSY

$$\{Q_+, \overline{Q_+}\} = H + P$$

$$\{Q_-, \overline{Q_-}\} = H - P$$

$$\{Q_+, Q_-\} = \bar{Z}$$

$$[F, Q_+] = Q_+ \qquad [F, \bar{Q}_-] = \bar{Q}_-$$

We will be interested in situations where two supersymmetries are unbroken:

$$U(\zeta) := Q_+ - \zeta^{-1}\overline{Q_-}$$

$$\{U(\zeta), \overline{U(\zeta)}\} = 2\left(H - \mathrm{Re}(\zeta^{-1}Z)\right)$$

Main Goal & Result

Goal: Say everything we can about the theory in the far IR.

Since the theory is massive this would appear to be trivial.

Result: When we take into account the BPS states there is an extremely rich mathematical structure.

Vacua and Solitons

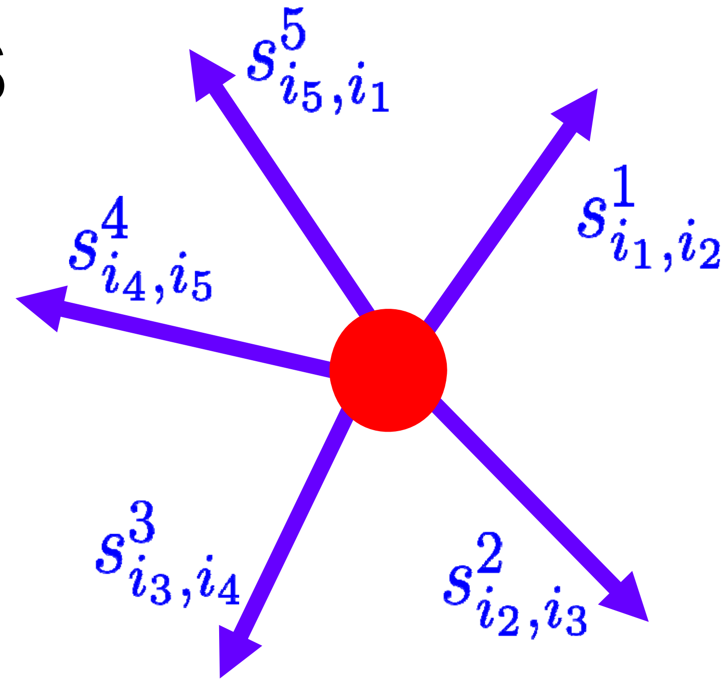
The theory has many vacua:

$$i, j, k, \cdots \in \mathbb{V}$$

There will be BPS states/solitons s_{ij} on $\mathbb{R}$

We develop a formalism – which we call the ``web-based formalism" -- which describes many things:

Interior Amplitudes



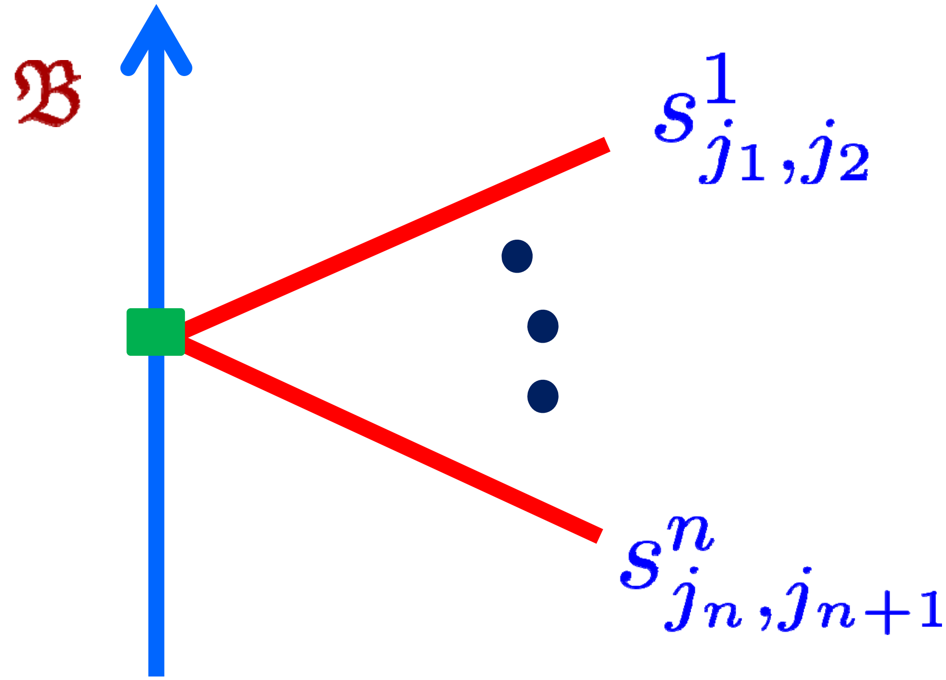
$$\beta(s^1, \ldots, s^n)$$

$$\sum_{\ell} \beta(\vec{s}_1) \cdots \beta(\vec{s}_\ell) = 0$$

BPS states have ``interaction amplitudes'' governed by an L∞ algebra

Branes/Half-BPS BC's

$$\mathcal{B}(s^1, \dots, s^n)$$

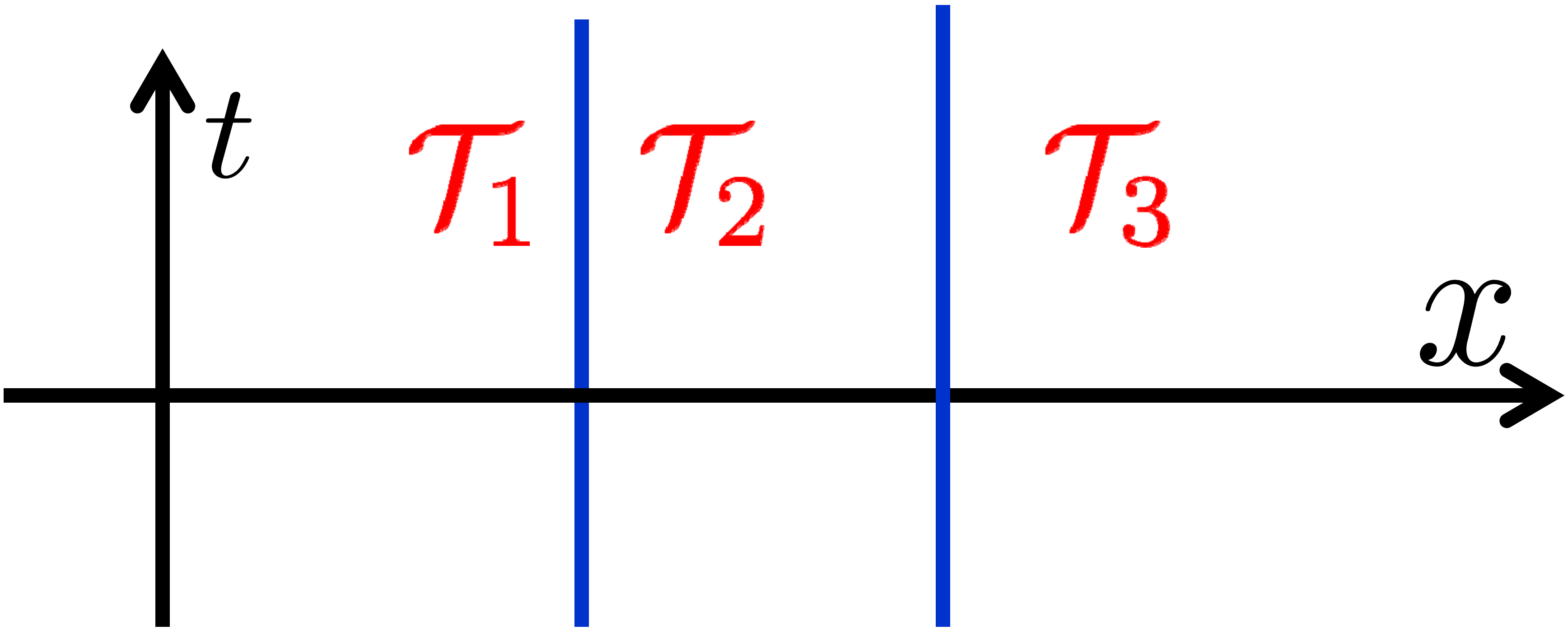


$$\sum_\ell \mathcal{B}(\vec{s}_1) \cdots \mathcal{B}(\vec{s}_\ell) = 0$$

BPS ``emission amplitudes'' are governed by an <u>A∞ algebra</u>

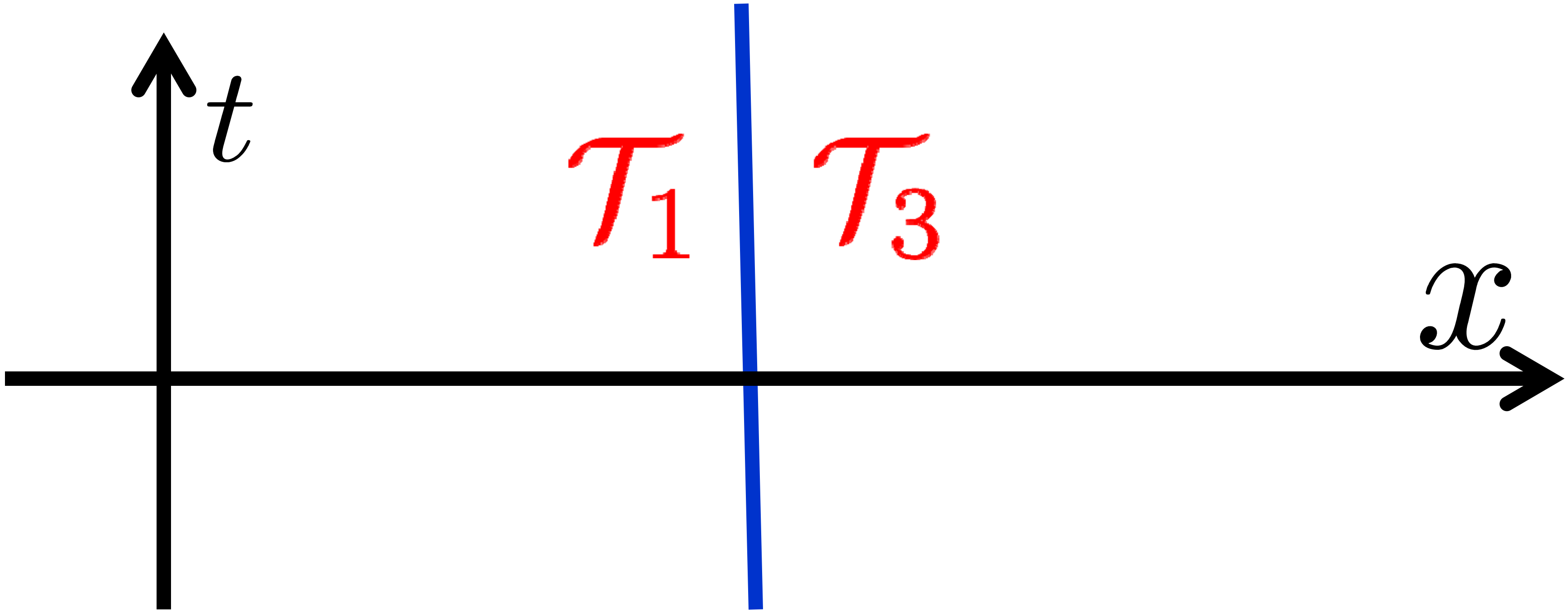
Interfaces

Given a pair of theories $\mathcal{T}_1, \mathcal{T}_2$ we construct supersymmetric interfaces



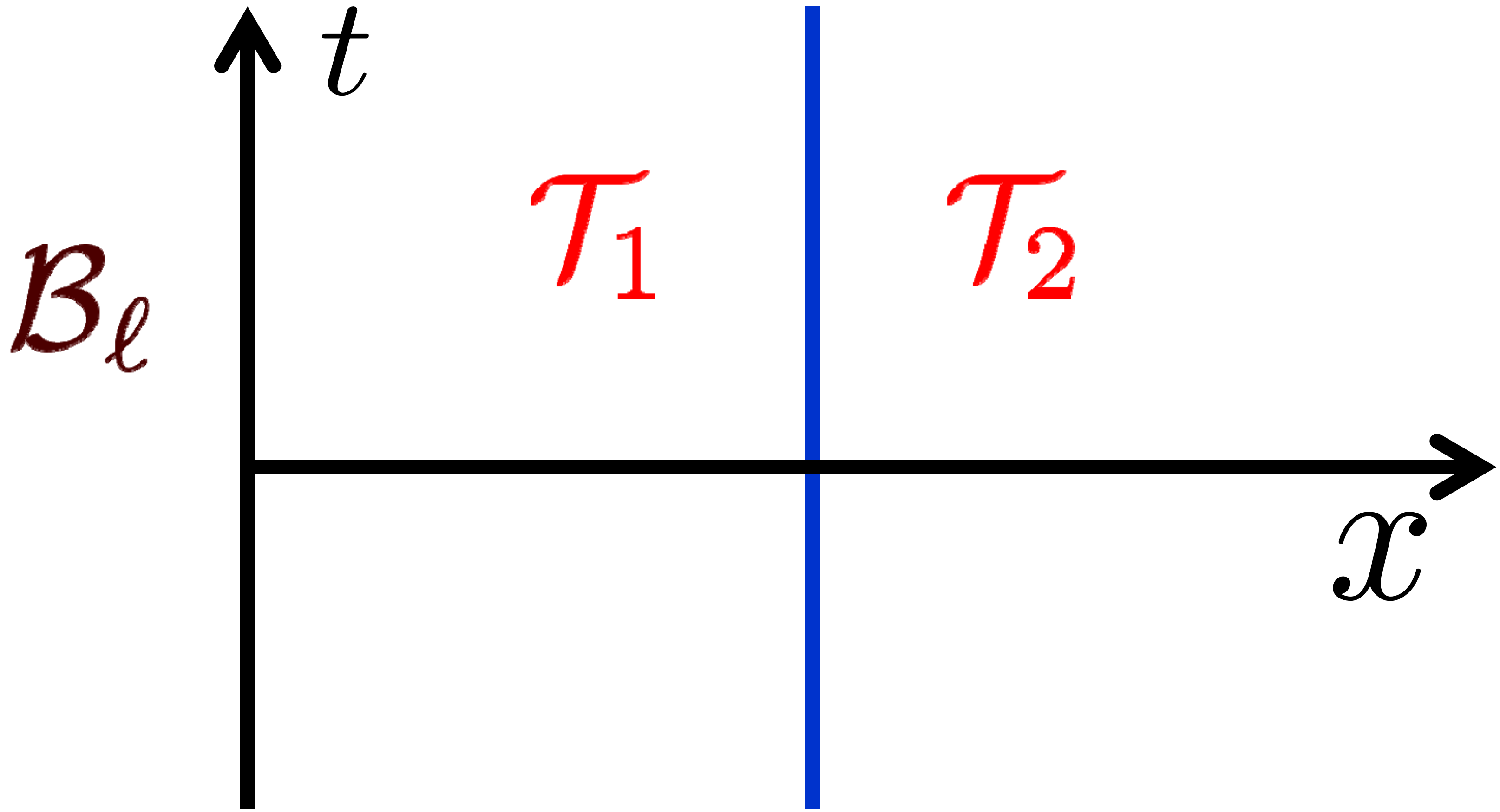
There is an (associative) way of ``multiplying'' interfaces to produce new ones

Interface OPE



We give a method to compute the product. It can be considered associative, once one introduces a suitable notion of ``homotopy equivalence'' of interfaces.

Mapping branes



Using interfaces we can ``map" branes in theory $\mathcal{T}_1$, to branes in theory $\mathcal{T}_2$.

This will be the key idea in defining a ``parallel transport'' of Brane categories.

