GGI LECTURE 4:
THE SPECTRUM-GENERATING
STOKES MATRIX: FLIPS, TWISTS
ε POPS
HITCHIN SYSTEMS & FLAT CONN'S

THE H.E.'S \implies

\mathbf{A} = \frac{R}{S} \varphi + \mathbf{A} + R\mathbf{S} \mathbf{\check{\varphi}}

IS FLAT.

NEAR REG. SING. POINT Z_i

\mathbf{A} \sim \left( \frac{R}{S} \frac{P}{Z^2} + \frac{\alpha}{2i} \right) \frac{dz}{Z - Z_i} + \left( R \mathbf{S} \frac{\check{P}}{Z^2} - \frac{\alpha}{2i} \right) \frac{d\bar{Z}}{Z - Z_i}

SO B.C.'S FIX MONODROMY OF \mathbf{A}.

AROUND Z_i:

\mathbf{M}_i \sim \begin{pmatrix} \mu_i & \mu_i^{-1} \\ \mu_i^{-1} & \mu_i \end{pmatrix}

\mu_i = \exp \left[ 2\pi i \left( \frac{1}{2} \mathbf{S} \mathbf{\check{R}} \mathbf{m}_i - \mathbf{m}_i^3 - \frac{1}{2} \mathbf{S} \mathbf{\check{R}} \mathbf{\check{m}_i} \right) \right]
THEOREM OF C. SIMPSON

IDENTIFY $M^5_\gamma$, $\gamma \in \mathbb{C}^*$, WITH MODULI OF FLAT $SL(2,\mathbb{C})$ CONNECTIONS WITH PRESCRIBED MONODROMY AT $z_i$.

MOREOVER, THE HOLONOMIC SYMPLECTIC FORM ON $M^5_\gamma$ HAS THE SIMPLE FORM:

$$\omega^5_\gamma = \int_C \text{Tr}(\delta A \delta A)$$
2. **FOCK-GONCHAROV COORD’S AND CLUSTER TMNS**

\[ U = \text{a flat } \text{SL}(2, \mathbb{C}) \text{ connection} \]

**WITH MONODROMY** \( M_i \) **AROUND** \( z_i \)

\[ M_i \sim \left( \frac{1}{\mu_i}, \mu_i \right) \]

**A. DECORATED TRIANGULATIONS**

**DEF:** A "DECORATED TRIANG." \( T \) IS AN IDEAL TRIANGULATION OF \( C \) WITH VERTICES AT \( z_i \); TOGETHER WITH A CHOICE OF MONODROMY EIGENVALUE \( \mu_i \) OR \( \frac{1}{\mu_i} \) AT EACH \( z_i \).
B. FLIPS AND POPS

DEFINE A GROUPOID: DECORATED
OBJECTS = TRIANGULATIONS

MORPHISMS ARE GENERATED BY FLIPS POPS

\[ \text{FLIP: } \sigma_E \text{ FOR } E \in \mathcal{E}(T) \]

\[ \text{Q}_E : \]
\[ \text{Q}_{E'} : \]

\[ \text{POP: } \pi_i \text{ FOR } z_i \in V(T) : \]

EXCHANGE: \[ \mu_i \leftrightarrow \mu_i^{-1} \]
RELATIONS ON FLIPS $\xi$ POPS

1. $\sigma_E^2 = 1$ AND $\pi_i^2 = 1$

2. POPS COMMUTE

3. $\sigma_E, \sigma_{E'}$ COMMUTE IF $Q_E, Q_{E'}$ DO NOT SHARE A TRIANGLE

4. IF $Q_E, Q_{E'}$ SHARE A TRIANGLE

LATER WE WILL ENHANCE OUR GROUPOID TO INCLUDE "LIMIT TRIANGLES" AND "TWISTS"
C. FG COORDINATES

GIVEN A DECORATED TRIANGULATION OF $C$, $\mathcal{F}_i \mathcal{G}$ DEFINE A COLLECTION OF FUNCTIONS ON $\mathcal{M}$:

