Hamburg Lectures on Spectral Networks Lecture 2

I didn't get to give the lecture...
Lecture 2: Defects, Gluing, and BPS States

1. An Important Special Case: Linear Conformal Quivers
2. General Description of Punctures
3. Gaiotto Gluing
4. Other Defect "operators"
5. BPS States: General Remarks
6. A Zoo Of Class S BPS States
7. Semiclassical Description of 4d BPS States & Hyperkähler Geometry
1. An Important Special Case

An important special case of the general discussion from the previous lecture is M-Theory on

\[ M^{11} \times C \times C \times \mathbb{R}^3 \]

\[ x^{0,1,2,3}, x^6 + ix^{10}, x^4 + ix^5, x^7, 8, 9 \]

\[ x^{10} \sim x^{10} + 2\pi \]

\[ ds^2 = dx^\mu dx_\mu + \left[ (dx^6)^2 + R(dx^{10})^2 \right] + \sum (dx^i)^2 \]

We follow an important paper of Witten: hep-th/9703166 and consider a system of \( K \) M5 branes with \( W_6 = M^{11} \times C \) at \( x^{4,5,7,8,9} = 0 \) together with \((n+1)\) transverse singly-wrapped M5's.
M5 wraps a holomorphic cone:

\[ \nu^{\frac{n}{\prod_{\alpha=0}^{n}} (t-t_\alpha)} \]

\[ t = e^{-(x^6+i x^{10})} \in \mathbb{C}^* \quad \nu = (x^4+i x^5)/R \]

LEET is a (2,0) theory on

\[ M^{1,3} \times (T^*C)_{\text{zero section}} \]

But with codimension 2 defects at \( t = t_\alpha \).

What do these do in the 4D field theory and in the Hitchin system?
To answer this question we reduce on the M-theory circle and get a configuration of branes.

At long distances \( \Rightarrow (x_6^\alpha - x_6^\alpha_{-1}) \)

The 4D effective gauge theory is a quiver gauge theory:

- In each interval \([x_6^\alpha, x_6^{\alpha+1}]\) the D4's give a \(U(k)\) gauge theory; see below.

- There are bifundamental hypermultiplets across each NS5

- \(k\) fundamentals from strings at end
The Coulomb branch vacua of this theory is defined by deformations of the above singular curve that do not change asymptotics at $\infty$.

$$\sum = \{ (t, v) \mid F(t,v) = 0 \}$$

$$\subset C \times C \cong T^*C$$

$$F(t,v) = \sqrt{v} \prod_{\alpha = 0}^{n} (t - t_{\alpha}) + \sum_{i=1}^{K} p_{i}(t) v^{k-i}$$

$k$ roots in $V = k$ sheets of

The $U(k)$ gauge theory as the D4's
Separate onto their Coulomb branches

For generic $p_i$, just one root goes to $\infty$ at $t \to t_\infty$

$$V_k(t) \sim \frac{1}{t-t_\alpha} \left( \frac{-p_i(t_\alpha)}{\prod_{\beta \neq \alpha} (t_\alpha-t_\beta)} + \ldots \right)$$

Also for $x^6 \to \pm \infty \iff \infty$

We should just have $k$ roots $\Rightarrow$

$\deg(p_i) \leq n+1 \implies \text{can also write}$

$$F(t, v) = \sum_{\alpha=0}^{n+1} t^{n+1-\alpha} q_{\alpha}(v) \quad \deg(q_{\alpha}) = k$$

Roots of $q_0(v)$: Roots in $v$ for $t \to \infty$

Have interpretation of fundamental string masses,

\[\text{Costs energy for these strings}\]
Let us understand the origin and nature of the U(k) gauge theory factors better.

Focus on one interval:

\[ X^6 \]

In the LEET theory on the D4's is 3+1 dimensional if we look at length scales \( \gg X^6 \). The DD strings inside each interval gives a separate U(k) SYM.