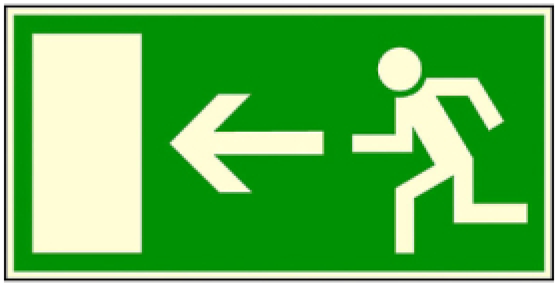
Categorification of 2d wall-crossing

If we have continuous path of theories
(e.g. a continuous family of LG superpotentials)
then we can construct half-supersymmetric interfaces between the theories.

When the path crosses marginal stability walls we construct interfaces which ``implement'' wall-crossing.

Half-susy interfaces form an A_∞ 2-category, and to a continuous family of theories we associate a flat parallel transport of brane categories.

The flatness of this connection implies, and is a categorification of, the 2d wall-crossing formula.




EMERGENCY
EXIT ONLY

Enough with vague generalities!

Now I will start to be more systematic.

First review d=2 N=(2,2) Landau-Ginzburg

Then review the relation to Morse theory.

The key ideas behind everything we do come from Morse theory.

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LG Models - 1

$\phi, \psi_\pm, \bar{\psi}_\pm, \ldots$ Chiral superfield

$W(\phi)$ Holomorphic superpotential

$$S = \int d\phi * d\bar{\phi} - |\nabla W|^2 + \cdots$$

Massive vacua are Morse critical points:

$$W'(\phi_i) = 0 \qquad W''(\phi_i) \neq 0$$

Label set of vacua: $\phi_i \in \mathbb{V}$

LG Models -2

More generally,…

(X,ω): Kähler manifold.

W: X $\longrightarrow \mathbb{C}$ Superpotential (A holomorphic Morse function)

$$\phi : D \times \mathbb{R} \to X$$

$$D = \mathbb{R}, [x_\ell, \infty), (-\infty, x_r], [x_\ell, x_r], S^1$$

Boundary conditions for ϕ

Boundaries at infinity:

$$\phi \to \phi_i \qquad \phi \to \phi_j$$

$$x \to -\infty \qquad x \to +\infty$$

Boundaries at finite distance: Preserve ζ-susy:

$$\phi|_{x_\ell, x_r} \in \mathcal{L}_{\ell,r} \subset X$$

$$\iota_{\mathcal{L}}^*(\lambda) = dk$$

(Simplify: ω=dλ)

$$\pm \mathrm{Im}(\zeta^{-1} W) \geq \Lambda$$

Fields Preserving ζ-SUSY

U(ζ)[Fermi] =0 implies the ζ-*instanton* equation:

$$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial \tau}\right)\phi^I = \zeta g^{I\bar{J}}\frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

Time-independent: ζ-*soliton* equation:

$$\frac{\partial}{\partial x}\phi^I = \zeta g^{I\bar{J}}\frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

Projection to W-plane

$$\frac{\partial}{\partial x}\phi^I = \zeta g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

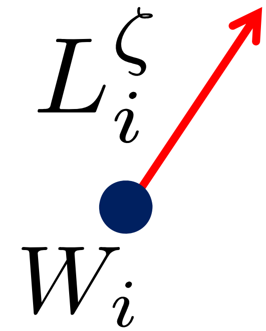
The projection of solutions to the complex W plane are contained in straight lines of slope ζ

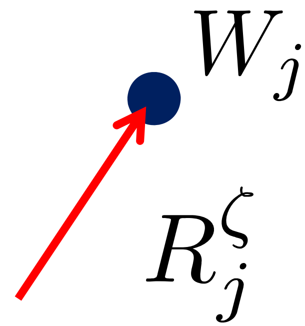
$$\frac{dW}{dx} = \frac{\partial W}{\partial \phi^I}\frac{\partial}{\partial x}\phi^I = \zeta \frac{\partial W}{\partial \phi^I} g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

$$W(x) - W(x_0) = \zeta \int_{x_0}^{x} |\nabla W|^2 dx'$$

Lefshetz Thimbles

If D contains x $\longrightarrow$ -∞ $\quad \phi \to \phi_i$





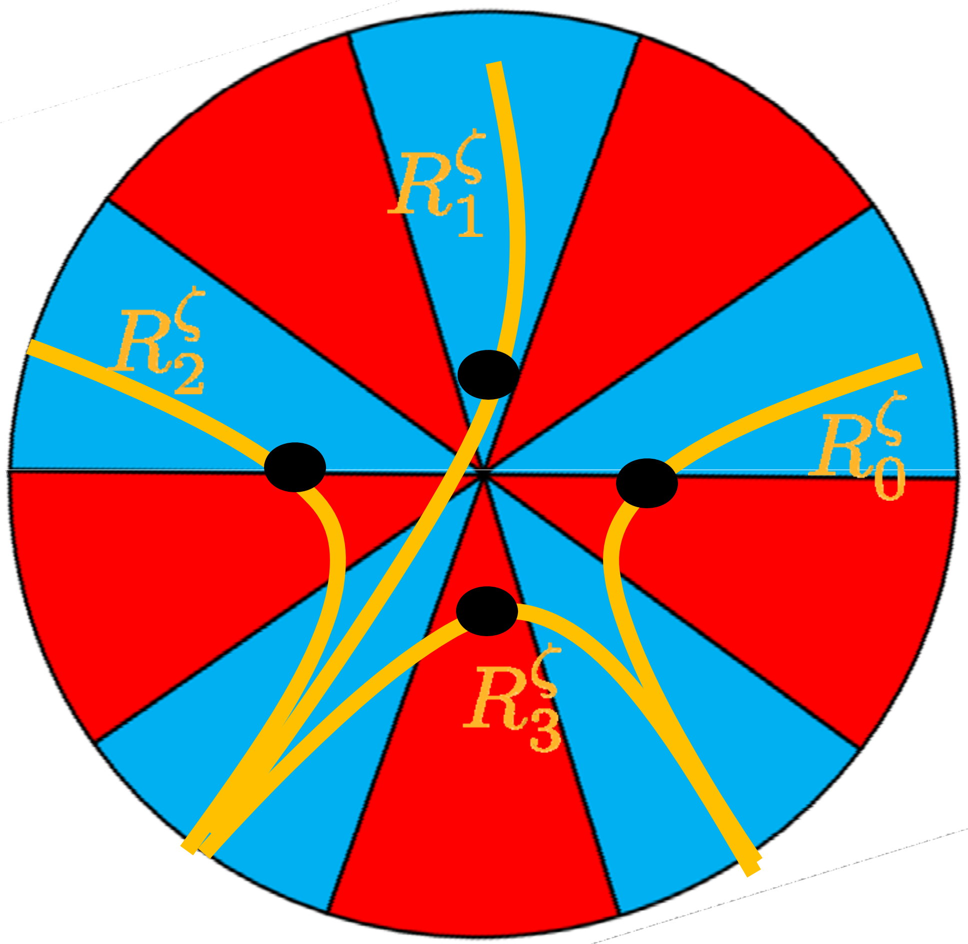
If D contains x $\longrightarrow$ +∞ $\quad \phi \to \phi_j$

Inverse image in X of all solutions defines left and right Lefshetz thimbles

They are Lagrangian subvarieties of X

Example:

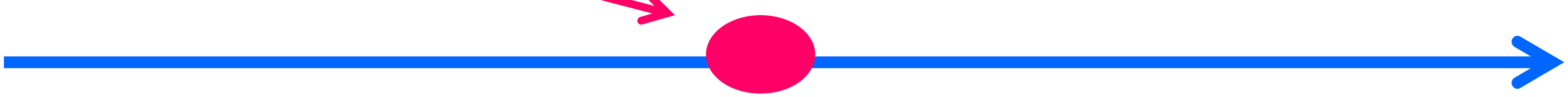
$$W = \phi - \frac{\phi^5}{5}$$



Solitons For D=$\mathbb{R}$

Scale set by W

$$\frac{\partial}{\partial x}\phi^I = \zeta g^{I\bar{J}}\frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$



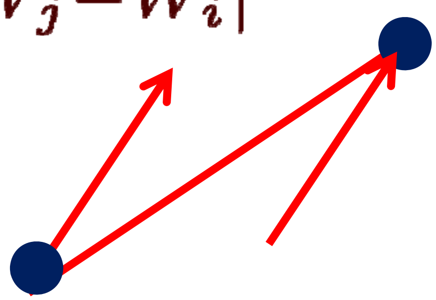
$\phi \cong \phi_i$ $\qquad$ $\phi \cong \phi_j$

For general ζ there is no solution.

$$\zeta = \zeta_{ji} := \frac{W_j - W_i}{|W_j - W_i|}$$

But for a suitable phase there is a solution

W_j



W_i

This is the classical soliton. There is one for each intersection (Cecotti & Vafa)

$$p \in L_i^{\zeta} \cap R_j^{\zeta}$$

(in the fiber of a regular value)

Near a critical point

$$W = W_i + \sum_I \tfrac{1}{2}\mu_I(\phi^I - \phi_i^I)^2$$

$$\phi^I = \phi_i^I + r^I \sqrt{\frac{\zeta\mu_I}{\kappa_I}} e^{\kappa_I x}$$

$$r^I \in \mathbb{R} \qquad |\kappa_I| = |\mu_I|$$

$$L_i^\zeta \qquad \forall I \;\; \kappa_I > 0$$

$$R_i^\zeta \qquad \forall I \;\; \kappa_I < 0$$

BPS Index

The BPS index is the Witten index:

$$\mu_{ij} := \mathrm{Tr}_{\mathcal{H}_{ij}^{BPS}} F(-1)^F$$

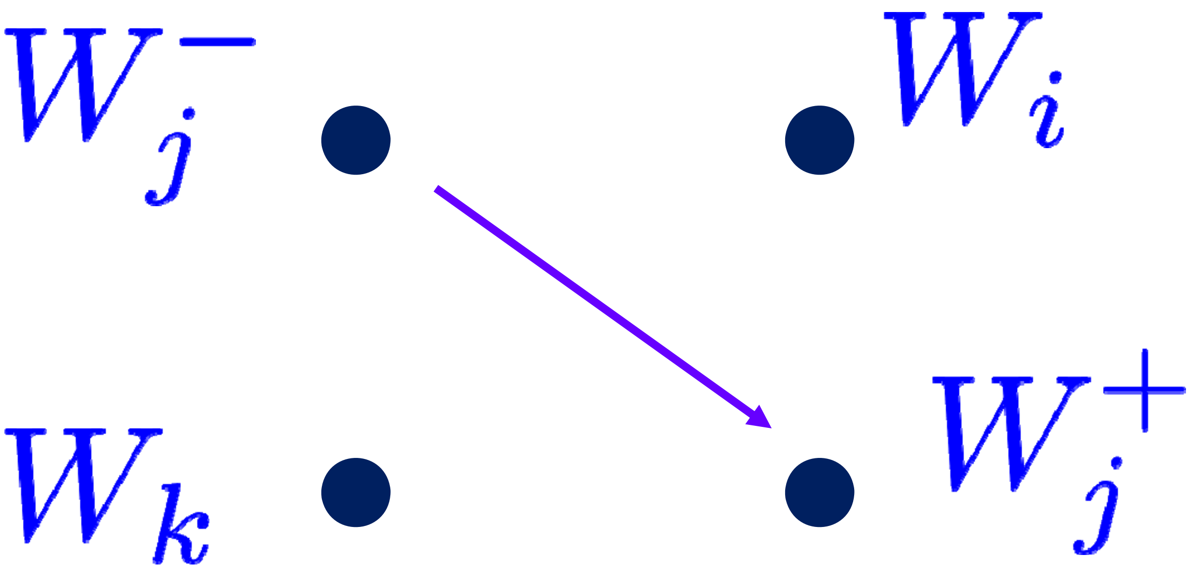
``New supersymmetric index'' of Fendley & Intriligator;
Cecotti, Fendley, Intriligator, Vafa; Cecotti & Vafa c. 1991

Remark: It can be computed with a signed sum over classical solitons:

$$\mu_{ij} = \sum_{p \in L_i^{\zeta} \cap R_j^{\zeta}} (-1)^{\iota(p)}$$

These BPS indices were studied by [Cecotti, Fendley, Intriligator, Vafa and by Cecotti & Vafa]. They found the wall-crossing phenomena:

Given a one-parameter family of W's:



$$\mu_{ik}^- \to \mu_{ik}^+ = \mu_{ik}^- + \mu_{ij}\mu_{jk}$$

One of our goals will be to ``categorify'' this wall-crossing formula.

That means understanding what actually happens to the ``off-shell complexes'' whose cohomology gives the BPS states.

We define chain complexes whose cohomology is the space of BPS states

Complex $(\mathbb{M}_{ij}, Q)$

$$\mathcal{H}^{\mathrm{BPS}}_{i,j} = H^*(\mathbb{M}_{ij}, Q)$$

Replace wall-crossing for indices:

$$\mu^{+}_{ik} = \mu^{-}_{ik} + \mu_{ij}\mu_{jk}$$

$$\left(\mathbb{M}^0_{ik} - \mathbb{M}^1_{ik}\right)^{+} =?$$

$$= \left(\mathbb{M}^0_{ik} - \mathbb{M}^1_{ik}\right)^{-}$$

$$+ \left(\mathbb{M}^0_{ij} - \mathbb{M}^1_{ij}\right) \otimes \left(\mathbb{M}^0_{jk} - \mathbb{M}^1_{jk}\right)$$

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SQM & Morse Theory
(Witten: 1982)

M: Riemannian; h: *M* $\longrightarrow \mathbb{R}$, Morse function

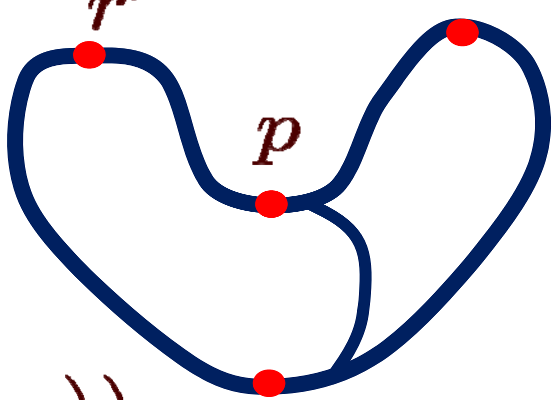
SQM: $q : \mathbb{R}_{\mathrm{time}} \rightarrow M \quad \chi \in \Gamma(q^*(TM \otimes \mathbb{C}))$

$$L = g_{IJ}\dot{q}^I\dot{q}^J - g^{IJ}\partial_I h \partial_J h$$

$$+g_{IJ}\bar{\chi}^I D_t \chi^J - g^{IJ} D_I D_J h \bar{\chi}^I \chi^J \; - R_{IJKL}\bar{\chi}^I \chi^J \bar{\chi}^K \chi^L$$

Perturbative vacua: $dh(m) = 0$

$\Longrightarrow \Psi(m)$



$$F(\Psi(m)) = \tfrac{1}{2}(d_{\uparrow}(m) - d_{\downarrow}(m))$$

Instantons & MSW Complex

Instanton equation: $$\frac{d\phi}{d\tau} = \pm g^{IJ}\frac{\partial h}{\partial \phi^J}$$