$$\chi : T \rightarrow \{ \chi^T_E \}_{E \in \mathcal{E}(T)}$$

**DEFINITION:**

$$Q_E : \begin{array}{c} Z_3 \\ E \\ Z_1 \end{array}$$

**CHOOSE FLAT SECTIONS $S_i$ OF SPECIFIED MONODROMY NEAR $Z_i$:**

$$\chi^T_E = -\frac{(S_1 \land S_2)(S_3 \land S_4)}{(S_2 \land S_3)(S_4 \land S_1)}$$
\( \chi_E^T := - \frac{(s_1 \land s_2)(s_3 \land s_4)}{(s_2 \land s_3)(s_4 \land s_1)} \)

- \( s_i \land s_j \in A^2_E = \text{LINE BUNDLE} \)
- \( \text{PARALLEL TRANSPORT TO ANY POINT } q \in \gamma_E \)
- \( \text{NORMALIZATION OF } s_i \text{ CANCELS} \)

**THEOREM:** \( (F^i \gamma C) \{ \chi_E^T \}_{i_E} \) PROVIDE HOLD COORDINATES ON OPEN SET \( U_T \) OF \( M \).

**D. COORDINATE TMN'S**

NOW DESCRIBE THE COORD.

TMNS AS WE CHANGE THE DECORATED TRIANGULATION \( T \rightarrow T' \).
**Transformation Under Flips**

Only the edges in red change:

\[
\chi_{E'}^{T'} = -\frac{s_4 s_1 s_2 s_3}{s_1 s_2 s_8 s_4} = \frac{1}{\chi_E^T}
\]

\[
\chi_{E_{12}}^{T'} = \chi_{E_{12}}^T (1 + \chi_E^T)
\]

"Cluster Transformations"
Transformation under Pops

A Pop at vertex $z_i$ changes the edge coordinates in red.

It is possible to write explicit formulae for the Pop transformation, but they are complicated....

Importantly!

It turns out that the product of all Pops $\prod_{i=1}^{V}$ is relatively simple...
E. SYMPLECTIC STRUCTURE

• USING THE SYMPLECTIC STRUCTURE \( \omega \), ONE CAN SHOW

\[
\{ X^T_E, X^T_{E'} \} = \langle E, E' \rangle X^T_E X^T_{E'},
\]

• TRANSFORMATIONS UNDER FLIPS \( \xi \) AND POPS ARE POISSON

Sketch proof.
3. WKB TRIANGULATIONS

A. MOTIVATION

RECALL THAT A KEY PROPERTY OF $\chi_y(s)$ ARE THE $s \to 0$ ASYMPTOTICS:

$$\lim_{s \to 0} \chi_y(s) e^{-\frac{\pi R}{s} Z_y(u)} \sim \text{FINITE}$$

$\Rightarrow$ WE NEED TO USE VERY SPECIAL TRIANGULATIONS FOR WHICH WE CAN PROVE SUCH ASYMPTOTICS.

IDEA: USE THE WKB APPROXIMATION TO DESCRIBE THE FLAT SECTIONS:

$$(d + A) s = 0$$
\[ A = \frac{R}{S} \varphi + A + R S \bar{\varphi} \]

For \( S \to 0 \), \( S \sim h \)

\[ (S \partial + R \varphi + \alpha \delta) S = 0 \]

Recall: \( \varphi \sim (\lambda - \lambda) \)

**WKB:**

\[ S \sim \exp \left( \frac{-R}{S} \int Z \sigma^3 \right) S_0 \]

\( S \to 0 \)

From this get asymptotics of FG coord.
B. WKB CURVES

However the WKB approx. is notoriously subtle.

Exponentially small corrections can grow in $z$ and invalidate computations for validity of WKB approx.