The Coulomb branch is obtained by the supersymmetric brane configuration:
Now naive reduction of the action for $D^4$

on $W_4 = M^{1/3} \times [x^6, x^6]$ gives

\[
\frac{1}{g_{\text{str}}} \int \text{Tr} \ F \wedge F \quad \longrightarrow \quad \frac{1}{g^2} \int \text{Tr} \ F \wedge F
\]

\[IM^{1/3} \times [x^6, x^6] \]

\[IM^{1/3} \times [x_\alpha, x_\alpha] \]

\[\frac{1}{g^2} = \frac{x^6 - x^6}{g_{\text{str}}} \sim \text{Re} \ (S_\alpha - S_{\alpha-1})
\]

(Recall $g_{\text{str}} = R$)

Now holomorphy dictates (Restoring factors of

$2\pi$ we were not

Careful about above)

\[-i\pi \tau \alpha = -i\pi \left( \frac{\theta \alpha}{2\pi} + \frac{4\pi i}{g^2} \right) = S_\alpha - S_{\alpha-1} \]
Actually this is slightly crude. The picture

\[ \text{NS}5 \quad \xrightarrow{\begin{array}{c} x_7 \to x_6 \\ \end{array}} \quad \text{D}4 \]

is only accurate at large scale. In fact the endpoint of the D4 defines a source for the Theory on the NS5 and

\[ \nabla^2 x^6 = \delta(\cdots) \]

So

\[ x^6 = \sum_{i=1}^{k} \log (v - v_i^n) \]

Put together with

\[ \frac{1}{g^2 m} \to \frac{x_\alpha - x_{\alpha-1}}{g^m} \]

This is a geometrization of the \( \beta \)-function equation. Taking into account the D4's from both sides gives:
In any case we should interpret the weak coupling limit $\left( \frac{t_{\alpha}}{t_{\alpha-1}} \right) \to 0$

This is a particular region of the complex structure moduli space of $\left( \mathbb{C} \mathbb{P}^1 : \{0, \infty\} \right) \setminus \{ t_1, \ldots, t_u \}$
Thus a weak coupling limit is associated with a particular degeneration of complex structure of the points $t_0, \ldots, t_n$. These will correspond to $S$-dual pictures of the same QFT.

Computing scalars $KE \Rightarrow \lambda = V \frac{dt}{t}$

Now from the nature of the roots of $F(t,v)$ we can deduce the behavior of the Higgs field:

$$\varphi(t) \sim \frac{dt}{t-t_\alpha} \begin{pmatrix} m_\alpha \cdots 0 \end{pmatrix} \quad t \to t_\alpha$$

$$\varphi(t) \sim \frac{dt}{t} \begin{pmatrix} v^{(0)} \vdots v^{(0)} \end{pmatrix} \quad t \to 0$$

$$\varphi(t) \sim \frac{dt}{t} \begin{pmatrix} v^{(\infty)} \vdots v^{(\infty)} \end{pmatrix} \quad t \to \infty$$

There are many ways to arrange the roots $t_0, \ldots, t_n$. These will correspond to $S$-dual pictures of the same QFT.
We can do a similar exercise with a 4D quiver gauge theory:

![Quiver Diagram]

The \( \beta \)-function at \( \alpha \)-th node is a positive multiple of

\[-2k_\alpha + k_{\alpha-1} + k_{\alpha+1} + d_\alpha \leq 0\]

Get more general punctures

\[\varphi \sim \frac{r}{t-t_\alpha}\]

\[Z(r) = \prod_{\beta=1}^{k} U(\beta)^{l_\beta}\]

A good way to encode this data is in terms of

\[\rho: \text{sl}(2) \rightarrow \text{sl}(K)\]

because that generalizes.

We also get irregular singular points

(Typical for asymptotically free theories.)
2. General Description Of Punctures

More generally, it is thought that there are \( \frac{1}{2} \) BPS codimension two defects in the \((2,0)\) theory.

So they are 4-diml objects that modify correlation functions - like boundary conditions. They are only characterized rather indirectly:

Recall that \( S(\mathcal{L})/ S'_{R_1} \times S'_{R_2} \) is \( N=4 \ d=4 \) SYM.
So we consider $S[\Sigma]$ on
$M^{1,2} \times S_{R_1} \times \hat{S}_{R_2}$

Reduction along $S_{R_2}$ is described at long distance by 5D SYM on $M^{1,2} \times S_{R_1} \times R_+$