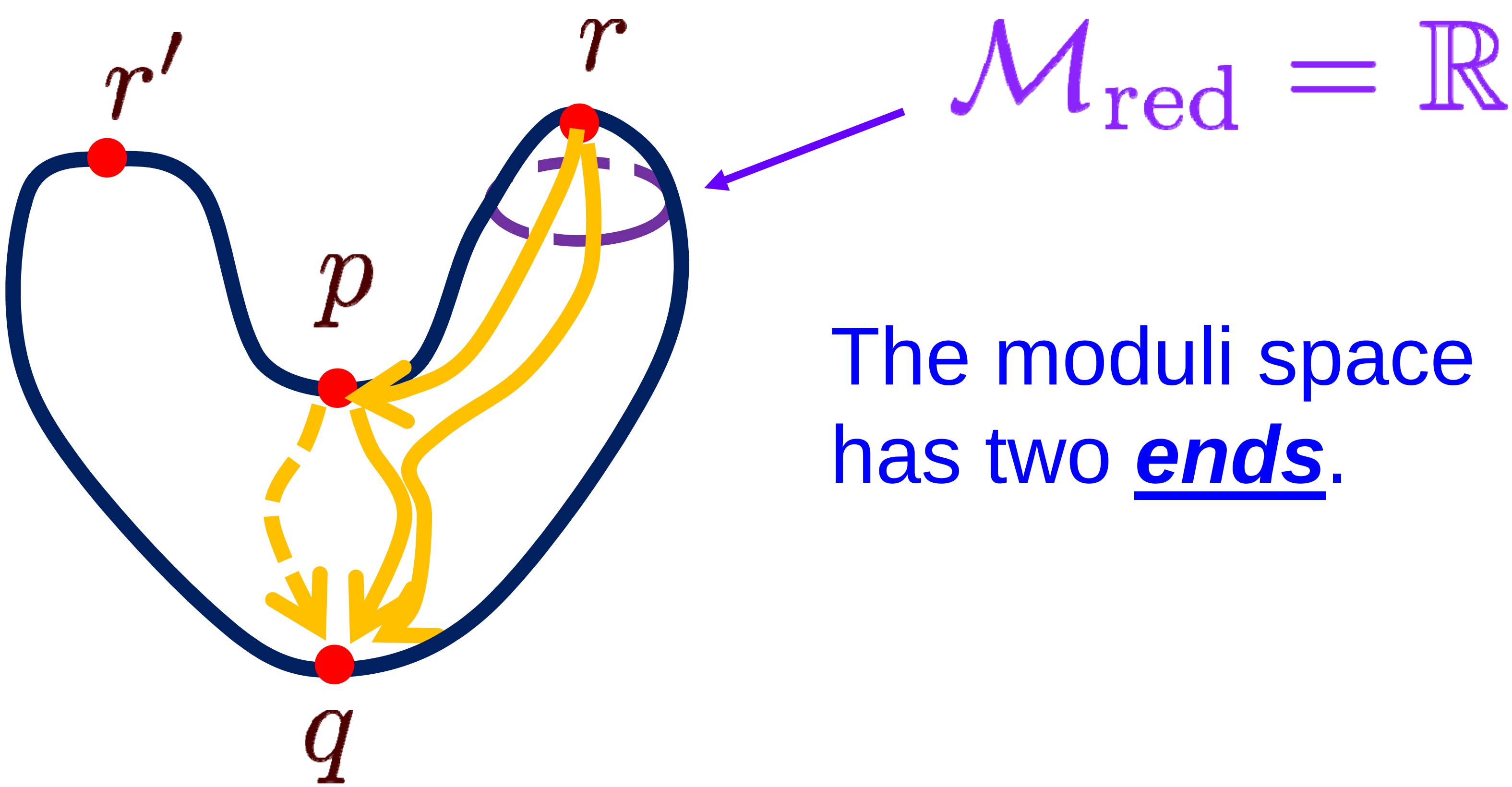
``Rigid instantons'' - with zero reduced moduli – will lift some perturbative vacua. To compute exact vacua:

MSW complex: $$\mathbb{M}^{\bullet} := \oplus_{p:dh(p)=0}\mathbb{Z}\cdot\Psi(p)$$

$$d(\Psi(p)) := \sum_{p':F(p')-F(p)=1} n(p,p')\Psi(p')$$

Space of groundstates (BPS states) is the *cohomology*.

Why $d^2 = 0$



The moduli space has two ***ends***.

Ends of the moduli space correspond to broken flows which cancel each other in computing $d^2 = 0$.

(A similar argument shows independence of the cohomology from h and g_{IJ}.)

1+1 LG Model as SQM

Target space for SQM:

$$M = \mathrm{Map}(D, X) = \{\phi : D \to X\}$$

$$D = \mathbb{R}, [x_\ell, \infty), (-\infty, x_r], [x_\ell, x_r], S^1$$

$$h = \int_D \left(\phi^* \lambda + \mathrm{Re}(\zeta^{-1} W) dx\right)$$

$$d\lambda = \omega \qquad \lambda = p dq$$

Recover the standard 1+1 LG model with superpotential: Two –dimensional ζ-susy algebra is manifest.

Two advantages of this view

1. Nice formulation of supersymmetric interfaces

2. Apply Morse theory ideas to the formulation of various BPS states.

Families of Theories

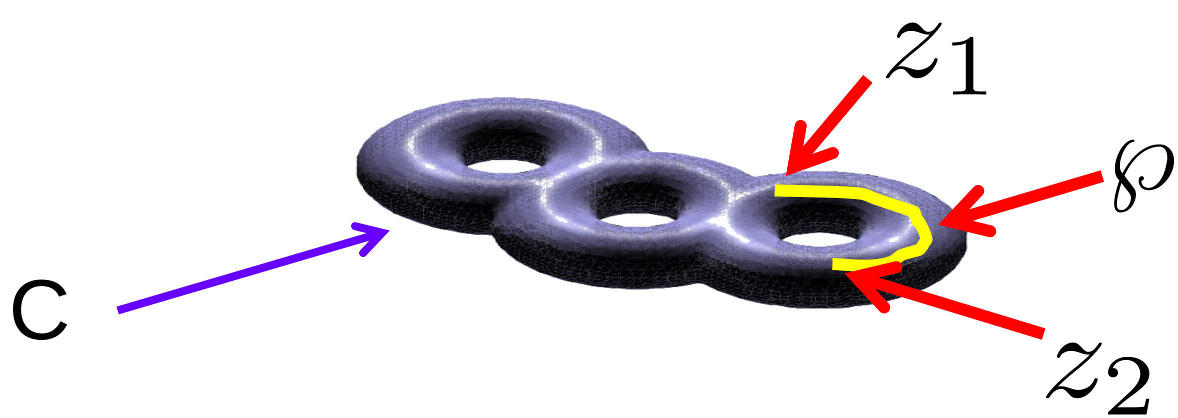
This presentation makes construction of half-susy interfaces easy:

Consider a *family* of Morse functions

$$W(\phi; z) \quad z \in C$$

Let $\wp$ be a path in C connecting z_1 to z_2.

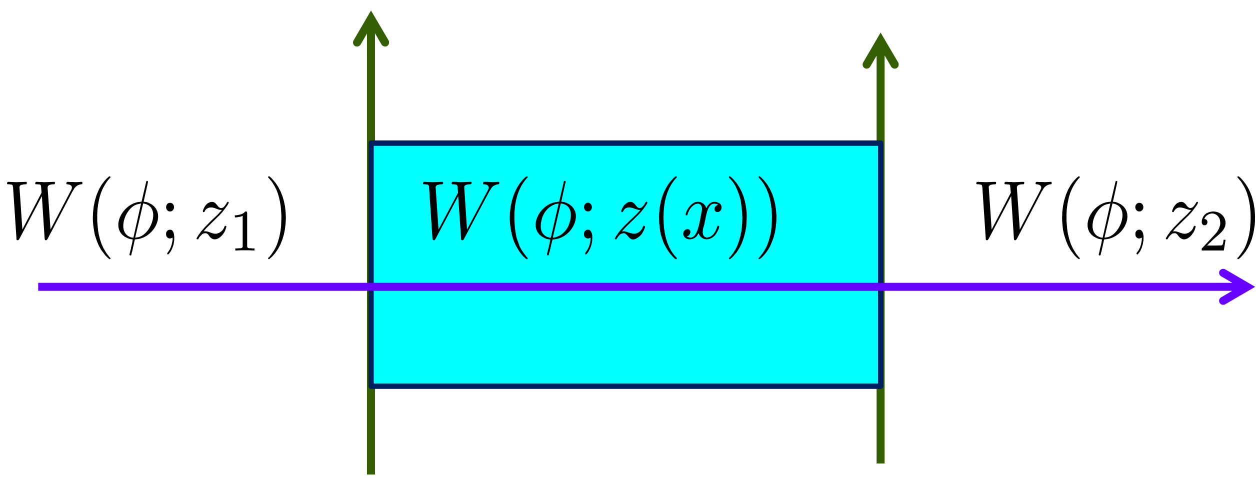
View it as a map z: $[x_l, x_r] \longrightarrow C$ with $z(x_l) = z_1$ and $z(x_r) = z_2$



Domain Wall/Interface

Using z(x) we can still formulate our SQM!

$$h = \int_D \phi^* (pdq) + \mathrm{Re}(\zeta^{-1} W(\phi; z(x)))dx$$



From this construction it manifestly preserves two supersymmetries.

MSW Complex

Now return to a single W. Another good thing about this presentation is that we can discuss ij solitons in the framework of Morse theory:

$$\frac{\delta h}{\delta \phi} = 0$$

Equivalent to the ζ-soliton equation

$$\mathbb{M}_{ij} = \oplus_{\text{solitons}} \mathbb{Z} \cdot \Psi_{ij}$$

(Taking some shortcuts here….)

$$D = \sigma^3 \mathrm{i} \frac{d}{dx} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{\zeta^{-1}}{2} W'' + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{\zeta}{2} \bar{W}''$$

$$F = -\frac{1}{2} \eta (D - \epsilon)$$

Instantons

Instanton equation $$\frac{d\phi}{d\tau} = -\frac{\delta h}{\delta \bar{\phi}}$$

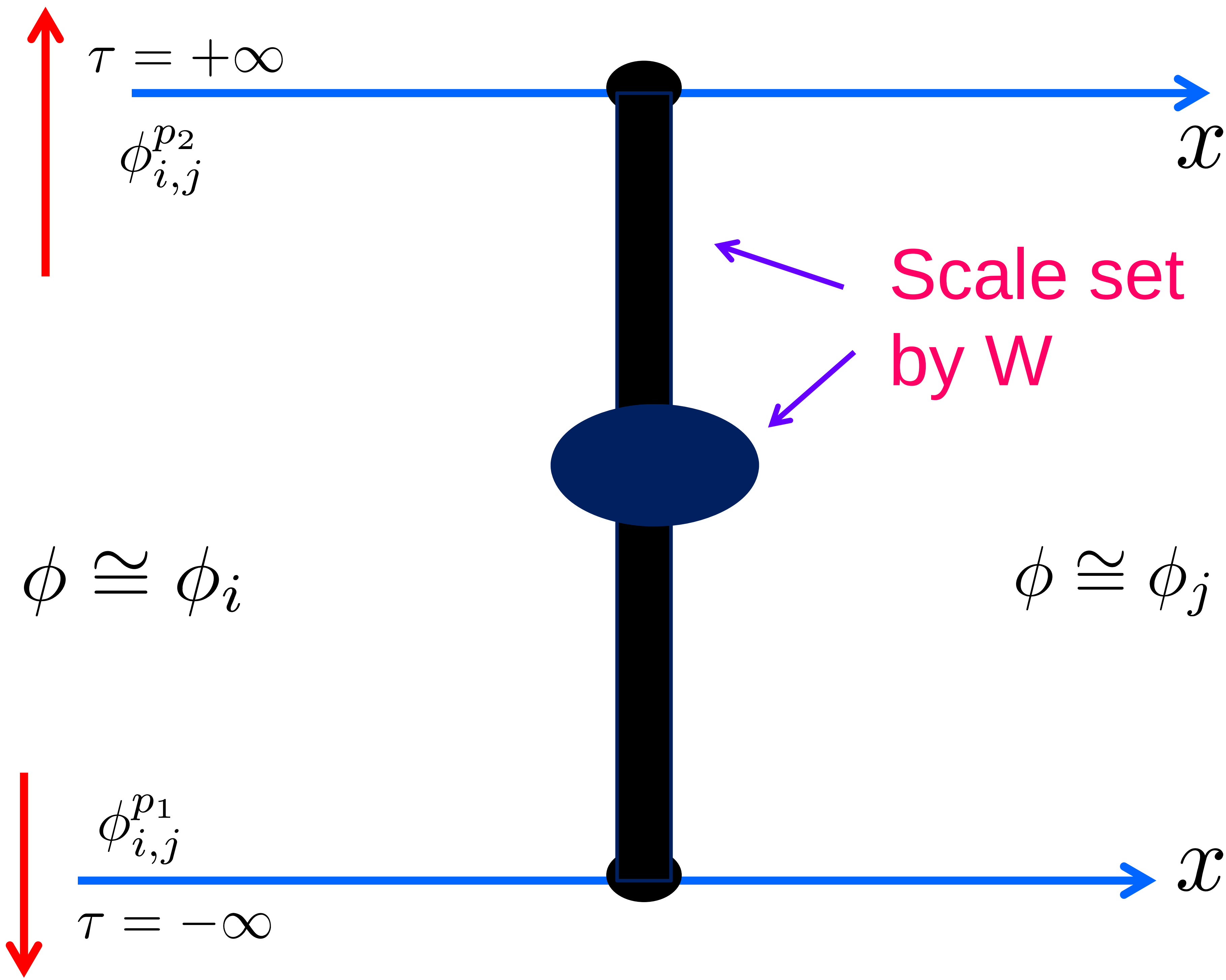
$$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial \tau}\right)\phi^I = \zeta g^{I\bar{J}}\frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

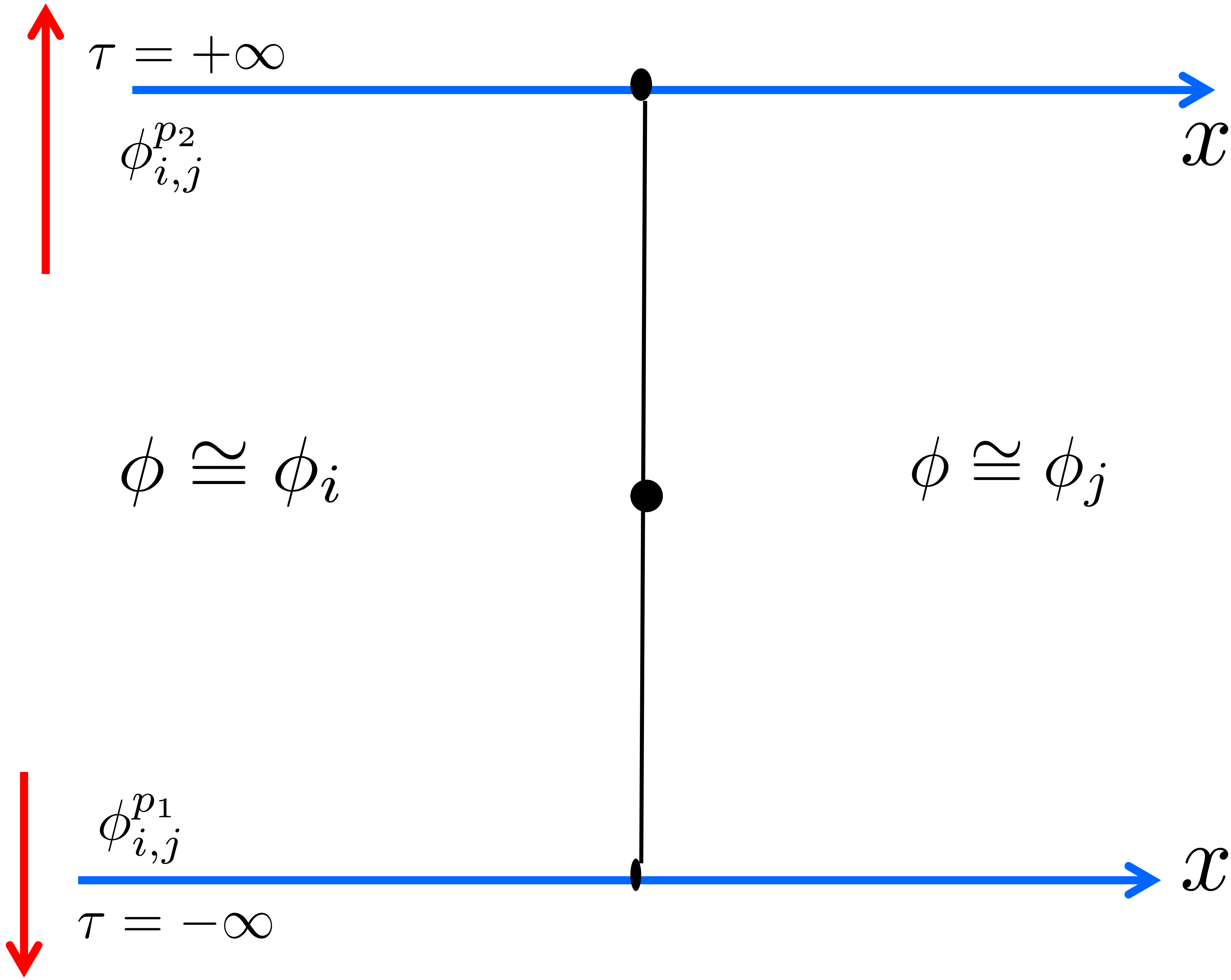
$$\bar{\partial}\phi^I = \zeta g^{I\bar{J}}\frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

At short distance scales W is irrelevant and we have the usual holomorphic map equation.