We must restrict to very special triangulations $T(\varphi, \lambda)$ whose edges are WKB curves

Def: WKB curve with angle $\varphi$: curve on $C$ with

\[ \langle \gamma, d \nu \rangle = \pm e^{i \varphi} \]

$\Rightarrow$ WKB foliation of $C$. 
**NOTE:** WKB CURVES GET TRAPPED BY SINGULARITIES:

\[ \lambda = \frac{m}{2} \frac{dz}{z} \implies z(t) = z_0 \exp \left( -\frac{e^{i\theta}}{m} t \right) \]

![Diagram of WKB curves with singularity](attachment:image.png)

THREE KINDS OF WKB CURVES:

- **GENERIC**: Both ends on \( z_i, z_j \)

- **SEPARATING**: Connects branch point \( w_a \) to singular point \( z_i \)

- **FINITE**: Closed, or both ends on turning points \( w_a, w_b \)
Note that our rule for BPS states was that \( \exists \theta_* \) for which there is a finite WKB curve.

**Important fact:** for generic values of \( \theta \) there are no finite WKB curves. But at special critical values of \( \theta \) there are finite WKB curves.

Recall that for finite WKB curves \( \langle \lambda, \delta_\theta \rangle = e^{2i\theta_*} \) with \( \theta_* = \text{arg} \ Z \).

So: the critical values are the phases \( \theta_* \) of BPS states.
C. Definition

To define our triangulation we first use the separating curves to split C into WKB cells.

Locally:

For generic \( \lambda, \Sigma \) it turns out there are only two kinds of cells:
For the WKB triangulation we choose a generic WKB curve in each cell:

Degenerate triangle:
**Choice of $\mu_i$**

Recall that we must define a "**decorated triangulation**".

$$(\lambda, \psi) \Rightarrow \text{**distinguished eigenvalue** of } M_i$$

"**Small flat section**: The flat section which decays along the WKB curve going into the singularity"

These are the sections for which we have good control in WKB appx.

More detail

Denote the resulting decorated triangulation $T(\psi, \lambda)$
D. Morphisms of WKB Triang's

Vary $\mathcal{V}$ $\Rightarrow$ (Homotopy class of)

$T(\mathcal{V}, \lambda)$ is unchanged

Except at critical values $\mathcal{V}_c$

Where finite WKB curves develop.

When varying $\mathcal{V}_c$

$T(\mathcal{V}, \lambda)$ jumps precisely at the values of phases of BPS states!
For generic $\lambda$ a jump in $T(\mathcal{J}, \lambda)$ only happens when a separating curve degenerates to a finite WKB curve joining turning points $W_a, W_b$.

Thus, for generic $\lambda$ there are only two kinds of jumps:

- Either $W_a \neq W_b$
- Or $W_a = W_b$
**HYPERMULTIPLE JUMP:** $w_a \neq w_b$

$\mathfrak{g} < \mathfrak{g}_c$

$\mathfrak{g} = \mathfrak{g}_c$

$\mathfrak{g} > \mathfrak{g}_c$

**THIS IS JUST A FLIP**
VECTOR MULTIplet JUMP: $w_a = w_b$

As $\vartheta \to \vartheta_c^-$, $T(\vartheta, \lambda)$ has an infinite sequence of flips $E_+ E_- E_+ E_- \cdots$
AT \ \mathcal{V} = \mathcal{V}_c

\mathcal{V} = \mathcal{V}_c

AT \ \mathcal{V} > \mathcal{V}_c

T_{-m}
Suitable combinations of $\chi_{E^+_m}, \chi_{E^-_m}$ have limits for $m \to \infty$:

$$\chi_{A_{\pm \infty}} = \lim_{m \to \infty} \chi_{E^+_m} \chi_{E^-_m}$$

$\Rightarrow$ Add new objects $T_{\pm \infty}$ to the groupoid, call them "limit triangulations".

New morphisms:

$$T_{+\infty} \rightarrow T_{-\infty}$$

called "twists"
4. DEFINING THE TWISTOR COORD'S

FINALLY, TO DEFINE $X^\nu_y(\cdot,5)$

WE ASSOCIATE TO $E \in \mathcal{E}(\Pi(\mathcal{O},\lambda))$

CERTAIN CYCLES $\gamma^\nu_E \in H_1(\Sigma,\mathbb{Z})$

need to motivate better the $\gamma^\nu_E$ and distinguish them from $\chi_y$.