Reducing along $S_{R_1}$ gives $d=4, N=4$ SYM on $M^{1,2}$

The defect is "defined" by the requirement that the boundary conditions on 3 of 6 scalars are

$$X_i \sim \frac{\phi(x_i)}{y} + \ldots$$
One can then argue that the induced singularity in the Hitchin system is

\[ \psi \sim \frac{r}{z} dz + \ldots \]

\( r \in \text{Nilpotent orbit: } \mathfrak{O}_{\rho^v} \)

Example \( \mathfrak{g} = \mathfrak{su}(k) \) \( \rho^v = \rho^1 \)

So \( \rho = 0 \iff [1, \ldots, 1] \)

\( \iff \rho^v = k \)

An interesting paper of Chacaltana-Distler-Tachikawa generalizes this statement to the claim that \( \rho \to \rho^v \) is known in Lie group theory as the "Spaltenstein map."

The defect has a global symmetry with Lie algebra \( \mathfrak{g}_D \)
The general class $S$ theory is then $S[\mathfrak{g}, C, D]$

$\mathfrak{g}$ - Lie algebra with all roots $\alpha^2 = 2$

$C$ - punctured Riemann surface

$D$ - collection of defects @ punctures

For suitable defects $D$ the theories are super conformal (in four dimensions) and have a manifold of couplings

$W_{g, D} = \text{complex structures on surface with labelled punctures.}$

Each defect contributes to global symmetry of 4D theory and

$\bigotimes_{D_a} f_{D_a} \subset \text{Global Symm}(S[\mathfrak{g}, C, D])$

might be larger
3. Gaiotto Gluing

Consider a weak coupling limit where the UV curve $C$ degenerates.

Describe this by a divisor $D \in \mathcal{M}_{g,n}$.

A description of the divisor is via the plumbing construction.

Identify $Z_1, Z_2 = q$.

$q$ a $1$ coordinate to $D$. 
There is an elegant description of the class $S$ theory in the limit $q \to 0$ due to D. Gaiotto. We consider Class $S$ theories associated to $C_1$ and $C_2$ with an extra puncture at $Z_1$ and $Z_2$ with full $SU(k)$ global symmetry. So the 4d theory

$$S[\{y, C_1, D_1 u \{D(2)\}\}] \times S[\{y, C_2, D_2 u \{D(2)\}\}]$$

has a global symmetry

$$SU(k) \oplus SU(k) \oplus \cdots$$

Now we gauge the diagonal $SU(k)_{\text{diag}}$ of the first two summands with gauge parameter

$$g = e^{2\pi i z}$$

Claim: This is the limiting class $S$-theory.
In this way one can reduce the theory to a pants decomposition:

\[ \text{pants decomposition} \]

Of \( VM \) for Lie algebra \( \mathfrak{g} \)

\[ = \] "trinion theory" with \( \mathfrak{g}_y \oplus \mathfrak{g}_y \oplus \mathfrak{g}_y \)

global symmetry

For \( \mathfrak{g}_y = \mathfrak{su}(2) \) the theory is a collection of hypermultiplets in the \( 2 \otimes 2 \otimes 2 \) representation. Otherwise some what mysterious – related to \( \mathfrak{W}_N \) algebras in the AGT correspondence.
Different torion decompositions correspond to different weak coupling limits of the theory and are related by $S$-duality.

The simplest case is an old observation about the geometrical interpretation of $S$-duality of $\mathcal{N}=4$ SYM (U(1)(k,0) ↪ $\mathbb{R}^{1,3}$ × $S^1_{R_1} × S^1_{R_2}$)

5D SYM for $U(K)$
on $\mathbb{R}^{1,3}$ × $S^1_{R_2}$

4D SYM for $U(K)$
on $\mathbb{R}^{1,3}$ with $\ell = \frac{R_2}{R_1}$
4. Other Defect "Operators"

The $(2,0)$ also has a class of 2-dimensional defect "operators." In the M-theory construction, they are associated with semi-infinite M2-branes ending on the M5.