(Leading a relation to the Fukaya-Seidel category.)

At long distances the theory is almost trivial since it has a mass scale, and it is dominated by the vacua of W.


$\tau = +\infty$
$\phi_{i,j}^{p_2}$
x
Scale set by W
$\phi \cong \phi_i$
$\phi \cong \phi_j$
$\phi_{i,j}^{p_1}$
x
$\tau = -\infty$



$\tau = +\infty$
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x
$\phi \cong \phi_i$
$\phi \cong \phi_j$
$\phi_{i,j}^{p_1}$
x
$\tau = -\infty$

BPS Solitons on half-line D:

Semiclassically:

Q_ζ -preserving BPS states must be solutions of differential equation

$$\frac{\partial \phi^I}{\partial x} = \zeta g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

$$\phi|_{x_\ell} \in \mathcal{L}$$

$$\phi \to \phi_j$$
$$x \to \infty$$

Classical solitons on the positive half-line are labeled by: $p \in \mathcal{L} \cap R_j^\zeta$

Quantum Half-Line Solitons

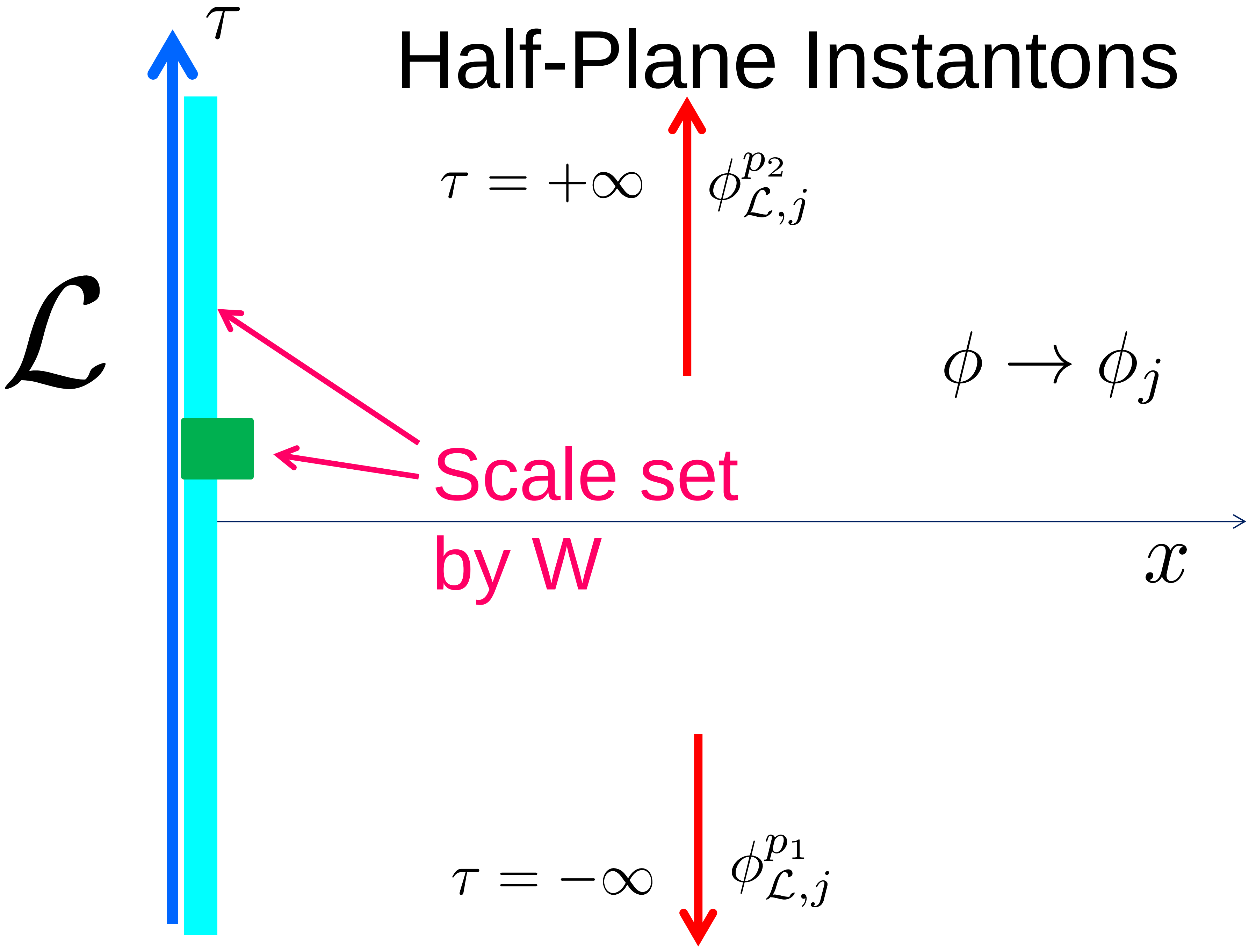
MSW complex: $\mathbb{M}_{\mathcal{L},j} = \oplus_p \mathbb{Z} \cdot \Psi_{\mathcal{L},j}(p)$

Grading the complex: Assume X is CY and that we can find a logarithm:

$$w = \mathrm{Im} \log \frac{\iota^*(\Omega^{d,0})}{\mathrm{vol}(\mathcal{L})}$$

Then the grading is by $f = \eta(D) - w$

Half-Plane Instantons



These instantons define the differential Q on the complex of approximate groundstates:

$$\mathbb{M}_{\mathcal{L},j} = \oplus_p \mathbb{Z} \cdot \Psi_{\mathcal{L},j}(p)$$

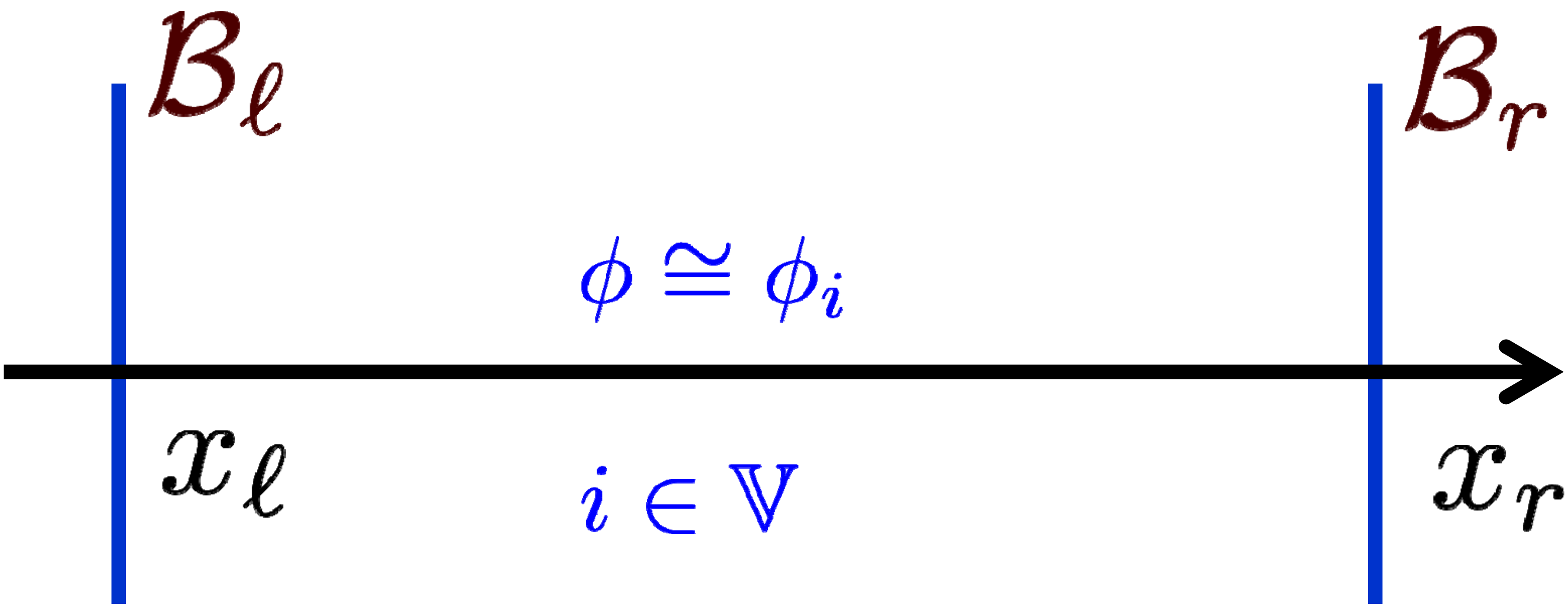
and the cohomology gives the BPS states on the half-line:

$$\mathcal{H}^{\mathrm{BPS}}_{\mathcal{L},j}$$

What is the space of BPS states on an interval ?

The theory is massive:

For a susy state, the field in the middle of a large interval is close to a vacuum:



Does the Problem Factorize?

For the Witten index: Yes

$$\mu_{\mathcal{B}_\ell,i} = \mathrm{Tr}_{\mathcal{H}^{\mathrm{BPS}}_{\mathcal{B}_\ell,i}}(-1)^F e^{-\beta H}$$

$$\boxed{\mu_{\mathcal{B}_\ell,\mathcal{B}_r} = \sum_{i\in\mathbb{V}} \mu_{\mathcal{B}_\ell,i}\cdot\mu_{i,\mathcal{B}_r}}$$

Naïve categorification?

$$\mathcal{H}^{\mathrm{BPS}}_{\mathcal{B}_\ell,\mathcal{B}_r} \overset{?}{\neq} \sum_{i\in\mathbb{V}} \mathcal{H}^{\mathrm{BPS}}_{\mathcal{B}_\ell,i}\otimes\mathcal{H}^{\mathrm{BPS}}_{i,\mathcal{B}_r}$$

No!

Solitons On The Interval

When the interval is much longer than the scale set by W the MSW complex is

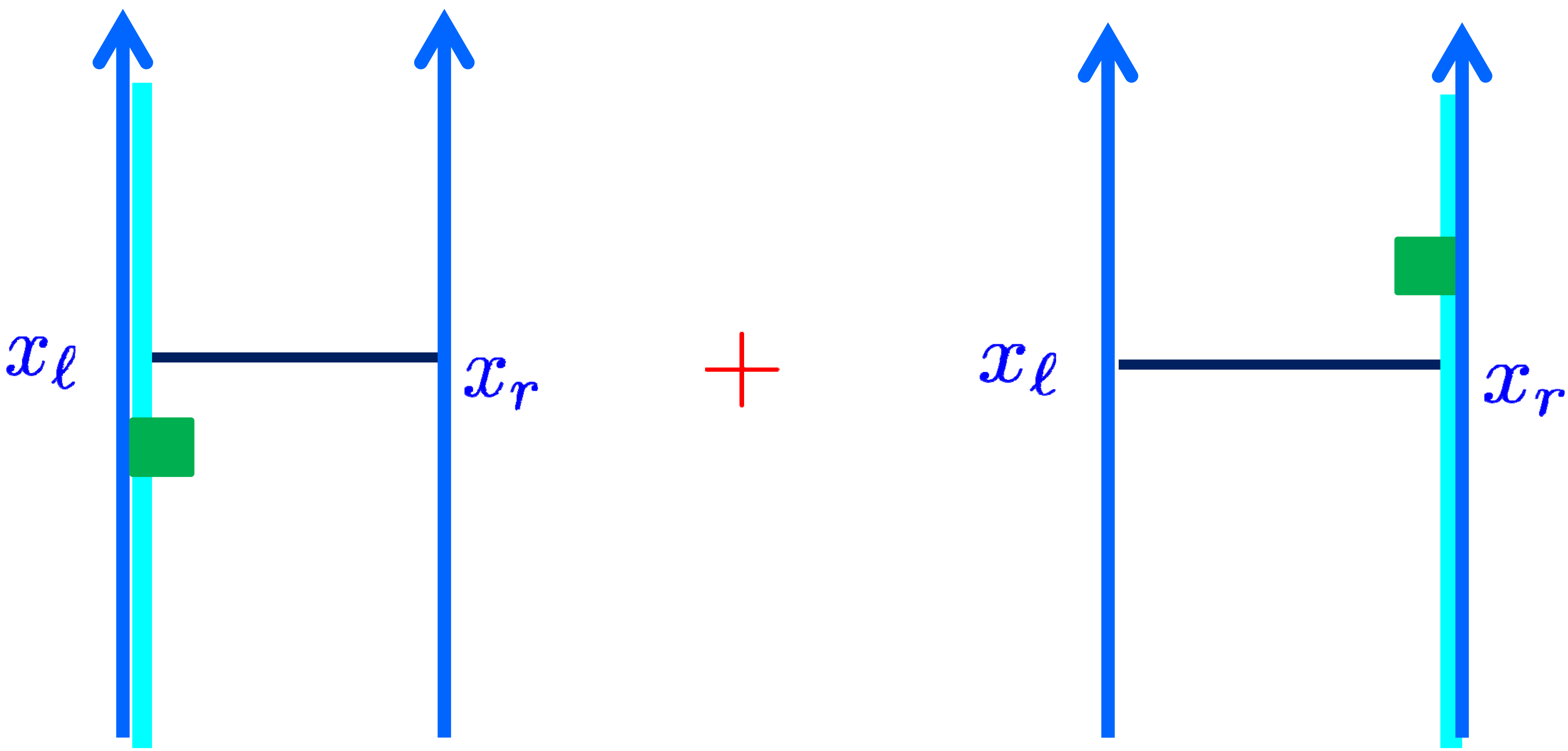
$$\mathbb{M}_{\mathcal{L}_\ell,\mathcal{L}_r} = \oplus_{i\in\mathbb{V}}\mathbb{M}_{\mathcal{L}_\ell,i}\otimes\mathbb{M}_{i,\mathcal{L}_r}$$

The Witten index factorizes nicely: $\mu_{\mathcal{L}_\ell,\mathcal{L}_r} = \sum_i \mu_{\mathcal{L}_\ell,i}\mu_{i,\mathcal{L}_r}$