RULE: ORIENT THE LIFTS $\hat{E}$ SO THAT $e^{-i\nu} \langle \lambda, \delta_t \rangle > 0$ : +VE

DEMAND $\langle \gamma^\nu_E, \hat{E} \rangle = +1$
THE \{ \gamma_E^\vartheta \}_{E \in E(T)} \text{ FORM A (POSITIVE) BASIS FOR } \Gamma.

NOW DEFINE:

\[ X_{\vartheta E}^E := X_{E}^T(\vartheta, \lambda) \]

\[ X_{\vartheta, \vartheta'} := X_{\vartheta}^\vartheta \cdot X_{\vartheta'}^\vartheta \]
**THEOREM 1:** IF $R \to \infty$ AND $S$ IS IN $H_{\theta}$, THEN

$$X_{y}^{\nu}(\cdot, \Sigma) \sim \exp \left( \frac{\pi R}{\Sigma} \mathbf{Z}_{y} + i \theta_{y} + \pi R s \bar{Z}_{y} \right)_{R \to \infty}$$

RECOVERS NEITZKE-PIOLINE SEMIFLAT TWISTOR COORDINATES.

**Proofs:**

$$S_{i} \sim \exp \left( \pm \frac{R}{\Sigma} \int_{Z_{i}}^{\infty} \right) \cdot \left\{ (1) \lor (i) \right\}$$

USE RELATION TO 2D SINE-GORDON
THEOREM 2: WITH RESPECT TO SYMPLECTIC STRUCTURE:
\[ \tilde{\omega}_\tau = \int C \text{Tr} \, S A A A \]
\[ \{ \chi^\omega \chi^\omega \} = \langle \omega, \omega \rangle \chi^\omega \]

THEOREM 3: AT SUFFICIENTLY LARGE R
\[ \chi_\gamma(\cdot, 5) = \chi_\gamma^{\theta = \text{arg} 5}(\cdot, 5) \]
Satisfy the 5 defining properties.
**Proof:**

(1) \( X_y(\cdot,\delta) \) holomorphic on \( M^5 \):
Fock & Goncharov

(2),(3) Follow easily from the definition

(48) For \( \delta \to 0 \) in the half-plane

\[
H_\theta = \begin{pmatrix}
1 & e^{i\omega} \\
0 & 1
\end{pmatrix}
\]

\[
\lim_{\delta \to 0} X_y^a(\delta) \exp\left(-\frac{\pi R}{\delta} Z_y\right) \text{ exists}
\]

Follows from WKB asymptotics

As with \( R \to \infty \)
(5) IF \( \psi = \psi_c \) IS THE PHASE OF A BPS STATE OF CHARGE \( \chi_0 \) THEN, DEFINING

\[
\chi^\pm_y = \lim_{\psi \to \psi_c^\pm} \chi^\psi_y
\]

\[
\chi^+_y = \chi^-_y (1 - \sigma(\chi_0)\chi^-_{y_0})^{\Omega(\chi_0)<\chi, \chi_0>}
\]

NOTE: \( \sigma(\chi_0) = +1 \), \( \Omega(\chi_0) = -2 \) VM

\( \sigma(\chi_0) = -1 \), \( \Omega(\chi_0) = +1 \) HM

1. FOR HM: CLUSTER TMN.

2. FOR VM: EXPLICIT COMPUTATION OF TWIST TMN:

\[
\chi^{T_+ \to \infty} \rightarrow \chi^{T_- \to \infty}
\]
5. $R \to \infty \text{ LIMIT } \frac{1}{\varepsilon} \text{ SINH-GORDON}$
6. WALL CROSSING

CHOOSE $\vartheta_- < \vartheta_+$ TO

DEFINE A CONVEX CONE IN COMPLEX

\[ V \]

\[ \arg -z = \vartheta_+ \]
\[ \arg -z = \vartheta_- \]

SUPPOSE WE FOLLOW A PATH $u_-$ TO $u_+$ SO THAT NO BPS RAY CROSSES $\arg (-z) = \vartheta_\pm$.