In the class S context there are then a number of things we can do:
<table>
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<tr>
<th>(2,0) defect dim.</th>
<th>Embedding in $\mathbb{R}^{11,3} \times \mathbb{C}$</th>
<th>$d=4$ Field Theory Interpretation</th>
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<td>$S \times { \mathbb{Z} }$</td>
<td>Surface defect $S_2$</td>
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<tr>
<td>2</td>
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<td>4</td>
<td>$\mathbb{R}^{11,3} \times { \mathbb{Z}_2 }$</td>
<td>$D_a$ used to define $S[\mathbb{L}_4, \mathbb{C}, { D_a }]$</td>
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<td>4</td>
<td>$S \times \mathbb{C}$</td>
<td>modifies $S_2$</td>
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We will focus on $S_2$ and $L(p)$ later when describing spectral networks.
5. BPS States: General Remarks

The term "BPS states" is used in physical mathematics in many different ways.

Often it has something to do with

- solitons
- magnetic monopoles
- holomorphic curves
- coherent sheaves and/or SPLAG's w/ flat v.b.
- objects in an A_a category

But in physics it ultimately means a "state," i.e. a positive trace-class operator \( p \) on a Hilbert space with \( \text{Tr}(p) = 1 \). It is always a pure state so \( p = \text{Rank one projector} \).

\[ p = |\psi\rangle \langle \psi| \] and \( \psi \in \mathcal{H} \) is in an irred. rep. of Poincaré group, an induced rep. for \( \text{SO}(d) \to \text{Poin}(1,d) \) with \( m^2 > 0 \)

What is the relation?

Correspondence Principle
I will explain with a little digression on BPS solitons in $N=(2,2)$ field theory in 1+1 dimensions.

\[\begin{align*}
\phi &: M^{1+1} \to \mathcal{X} = TR \\
U(\phi) &= (\phi^2 - v^2)^2 \text{ or more generally a potential like}
\end{align*}\]

\[\begin{align*}
H &= \int_\mathbb{R} dx \left( (\Box \phi)^2 + U(\phi) \right) < \infty
\end{align*}\]

Induces topology on $C(\mathbb{R} \to \mathcal{X})$;
Connected components are determined by 4 boundary conditions

\[\begin{align*}
\phi(X) &\to \text{element of } \{\phi_+, \phi_-\} \\
X &\to \pm \infty
\end{align*}\]
Phase space $P = \prod_x P_x$
and we separately quantize the 4 connected components.
If $x = (\phi_-, \phi_+)$ then the field looks like:

energy density for minimal energy field config. with these bosons is

$\Rightarrow$ Soliton - behaves like a particle.
Now in free field theory there are "coherent states" - quantum states that correspond to well-defined classical field configurations for $\hbar \to 0$.

Harmonic Oscillator \( z \in \text{PhaseSpace} = \mathbb{R}^2 \)

\[
\psi_z = e^{-\frac{1}{2}z^2} e^{za^+} |0\rangle
\]

Compute

\[
\langle \psi_z | p(a) | \psi_z \rangle \\
\langle \psi_z | q(a) | \psi_z \rangle
\]

find just like in classical mechanics.

In weakly coupled field theory
to a classical field config. \( \phi_{\text{sol}}(x) \)

we try to construct a state \( |\psi_{\text{sol}}\rangle \)

So that

\[
\langle \psi_{\phi_{\text{sol}}} | \hat{\phi}(x) | \psi_{\phi_{\text{sol}}} \rangle = \phi_{\text{sol}}(x)
\]
In general there can be important quantum corrections to this story — for example, even computing the exact energy of a coherent eigenstate of the Hamiltonian \( | \Psi_{\text{sol}} \rangle \) is in general out of reach.

However in field theories with extended supersymmetry we can do better.

**Susy** \( \{ Q, Q^+ \} = H \)

**Extended Susy** \( \{ Q_i, Q_j^+ \} = \delta_{ij} H \)

\( i, j = 1, \ldots, N \)

But now \( \{ Q_i, Q_j \} \neq 0 \) is possible

e.g. for \( N=2 \) \( \{ Q_1, Q_2 \} = \hat{Z} \)

\[ \left[ \hat{Z}, H \right] = 0 \]

is a consistent susy operator algebra.
It turns out that in these theories, when we quantize $\mathcal{P} = \frac{1}{\alpha} \mathcal{P}_x$
the $\hat{Z}$ operator becomes a scalar that just depends on the component $\alpha$

$$\{Q_1, Q_2\} = \mathcal{Z}_\alpha \cdot 1$$

In our soliton case $\alpha$ is ordered pair of classical vacua $\phi_\pm$

More generally $\alpha$ is typically a Chern class or an element of a K-theory lattice.
Write $Z_\alpha = e^{i\alpha}[Z]$.