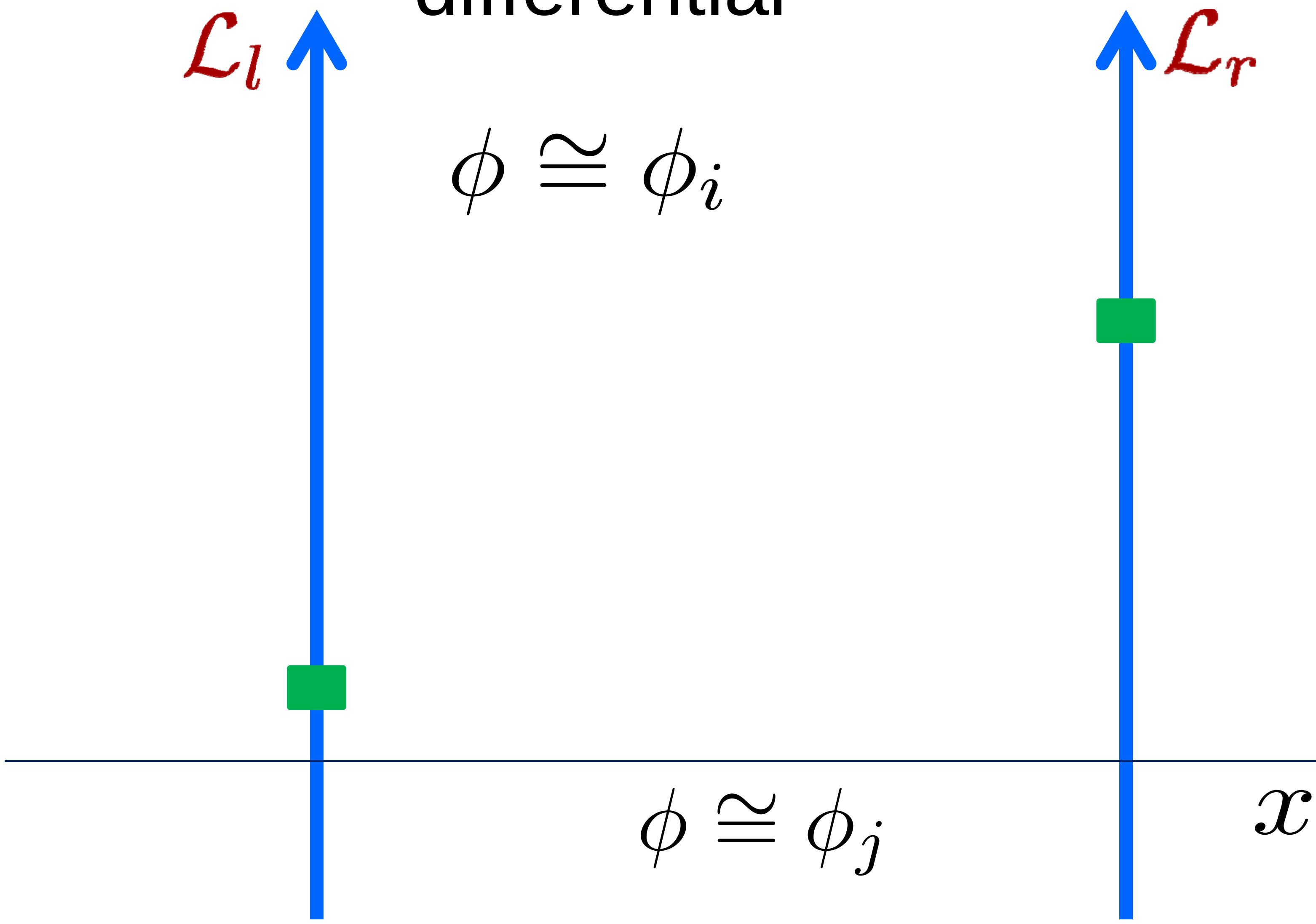
But the differential $d_{\mathcal{L}_\ell,i}\otimes 1 + 1\otimes d_{i,\mathcal{L}_r}$

is too naïve !

$$\sum_i \left(d_{\mathcal{L}_\ell, i} \otimes 1 + 1 \otimes d_{i, \mathcal{L}_r} \right)$$



Instanton corrections to the naïve differential



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The Boosted Soliton - 1

We are interested in the ζ-instanton equation for a fixed generic ζ

We can still use the soliton to produce a solution for phase ζ

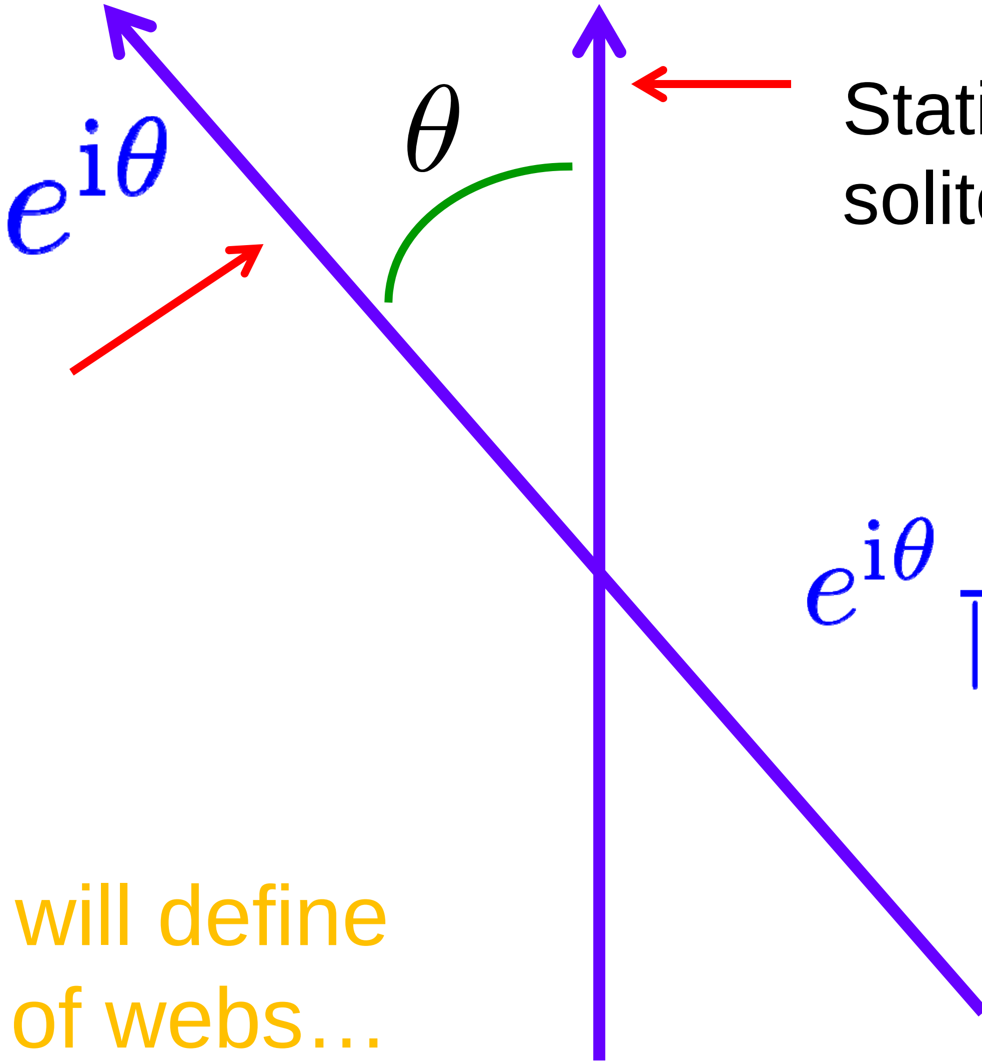
$$\phi_{ij}^{\mathrm{inst}}(x,\tau) := \phi_{ij}^{\mathrm{sol}}(\cos\theta x + \sin\theta\tau)$$

$$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial \tau}\right)\phi_{ij}^{\mathrm{inst}} = e^{\mathrm{i}\theta}\zeta_{ji}\frac{\partial \bar{W}}{\partial \bar{\phi}}$$

Therefore we produce a solution of the instanton equation with phase ζ if

$$\zeta = e^{\mathrm{i}\theta}\zeta_{ji} \qquad\qquad \zeta_{ji} := \frac{W_j - W_i}{|W_j - W_i|}$$

The Boosted Soliton -2



Stationary soliton

$\phi \cong \phi_j$

``Boosted soliton''

$\phi \cong \phi_i$

$$e^{i\theta}\frac{W_{ji}}{|W_{ji}|} = \zeta$$

These will define edges of webs…

The Boosted Soliton - 3

Put differently, the stationary soliton in ***Minkowski*** space preserves the supersymmetry:

$$Q_+ - \zeta_{ij}^{-1}\overline{Q_-}$$

So a boosted soliton preserves supersymmetry :

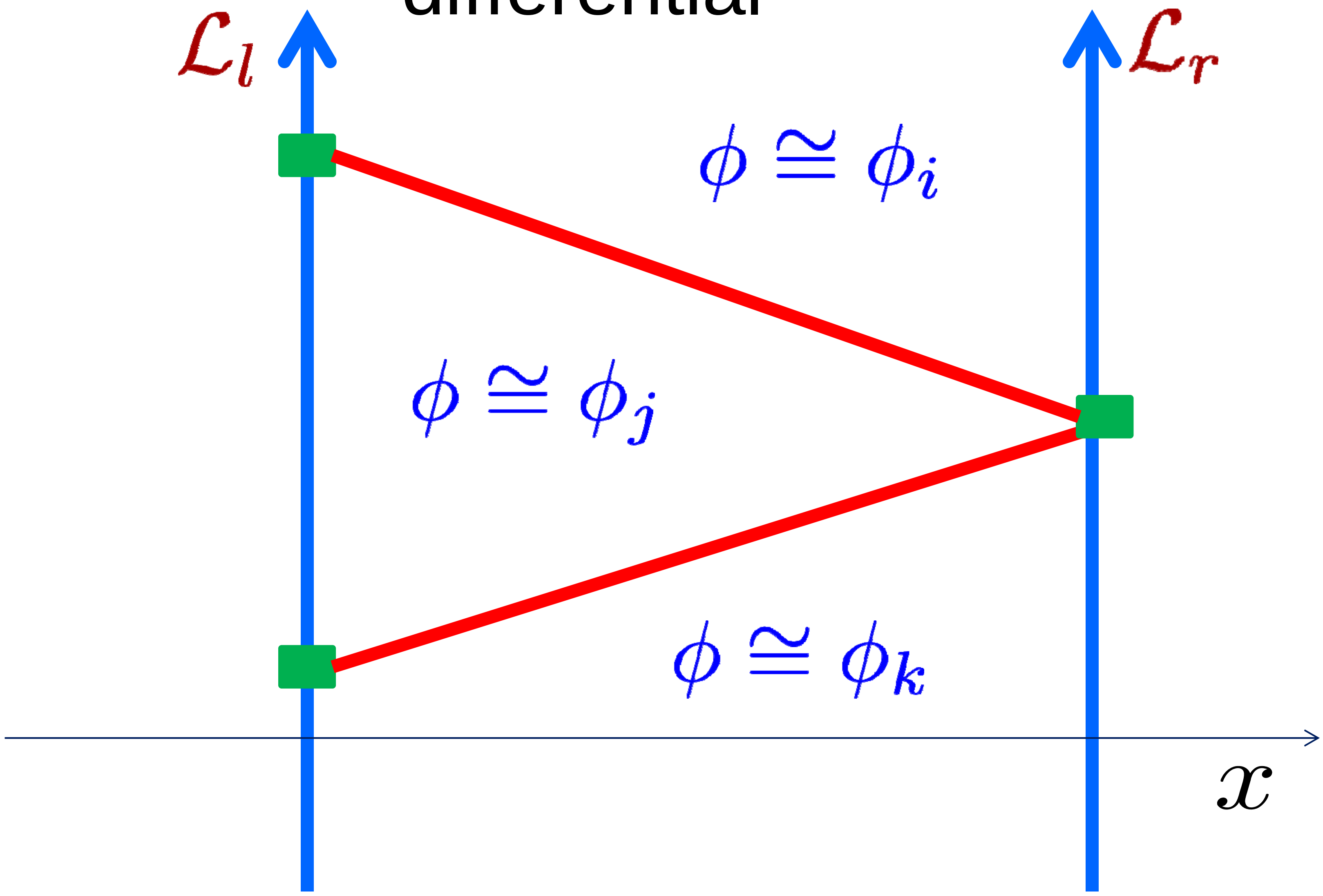
$$e^{\beta/2}Q_+ - \zeta_{ij}^{-1}e^{-\beta/2}\overline{Q_-}$$

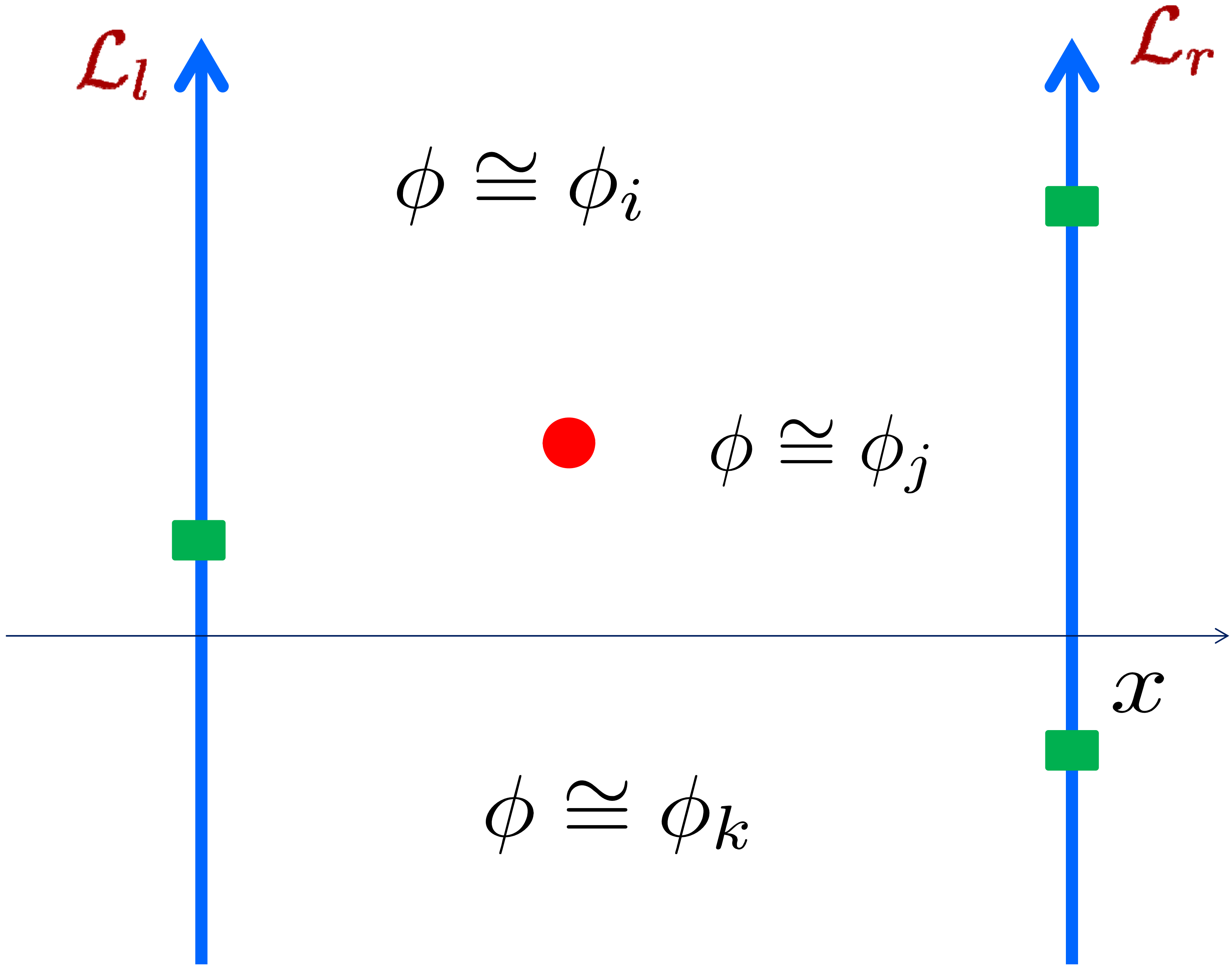
β is a real boost. In ***Euclidean*** space this becomes a rotation:

$$e^{\mathrm{i}\theta/2}Q_+ - \zeta_{ij}^{-1}e^{-\mathrm{i}\theta/2}\overline{Q_-}$$