THEN $T(\vartheta_\pm, \lambda_-)$ SMOOTHLY EVOLVES TO $T(\vartheta_\pm, \lambda_+)$
On the other hand, evolving $\mathcal{I}_-$ to $\mathcal{I}_+$ at fixed $\lambda$ produces a sequence of flips, twists, and pops.

**Fact:** All pops occur in degenerate triangles, and the induced transformation is 1 for such pops.

Therefore $\chi^{\mathcal{I}_+}$ is related to $\chi^{\mathcal{I}_-}$ via the image of

$$\prod \sigma_{\mathcal{I}_+}^{\mathcal{I}_-} \nu_{\mathcal{I}_+}^{\mathcal{I}_-} \nu_{\mathcal{I}_-} \nu_{\mathcal{I}_c} < \nu_{\mathcal{I}_c} < \nu_{\mathcal{I}_+} E_c$$

i.e.

$$\prod \kappa_{\gamma}^{\Omega(\gamma, \lambda)} = A_{\mathcal{V}}$$

$\theta_- < \arg(-z_\gamma) < \theta_+$
BUT THERE IS NO DISCONTINUITY IN $\chi^T(\vartheta, \lambda_-) \rightarrow \chi^T(\vartheta, \lambda_+)$

\[
\begin{align*}
\Pi K_\vartheta^\Omega(\vartheta, \lambda_-) &\quad \rightarrow \quad \Pi K_\vartheta^\Omega(\vartheta, \lambda_+) \\
\chi^T(\vartheta, \lambda_-) &\quad \rightarrow \quad \chi^T(\vartheta, \lambda_+) \\
\vartheta &\quad \rightarrow \quad \lambda
\end{align*}
\]

\[
\prod K_\vartheta^\Omega(\vartheta, \lambda_-) = \prod K_\vartheta^\Omega(\vartheta, \lambda_+)
\]
7. MOVIES ¹ ² EXAMPLES
8. Determining the BPS Spectrum

Now let us vary \( \vartheta \) to \( \vartheta + \pi \).

We capture all the BPS states

\[
\prod \sigma_{\gamma} v_{E_c} \rightarrow \prod K_{\gamma}^{\mathcal{Q}(\delta, \lambda)}
\]

\( \vartheta < -\arg Z < \vartheta + \pi \)

On the other hand,

\( T(\vartheta, \lambda) \) and \( T(\vartheta + \pi, \lambda) \)

only differ by simultaneously popping all the vertices!!
SPECTRUM-GENERATING STOKES MATRIX

While the change $\chi_E^T$ for popping one vertex is complicated, it turns out that popping all vertices leads to a rather simple formula!

\[
\tilde{\chi}_{ac}^T \chi_{ac}^T = \frac{(1 + A_{ab})(1 + A_{cd})}{(1 + A_{bc})(1 + A_{da})}
\]

To give a formula for $A_{p,p_2}$:
\[ A_{p_1, p_2} = \frac{1}{1 - \mu_{p_1}^2} \cdot \frac{1}{1 - \mu_{p_2}^2} \cdot \chi_{p_1, p_2} \cdot \left( 1 + \sum_{k=1}^{N_{p_1} - 1} \prod_{j=1}^{k} \chi_{p_1, j} \right) \cdot \left( 1 + \sum_{k=1}^{N_{p_2} - 1} \prod_{j=1}^{k} \chi_{p_2, j} \right) \]
To find the BPS spectrum

The transformation

\[ S : \chi_i \rightarrow \tilde{\chi}_i \]

\[ \tilde{\chi}_i = \chi_i \left( \frac{1 + A_{a_0(i)}}{1 + A_{b_c(i)}} \right) \]

Has a unique decomposition

of the form:

\[ S = \prod K_{y_i}^{\Omega(y, \lambda)} \]

\[ \forall \arg Z \in \theta + \pi \]

This determines the \( \Omega(x, u) \)
Conclusion: Future Directions

1. We have some ideas about how to go to rank $k > 2$.

2. Relation to integrable systems (e.g. the integral equation for $X_y$ is a version of the TBA.)

3. Supergravity

4. New modular functors