When working out the induced rep^h of the super-Poincaré algebra you first quantize the Clifford algebra.
\[ \{ Q_i, Q_i^+ \} = \delta_{i,j} M \]
\[ \{ Q_1, Q_2 \} = \mathbb{Z} \]

Diagonalize the quadratic form:
\[ Q_1 = Q_1 - e^{i \varphi_x} Q_2 \]
\[ Q_2 = Q_1 + e^{-i \varphi_x} Q_2 \]

\[ \{ Q_1, Q_1^+ \} = 2 (M - |Z_a|) \Rightarrow M \geq |Z_a| \] (Bogomolnyi bound)

\[ \{ Q_2, Q_2^+ \} = 2 (M + |Z_a|) \]

\[ M > |Z_a| \Rightarrow \text{minimal Clifford rep. } C \]
\[ M = |Z_a| \Rightarrow \text{minimal Clifford rep. } C \]

Unitarity \( \Rightarrow \) \( Q_1 = Q_1^+ = 0 \) in rep \( n \)

**Def:** \( \mathcal{H}_{\text{BPS}} = \{ \psi \mid H\psi = |Z_a\rangle \langle \psi | \} \)
$H, Z$ are functions of parameters (such as $u \in B$ in the Coulomb branch)

In order to count BPS states in a stable way, introduce an operator ("chago") so that

$$\left[ Q_0, Q_{1,2}^+ \right] = Q_{1,2}^+$$

$$\text{Tr} \times \delta = \begin{cases} (1+x)^2 & \text{long rep } C \otimes \hat{C} \\ 1+x & \text{short rep} \end{cases}$$

$$\left. \frac{d}{dx} \left[ \text{Tr} (x \delta) \right] \right|_{x=-1} = \sum_0$$

$$\left. \frac{d}{dx} \left[ \text{Tr} _x \delta \right] \right|_{x=-1} = \Omega(\alpha)$$

"Counts" BPS States.
6. A Zoo Of Class S BPS States

In Class S with defects there are several kinds of BPS states relevant to our story.

- 4d BPS particles

\[ \alpha \rightarrow \chi \in \Gamma_u = \text{electromagnetic charge lattice} \]

(subquotient of) \( H_1(\Sigma, \mathbb{Z}) \)

\[ Z_\chi = \oint \lambda \]

\[ \Omega(\theta; u) \text{ piecewise constant in } u \text{ satisfies KSWCF} \]

- 4d Framed BPS States

Defined in presence of line defect \( k(p, S) \)

\( S = \text{phase used in defining the line defect} \)

BPS particles can bind.
\( \alpha \in \Gamma_L = \text{torsor for } \Gamma \)

\[
\overline{\omega}(\sigma^j, \cdot) : \Gamma_L \rightarrow \mathbb{Z}
\]

Satisfies a simpler WCF (\( \Rightarrow \) KSWCF)

Used in Darboux expansion of \( v\nu \)'s

\[
\langle \mathcal{L}(P, S) \rangle = \sum_{\delta \in \Gamma_L} \overline{\omega}(\sigma, S, \delta) y_{\delta}
\]

- **Canonical Surface Defect Soliton**

Degeneracies

\[
\sum
\]

\[
\Sigma \xrightarrow{z^{ij}} \mathcal{S}_z
\]

\[
\Gamma^i_j(Z, Z) = \left\{ \zeta \in C^1(\Sigma, \mathbb{Z}) \mid \delta \zeta = \frac{Z^i}{Z} - \frac{Z^j}{Z} \right\}
\]

Pre-images \( \xleftrightarrow{\text{vacua for } \mathcal{S}_z} \)

\[
\Gamma(Z, Z) = \bigcup_{i, j} \Gamma^i_j(Z, Z) \ni \alpha
\]
\[ Z_2 = \oint \lambda \sim \text{Difference of critical values of superpotential} \]

Degeneracies \( \mu(x) = \text{Tr} \ F(-1) F \)
in theory of surface defect.