And for suitable θ this will preserve ζ-susy

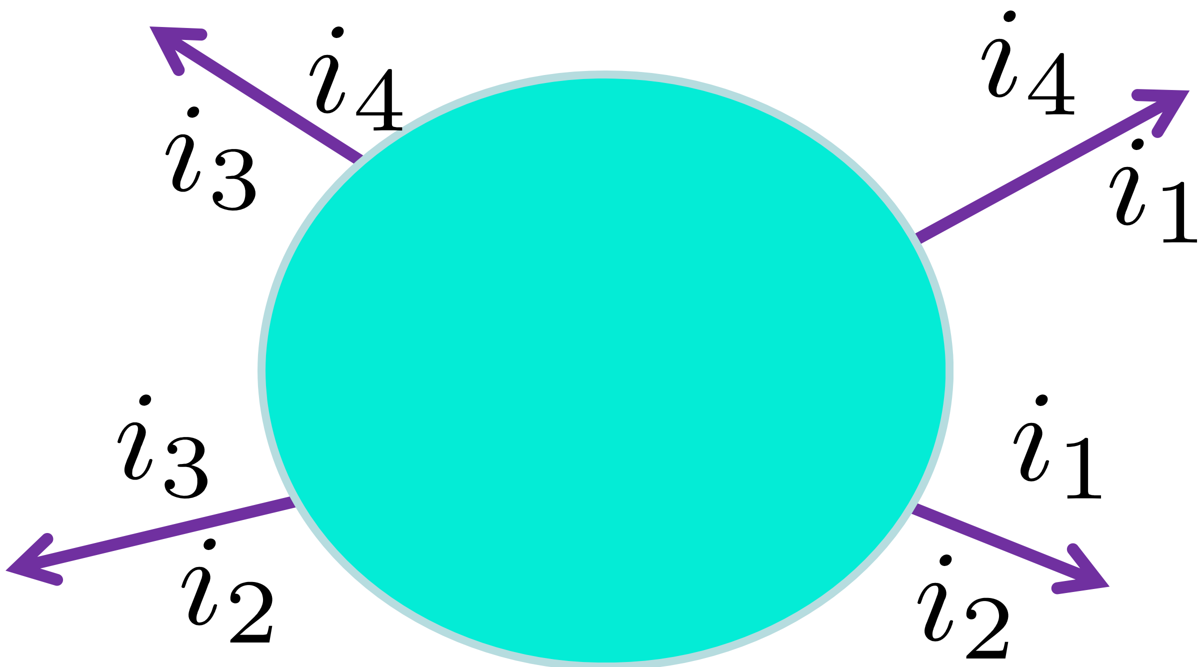
More corrections to the naïve differential





$\mathcal{L}_l$
$\mathcal{L}_r$
$\phi \cong \phi_i$
$\phi \cong \phi_j$
x
$\phi \cong \phi_k$

Path integral on a large disk



Choose boundary conditions preserving ζ-supersymmetry:

Consider a cyclic ``fan of solitons''

$$\mathcal{F} = \{\phi^{\mathrm{inst}}_{i_1 i_2}, \cdots, \phi^{\mathrm{inst}}_{i_n i_1}\}$$

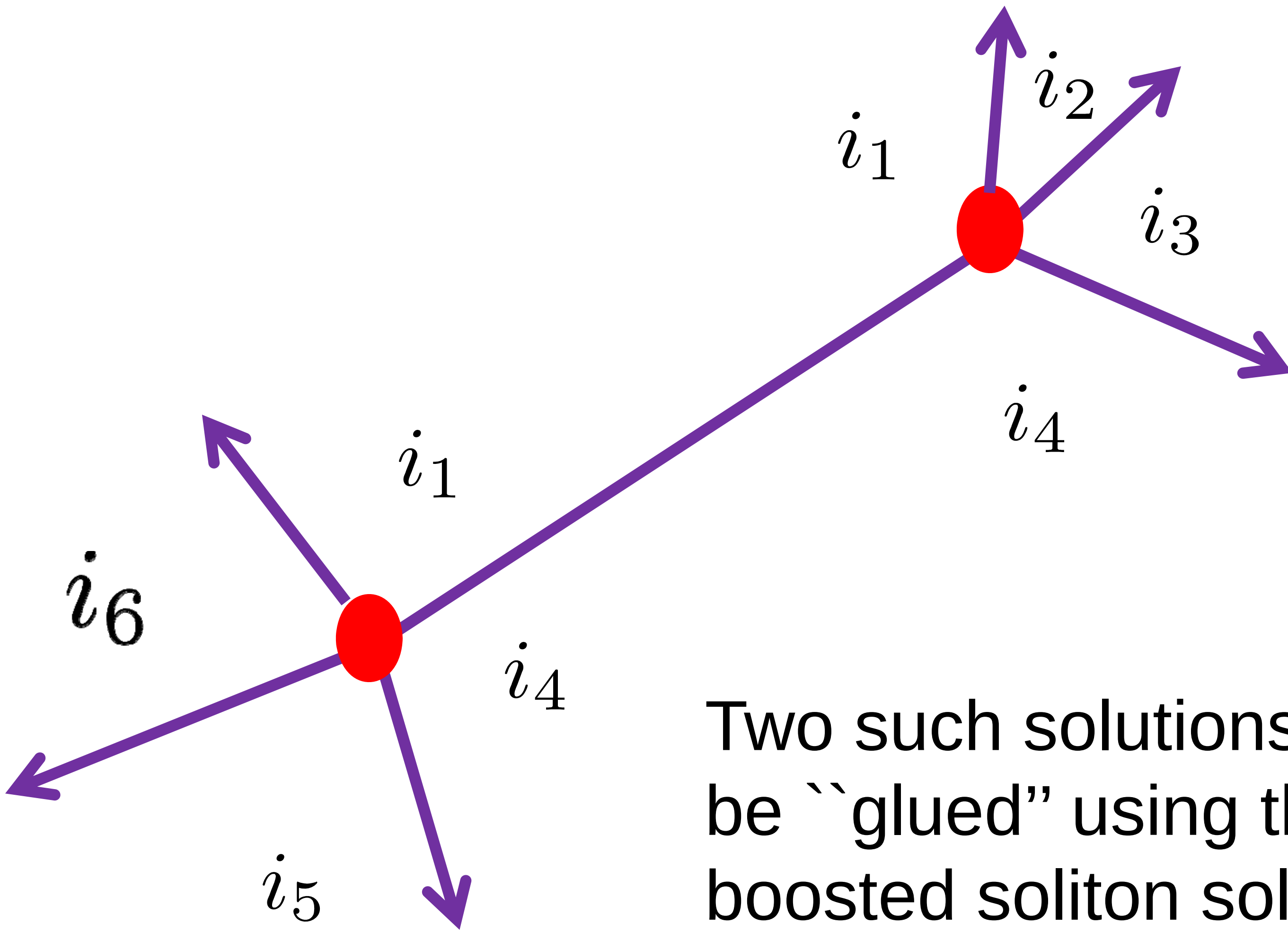
Localization

The path integral of the LG model with these boundary conditions (with A-twist) localizes on moduli space of ζ-instantons:

$$\mathcal{M}(\mathcal{F})$$

We assume the mathematically nontrivial statement that, when the index of the Dirac operator (linearization of the instanton equation) is positive then the moduli space is nonempty.

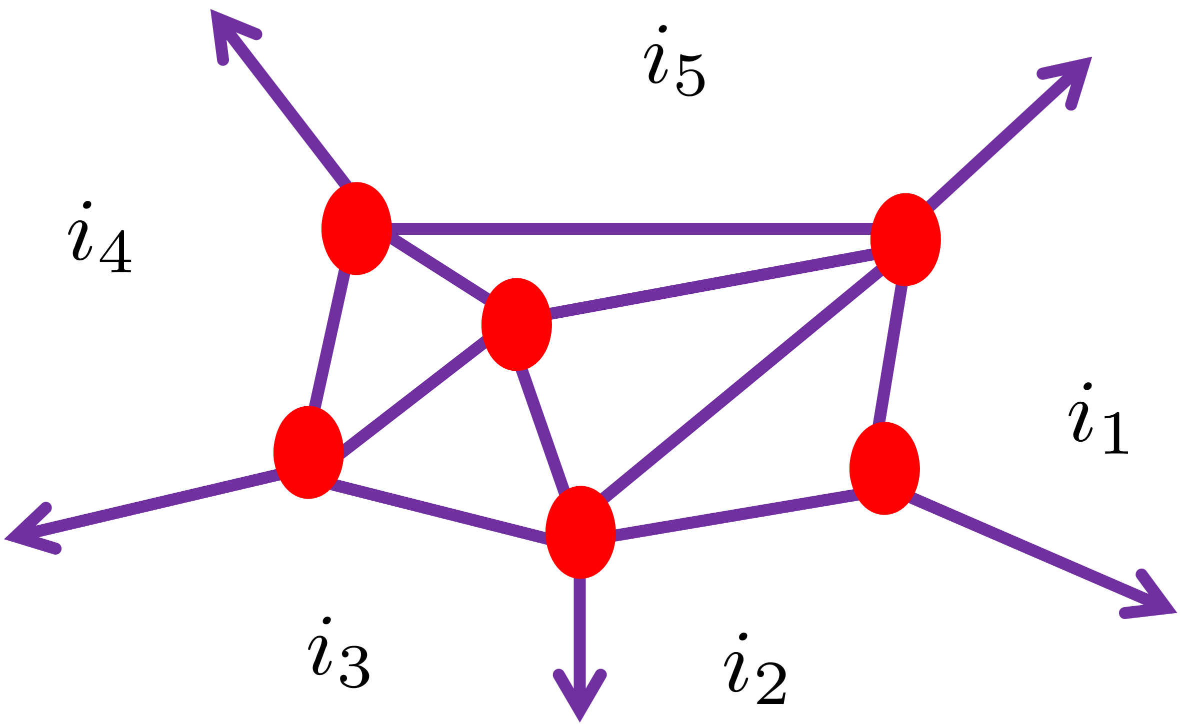
Gluing



Two such solutions can be ``glued'' using the boosted soliton solution -

Ends of moduli space

This moduli space has several "ends" where solutions of the ζ-instanton equation look like



We call this picture a ζ - *web*: $\mathfrak{w}$

ζ-Vertices

The red vertices represent solutions from the ***compact*** and ***connected*** components of

$$\mathcal{M}(\mathcal{F})$$

The contribution to the path integral from such components are called ``interior amplitudes.'' In the A-model for the zero-dimensional moduli spaces they count (with signs) the solutions to the ζ-instanton equation.

Path Integral With Fan Boundary Conditions

Just as in the Morse theory proof of $d^2=0$ using ends of moduli space corresponding to broken flows, here the broken flows correspond to webs $\mathfrak{w}$

Label the ends of $\mathcal{M}(\mathcal{F})$ by webs $\mathfrak{w}$. Each end contributes $\Psi(\mathfrak{w})$ to the path integral:

The total wavefunction is Q-invariant

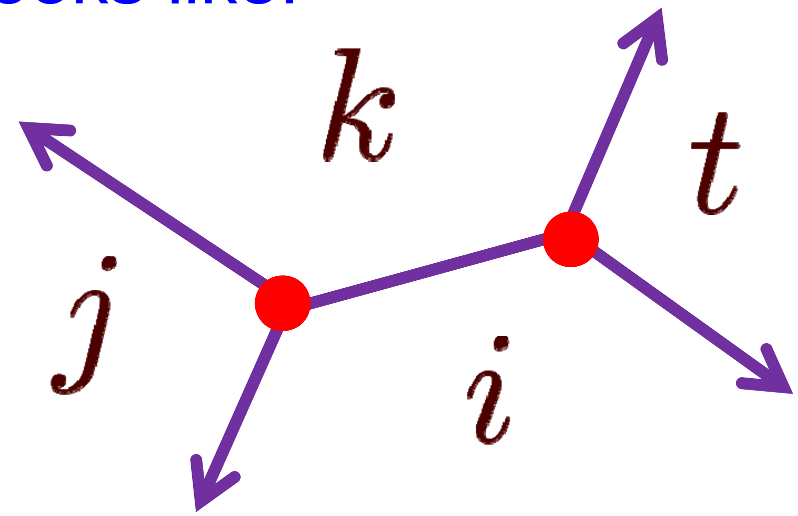
$$Q \sum_{\mathfrak{w}} \Psi(\mathfrak{w}) = 0$$

The wavefunctions $\Psi(\mathfrak{w})$ are themselves constructed by gluing together wavefunctions $\Psi(\mathfrak{r})$ associated with ζ-vertices $\mathfrak{r}$

⟹ L_∞ identities on the interior amplitudes

Example:

Consider a fan of vacua {i,j,k,t}. One end of the moduli space looks like:



The red vertices are path integrals with rigid webs. They have amplitudes β_{ikt} and β_{ijk}.

$$\mathcal{M} = \mathbb{R}^2_{transl} \times \mathbb{R}^+_{length} \, ?$$

Ends of Moduli Spaces in QFT

In LG theory (say, for X= $\mathbb{C}^n$) the moduli space cannot have an end like the finite bndry of $\mathbb{R}_+$

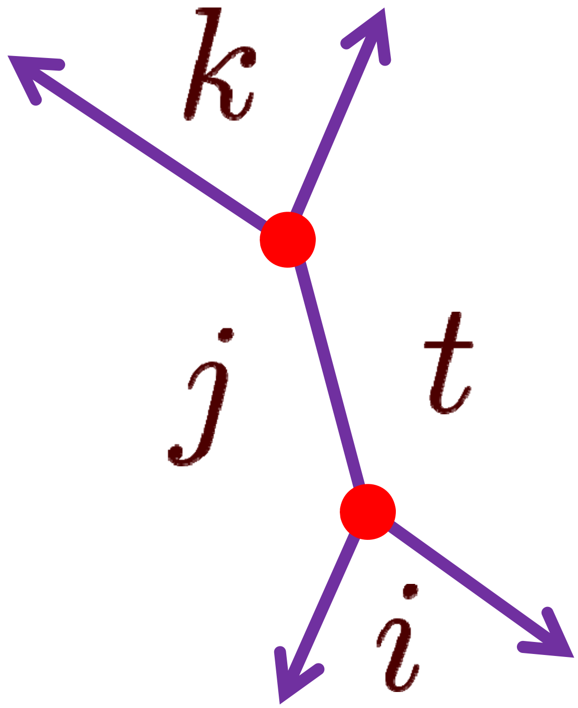
In QFT there can be three kinds of ends to moduli spaces of the relevant PDE's:

UV effect: Example: Instanton shrinks to zero size; bubbling in Gromov-Witten theory

Large field effect: Some field goes to ∞

Large distance effect: Something happens at large distances.

None of these three things can happen at the finite boundary of $\mathbb{R}_+$. So, there must be another end:



Amplitude: $\beta_{jkt}\beta_{ijt}$

The boundaries where the internal distance shrinks to zero must cancel leading to identities on the amplitudes like:

$$\beta_{ijk}\beta_{ikt} - \beta_{jkt}\beta_{ijt} = 0$$

This set of identities turns out to be the Maurer-Cartan equation for an L_∞ - algebra.

This is really a version of the argument for $d^2 = 0$ in SQM.

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Knot Homology -1/5

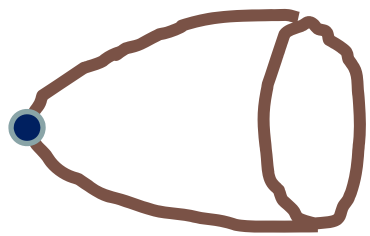
(Approach of E. Witten, 2011)

Study (2,0) superconformal theory based on Lie algebra $\mathfrak{g}$

$$\mathbb{R} \times M_3 \times D$$

TIME

D: p

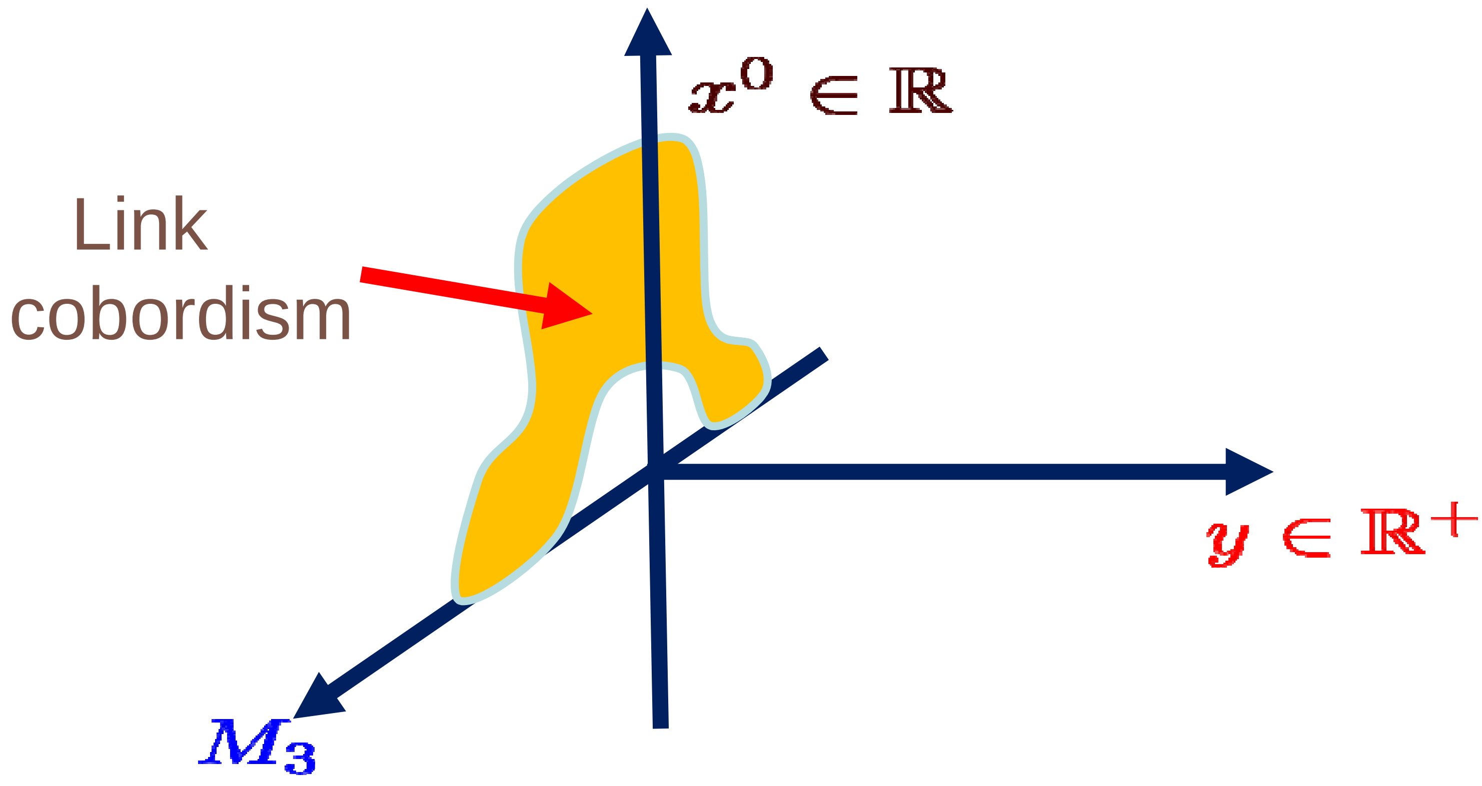


M_3: 3-manifold containing a surface defect at $\mathbb{R}$ x L x {p}

More generally, the surface defect is supported on a link cobordism $L_1 \rightarrow L_2$:

Knot Homology – 2/5

Now, KK reduce by U(1) isometry of the cigar D with fixed point p to obtain 5D SYM on $\mathbb{R} \times M_3 \times \mathbb{R}_+$



Knot Homology – 3/5

Hilbert space of states depends on M_3 and L:

$$\mathcal{H}_{\mathrm{BPS}}(M_3, L)$$

is identified with the knot homology of L in M_3.

This space is constructed from a chain complex using infinite-dimensional Morse theory on a space of gauge fields and adjoint-valued differential forms.

Knot Homology 4/5

Equations for the semiclassical states generating the MSW complex are the Kapustin-Witten equations for gauge field with group G and adjoint-valued one-form ϕ on the four-manifold $M_4 = M_3 \times \mathbb{R}^+$

$$F - \phi^2 + t(d_A\phi)^+ - t^{-1}(d_A\phi)^- = 0$$
$$d_A * \phi = 0$$

Boundary conditions at y=0 include Nahm pole and extra singularities at the link L involving a representation $R^\vee$ of the dual group.

Differential on the complex comes from counting ``instantons'' – solutions to a PDE in 5d written by Witten and independently by Haydys.

Knot Homology 5/5

In the case $M_3 = C \times \mathbb{R}$ with coordinates (z, x^1) these are precisely the equations of a ***gauged Landau-Ginzburg model*** defined on 1+1 dimensional spactime (x^0, x^1) with target space

$$\mathcal{X} : \mathcal{A} = A + i\phi \quad \tilde{M}_3 := C \times \mathbb{R}_+$$

$$\mathcal{G} = \mathrm{Map}(\tilde{M}_3, G^c)$$

$$W(\mathcal{A}) = \int_{\tilde{M}_3} \mathrm{Tr}(\mathcal{A} d\mathcal{A} + \tfrac{2}{3}\mathcal{A}^3)$$

Gaiotto-Witten showed that in some situations one can reduce this model to an ungauged LG model with finite-dimensional target space.

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Theories of Class S

(Slides 77-91 just a reminder for experts.)

Begin with the (2,0) superconformal theory based on Lie algebra $\mathfrak{g}$

Compactify (with partial topological twist) on a Riemann surface C with codimension two defects D inserted at punctures $\mathfrak{s}_n \in$ C.

Get a four-dimensional QFT with d=4 N=2 supersymmetry S[$\mathfrak{g}$,C,D]

Coulomb branch of these theories described by a Hitchin system on C.

UV Curve

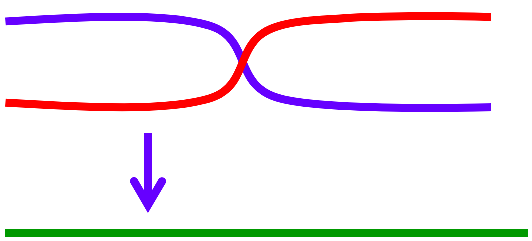
Seiberg-Witten Curve

$$\Sigma : \det(\lambda - \varphi(z, \bar{z})) = 0 \subset T^*C$$

$\lambda = p dq \qquad \lambda|_{\Sigma}$ SW differential

For $\mathfrak{g}$ =su(K) $\quad \pi : \Sigma \to C$

Σ



C

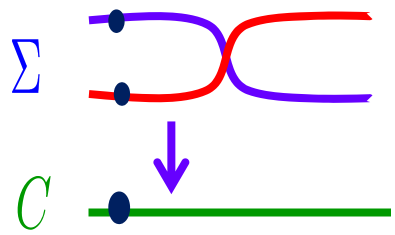
is a K-fold branched cover

$$\lambda^K + \lambda^{K-2}\phi_2(z) + \cdots + \phi_K(z) = 0$$

Canonical Surface Defect in S[$\mathfrak{g}$,C,D]

For $z \in C$ we have a ***canonical surface defect*** $\mathbb{S}_z$

It can be obtained from an M2-brane ending at $x^1=x^2=0$ in $\mathbb{R}^4$ and z in C

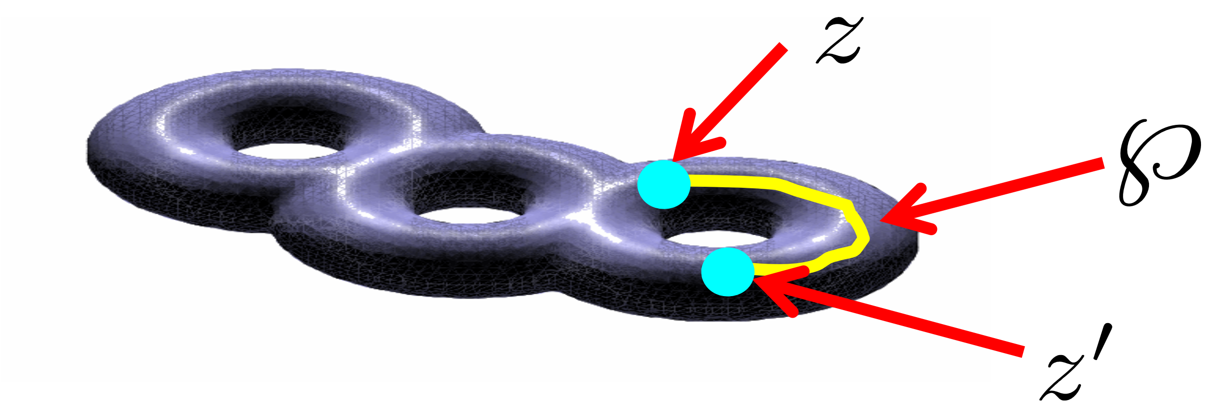


This is a 1+1 dimensional QFT localized at $(x^1,x^2)=(0,0)$ coupled to the ambient four-dimensional theory. At a generic point on the Coulomb branch it is massive.

In the IR the different vacua for this M2-brane are the different sheets in the fiber of the SW curve over z.

Susy interfaces for S[$\mathfrak{g}$,C,D]

Interfaces between $\mathbb{S}_z$ and $\mathbb{S}_{z'}$ are labeled by ***open paths*** $\wp$ on C

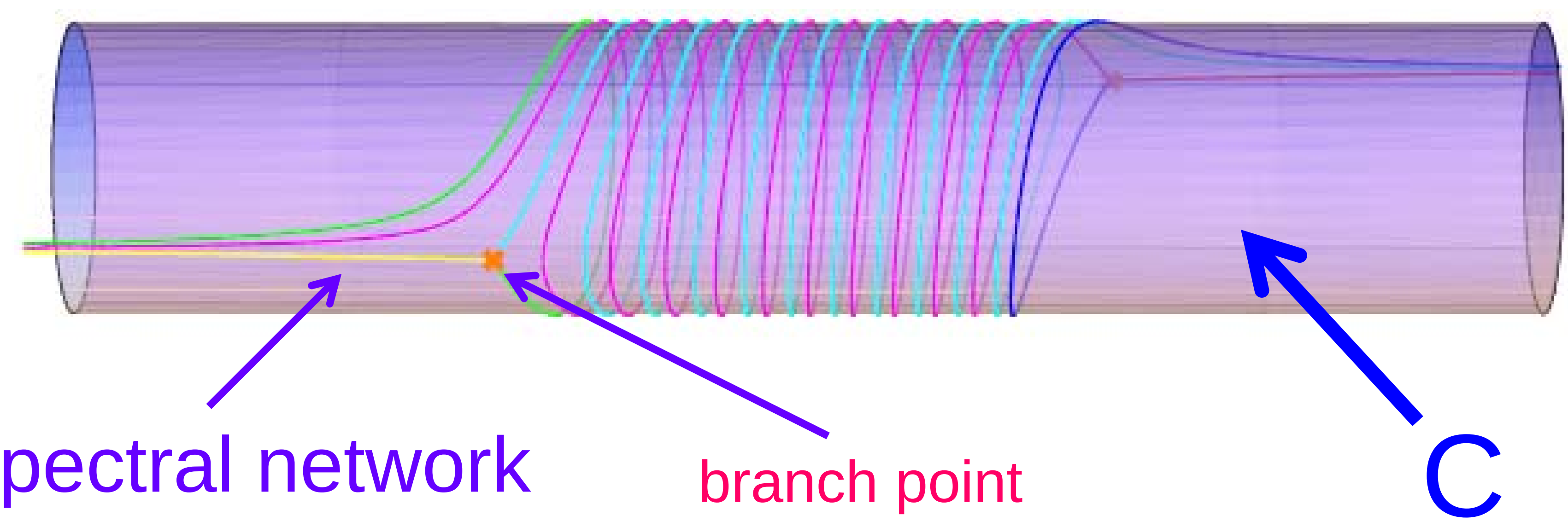


This data, together with an angle ϑ defines a susy interface $L_{\wp,\vartheta}$.

Spectral networks

(D. Gaiotto, G. Moore, A. Neitzke)

Spectral networks are combinatorial objects associated to a branched covering of Riemann surfaces $\Sigma \longrightarrow$ C with λ



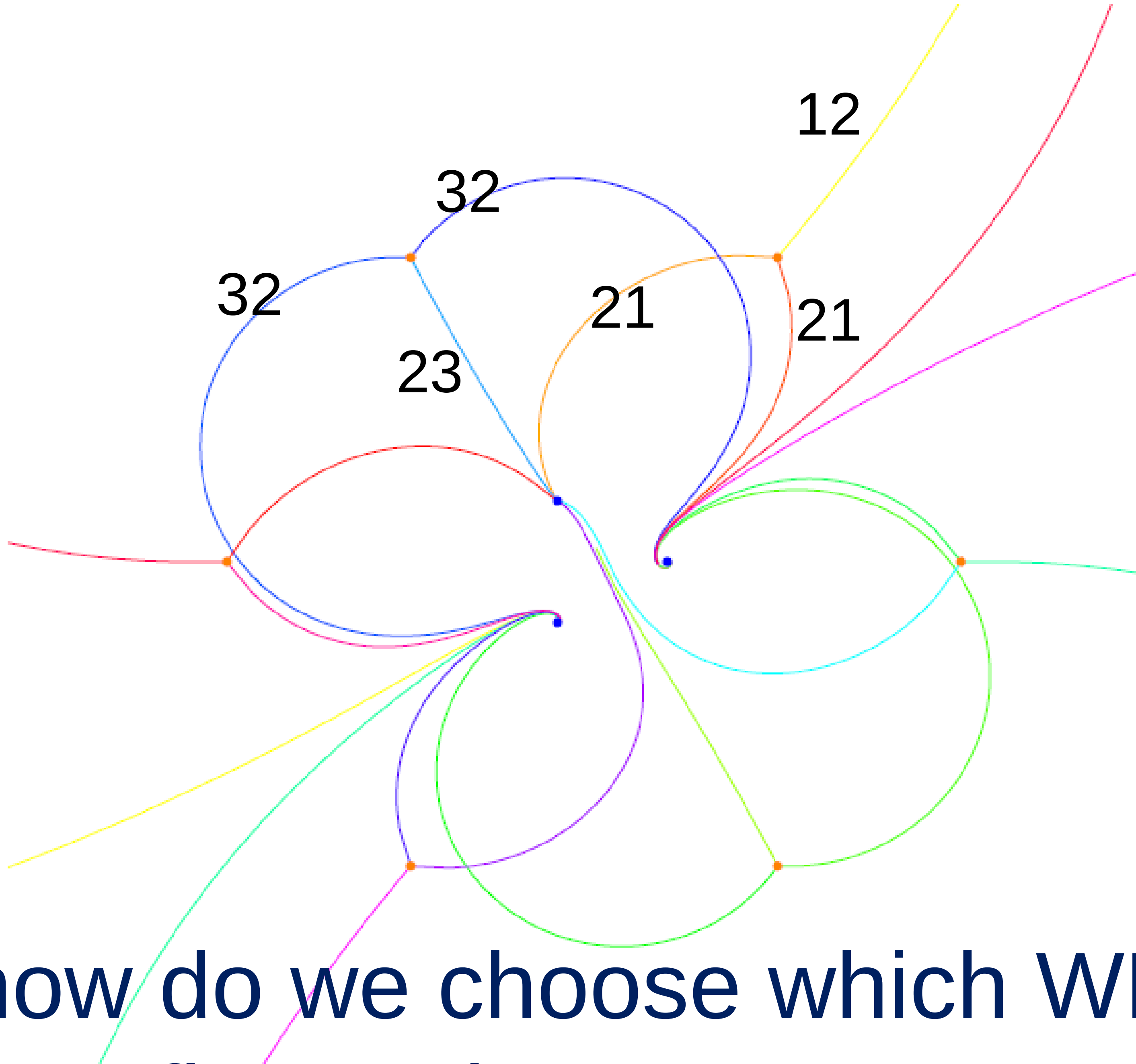
S-Walls

Spectral network $\mathcal{W}_\vartheta$ of phase ϑ is a graph in C.