- Framed Degeneracies For Interfaces

\[ S_z : \text{1+1D theory whose couplings depend on } z. \]

Imagine varying couplings with the 1D spatial variable:

\[ x \]
\[ t \]

\[ S_{z_1} \]
\[ S_{z_2} \]
$Z(x)$ describes a path in $C$

At long distance we have a line defect inside the surface defect

and we have analogs of framed BPS states

$$\alpha \in \Gamma_1 \left( \mathcal{L}_1, \mathcal{L}_2 \right) = \bigcup_{i,j} \Gamma_{ij} \left( \mathcal{L}_1, \mathcal{L}_2 \right)$$

$$\Gamma_{ij} \left( \mathcal{L}_1, \mathcal{L}_2 \right) = \left\{ \bar{\alpha} \in C \left( \mathcal{L}_1, \mathcal{L}_2 \right) \mid \partial \bar{\alpha} = \mathcal{L}_1^{(i)} - \mathcal{L}_2^{(j)} \right\}$$

$$\bar{\alpha} \sim \bar{\alpha} + \partial \sigma$$

$$Z_{\alpha} = \sum_{\alpha} \Omega \left( \rho, \mathcal{S}, \cdot \right) \rightarrow \mathbb{Z}$$
7. Semiclassical Description

There are many ways to define the BPS degeneracies. One nice way applies to Lagrangian $d=4$ $N=2$ theories. So, these are defined by the data:

- $G$ - compact s.s. Lie group
- $\mathcal{R}$ - quaternionic representation

We need to work "at infinity" in $\mathcal{B}$ in regions corresponding to weak coupling (Recall $S_2(Yiu)$ is piecewise constant, jumping only on real cod. 1 walls of marginal stability.)

In these regions we have a canonical duality frame:

$$\Gamma \simeq \Gamma_{mg} \oplus \Gamma_{el}$$

$$\Gamma_{mg} \simeq \Lambda \text{cocharacter} (G)$$

$$\Gamma_{el} \simeq \Lambda \text{weight} (G)$$
\[ \gamma = \gamma_m \oplus \gamma_e \]

\( \gamma_m = \text{magnetic charge, determines a magnetic monopole moduli space} \)

\[ U_{\text{mag mon}} (\gamma_m, X_\alpha) = \{ F = * \nabla \chi \text{ on } \mathbb{R}^3 \} \]

\[ X_\alpha = \text{Re}(\xi^{-1}_a) \in \mathfrak{t} \]

asymptotic Higgs field.

We have to write

\[ U_{\text{mag mon}} = \mathbb{R}^3 \times \frac{\mathbb{R} \times M_{\text{strong. cent.}}}{\mathbb{Z}} \]

leading to a lot of technical headaches.

Now \( \mathbb{R} \) determines (via a universal construction) a hyper-holomorphic bundle

\[ E_{\mathbb{R}} \to U_{\text{mag mon}}. \]
We then consider the Dirac operator $\mathcal{D}$ coupled to $E_R \rightarrow \mathcal{U}_{\text{magmon}}$.

(Actually, it is not exactly the D.O. Rather we add Clifford mult. by a hyperhalo, v.f. determined by $\gamma = \text{Im} \gamma_0$.)

Suitably separating out the center of mass, the Hilbert space of BPS states with magnetic charge $\gamma_m$ is just

$$\text{Ker}_{L^2} \mathcal{D} \bigg| = \left\{ \text{BPS states with } \gamma = \gamma_m \oplus * \left( \mathcal{U}_{\text{strong}.\text{cent.}} \right) \right\}$$

Now $T \subset G$ has a hyper-halo action on $\mathcal{U}_{\text{strong}.\text{cent.}}$, lifts to $SO_R$ and commutes with the Dirac operator. The isotypical $\chi_{ee} \wedge \omega_t(G)$ space gives

$$\mathcal{H}_{\text{BPS}} \cong \left( \text{Ker}_{L^2} \mathcal{D} \bigg| \right)_{\mathcal{U}_0}$$
This space is also a representation of the rotation action $SU(2)$ on $\mathbb{R}^3$ and

$$S_2(\mathfrak{m} \oplus \mathfrak{p}_e) = \text{Tr}_{\mathfrak{m} \oplus \mathfrak{p}_e} (-1)^{2J_3},$$

There is a very similar description of framed BPS degeneracies $\Sigma$ using singular monopole --

See my papers with

D. van den Bleeken
D. Brennan
A. Royston

for more details.