Edges are made of WKB paths:

$$\langle \lambda_i - \lambda_j , \partial_t \rangle = e^{i\vartheta}$$

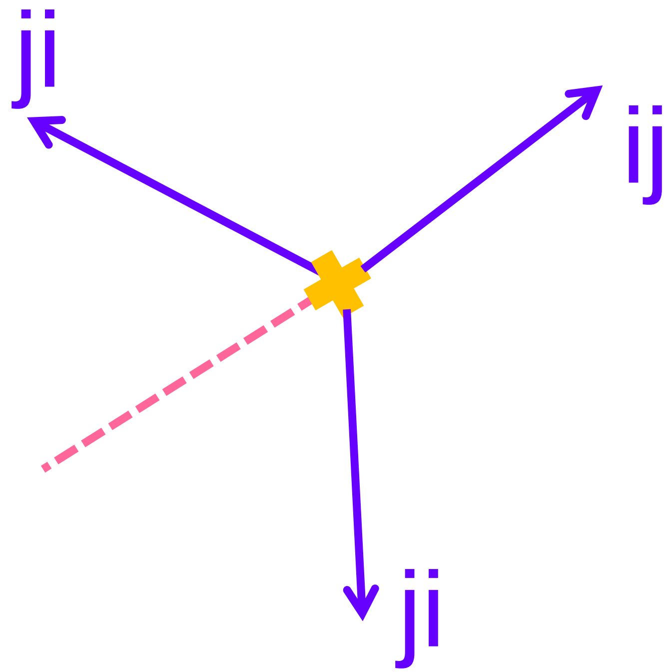
The path segments are ``***S-walls of type ij***''



But how do we choose which WKB paths to fit together?

Evolving the network -1/3

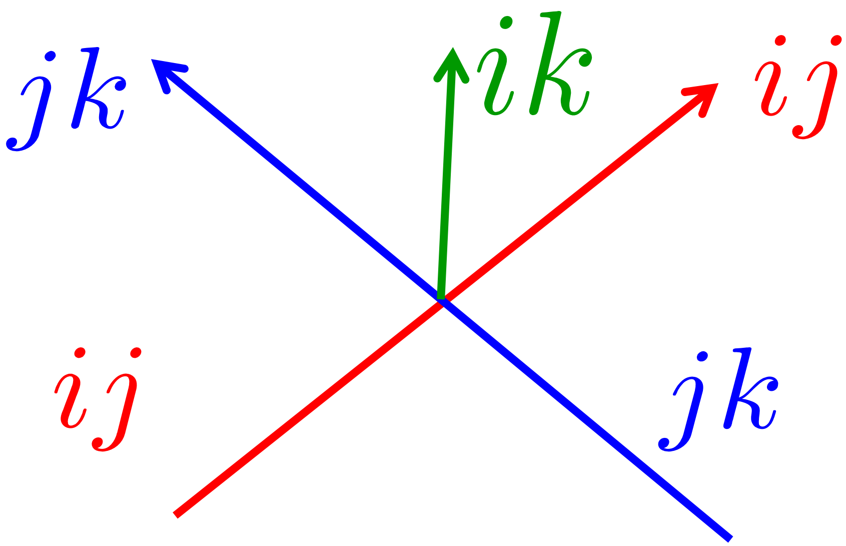
Near a (simple) branch point of type (ij):

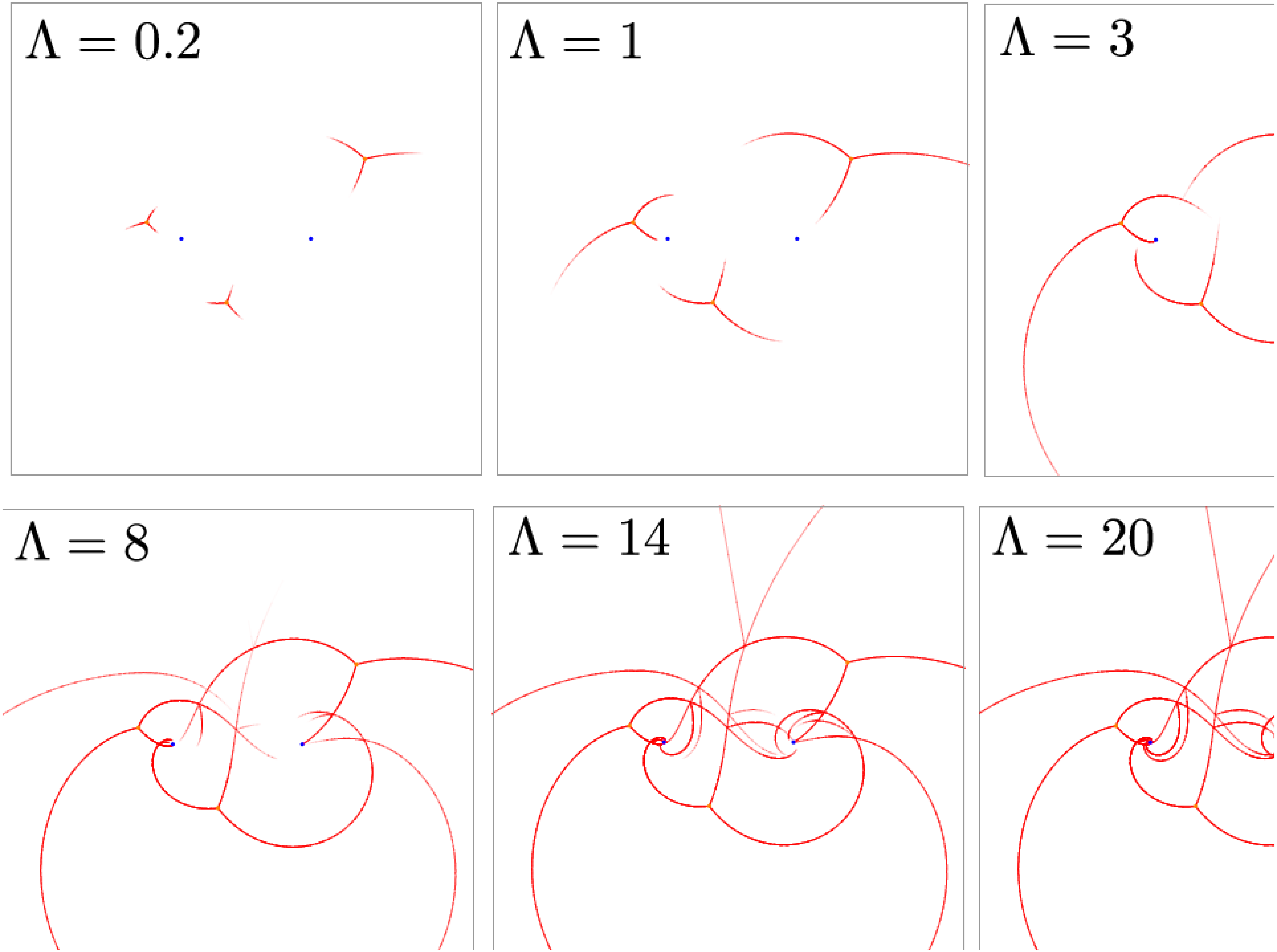


$$\int \lambda_i - \lambda_j \sim z^{3/2}$$

Evolving the network -2/3

Evolve the differential equation.
There are rules for how to continue
when S-walls intersect. For example:



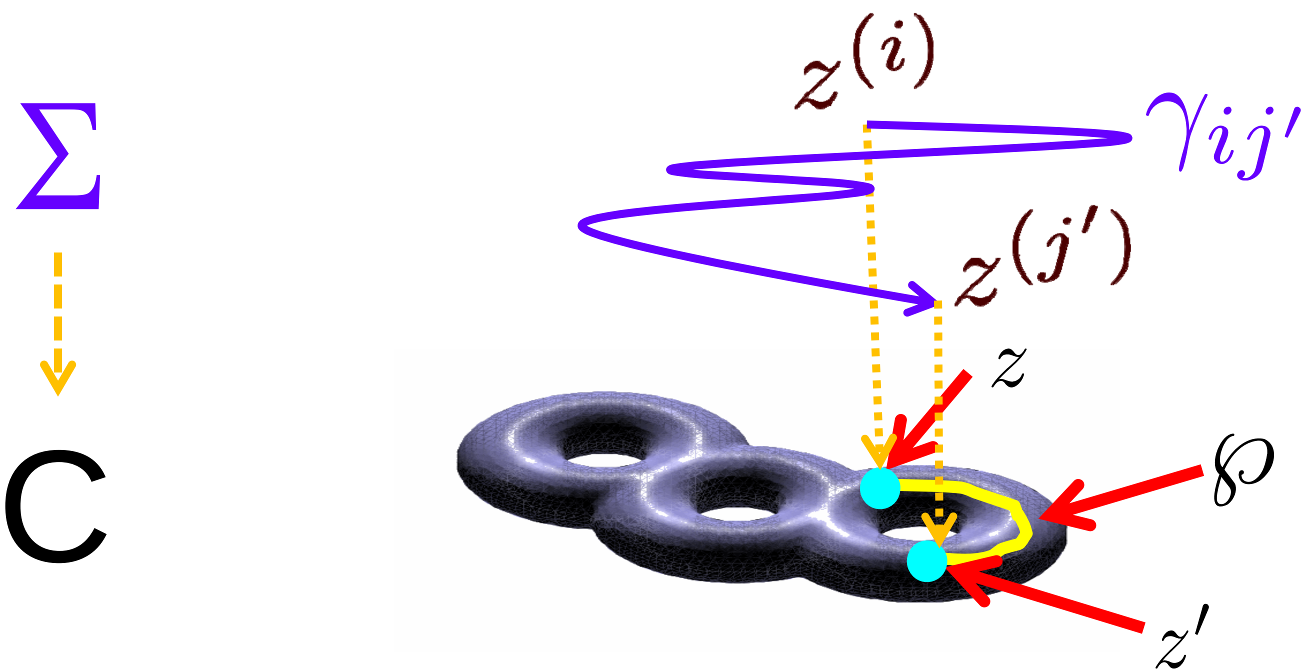


$\Lambda = 0.2$
$\Lambda = 1$
$\Lambda = 3$
$\Lambda = 8$
$\Lambda = 14$
$\Lambda = 20$

Formal Parallel Transport

Introduce the generating function of framed BPS degeneracies:

$$F(\wp,\vartheta) := \sum_{\Gamma_{ij'}} \overline{\underline{\Omega}}(L_{\wp,\vartheta},\gamma_{ij'}) X_{\gamma_{ij'}}$$



Homology Path Algebra

To any relative homology class
$a \in H_1(\Sigma,\{x_i, x_{j'}\}; \mathbb{Z})$ assign X_a

$$X_a X_b := \begin{cases} X_{a+b} & a, b \text{ composable} \\ 0 & \text{else} \end{cases}$$

X_a generate the "homology path algebra" of Σ

Four (Defining) Properties of F

1 $$F(\wp,\vartheta)F(\wp',\vartheta) = F(\wp\wp',\vartheta)$$

(``Parallel transport'')

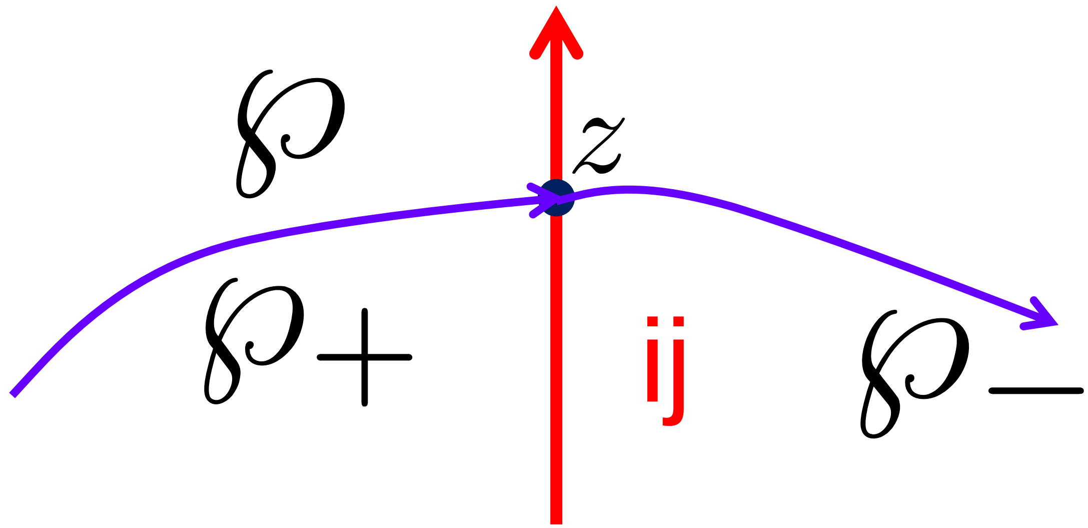
2 Homotopy invariance $$F(\wp_1,\vartheta) = F(\wp_2,\vartheta)$$

(``Flat parallel transport'')

3 If $\wp$ does NOT intersect $\mathcal{W}_\vartheta$: $$F(\wp,\vartheta) = \sum_{i=1}^{K} X_{\wp^{(i)}}$$

4 If $\wp$ DOES intersect $\mathcal{W}_\vartheta$: ``Detour rule''

Detour Rule



$$F(\wp,\vartheta) = \sum_{s=1}^{K} X_{\wp^{(s)}}$$

$$+ \sum_{\gamma_{ij}} \mu(\gamma_{ij}) X_{\wp_+^{(i)}} X_{\gamma_{ij}} X_{\wp_-^{(j)}}$$

Detour Rule = Wall-crossing formula for $\overline{\underline{\Omega}}(L_{\wp,\vartheta}, \gamma_{ij'})$

Theorem: These four conditions completely determine both F($\wp$,ϑ) and μ

One can turn this formal transport into a rule for pushing forward a flat GL(1,$\mathbb{C}$) connection on Σ to a flat GL(K,$\mathbb{C}$) connection on C.

``Nonabelianization map''

We want to categorify the parallel transport F($\wp$,ϑ) and the framed BPS degeneracies: $\overline{\underline{\Omega}}(L_{\wp,\vartheta}, \gamma_{ij'})$

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Summary – 1/2

1. Instantons effects can be thought of in terms of an ``effective theory'' of BPS particles.

2. This naturally leads to L_∞ and A_∞ structures.

3. Naïve categorification can fail. (Example of the BPS states on the interval and half-lines.)

4. We expect these algebraic structures to be universal identities for massive 1+1 N=(2,2) QFT.

(Because the web formalism can be formulated at this level of generality.)

Summary – 2/2

5. When there are paths of Landau-Ginzburg theories, one can define supersymmetric interfaces. Colliding these interfaces with the boundaries gives a map of branes.

6. This defines a notion of flat parallel transport of the A_∞ category of branes. Existence of this transport categorifies 2d wall-crossing.

Some Open Problems

1. What is the relation of interior amplitudes to S-matrix singularities?

2. Generalization to 2d4d systems: Categorification of the 2d4d WCF.

(Under discussion with T. Dimofte.)

3. Is the formalism at all useful for knot homology?