GGI LECTURE 3:

M5-BRANES, HITCHIN SYSTEMS,
& BPS STATES

KEY REFS:

1. KLEMM, LERCHE, MAYR, VAFA, WARNER
"SELF-DUAL STRINGS ..." hep-th/9604034

2. E. WITTEN, "SOLUTIONS OF FOUR-DIMENSIONAL
FIELD THEORIES VIA M-THEORY,"
hep-th/9703166
1. Introduction

For a large class of $d=4, \mathcal{N}=2$ theories the moduli space $(\mathcal{M}, g)$ is also the moduli space of a Hitchin system (with singularities) on $\mathcal{P}$.

This generalizes a result of Cherkis + Kapustin and follows from the construction of $d=4, \mathcal{N}=2$ theories using M5-branes.

The aim of this lecture is to explain the connection to Hitchin moduli space, and how BPS states look like in this description.
2. M5 ON A RIEMANN SURFACE

A. $M$-THEORY / $\mathbb{T}R^{1,3}_{0123} \times C \times C \times R^3_{6,10 \ 4,5 \ 7,8,9}$

**METRIC:**

$$ds^2 = \sum_{\mu=0,1,2,3} dx^\mu dx_\mu + \sum_{i=45,789} (dx^{i})^2 + R_{ii}^{2} [(dx^6)^2 + (dx^{10})^2]$$

$x^6, x^{10}$ DIMENSIONLESS.

NOW DRAW PICTURES

$\mathbb{T}R^{1,3}_{0123}$

$x^4 + i x^5 := V$

$x^6 + i x^{10} := S$

EVERYTHING AT A SINGLE POINT IN $R^3_{7,8,9}$
SO SUPPRESS IT.
**CONSIDER M5 CONFIGURATION**

WRAPS A HOLO. CURVE SO PRESERVES 8 SUPERSYMMETRIES

\[ C \cong C^* \quad t := e^{-(x^6 + ix^{10})} \in C^* \]
\[ v := (x^4 + ix^5)/\ell \]

**M5 ON CURVE:**

\[ \nu^k \frac{T_T}{\ell-t_\alpha}(=t_\alpha) = 0 \quad \alpha = 0 \]
Projection to IIA Sugra

\[ x^4 \rightarrow x^5 \quad \uparrow \quad x^6 \quad \rightarrow \quad \mathbb{R}^{1,3} \]

\[ \alpha = 0 \quad \alpha = 1 \quad \alpha = n \]

NS5

\[ k \]

The D4's in each interval can split apart separately maintaining supersymmetry:
• Take a limit of low energy and gravitational decoupling

\[ \Rightarrow \text{replace M5 by SU(k)} \]

(2,0) superconformal theory

• At long distances compared to \( R_{\alpha}(x^\alpha - x^\alpha_{-1}) \) we have an effective 4D gauge theory:

4D IR: describes linear quiver

\[
\begin{array}{c}
\mathbf{K} \rightarrow \mathbf{K} \rightarrow \mathbf{K} \\
\hdashline
\cdots \\
\mathbf{K} \rightarrow \mathbf{K}
\end{array}
\]

\( n \)

• The picture is a picture of the brane config. describing the low energy dynamics on the Coulomb branch:
Lifting back up to M-theory we are deforming the curve $V^{k_{\alpha}}_{\alpha=0} (t-t_{\alpha}) = 0$ and wrapping a single M5-brane on that curve:

$$\Sigma = \{(t,v) \mid F(t,v) = 0 \}$$

$$\subset \mathbb{C}_{6+1|0} \times \mathbb{C}_{4+1|5} \cong \mathbb{C}^* \times \mathbb{C}$$

The different roots in $V$ at fixed $t$ describe the positions of the $k$ different D4's $\Rightarrow$ add polynomial in $V$ of degree $< k$:

$$F(t,v) = V^k \prod_{\alpha=0}^{n} (t-t_{\alpha}) + \sum_{i=1}^{K} p_i(t)V^{k-i} = 0$$
At generic $t$ there are $k$ roots. At $t \to t_c$ leading coeff. degenerates.

For generic $p_i$ one root $V(t)$ goes to $\infty$:

$$V_*(t) \sim \frac{1}{t-t_c} \left( \frac{-p_i(t_c)}{t_c(t_c-t_\beta)} \right)^{1-}$$

The other $(k-1)$ roots remain finite.

$$\sum_{k:1} \rightarrow C \quad \text{is a branched cover. Think of the branches as sheets of singly-wrapped M5. The divergent root at $t_c$ is the transverse M5.}$$
In addition, for \( x^6 \to \pm \infty \), i.e. \( t \to 0 \} \quad t \to \infty \)

We want precisely \( k \) roots

\[ \Rightarrow \deg \left( P_i \right) \leq n+1 \]

\[ \Rightarrow \text{we can also write:} \]

\[ F(t,v) = \sum_{\alpha=0}^{n+1} t^{n+1-\alpha} q_\alpha(v) = 0. \]

\[ \deg \left( q_\alpha \right) = k. \]

Roots of \( q_0(v) \): Positions for \( t \to \infty \), etc.
LC. WEAK COUPLING LIMIT

TO GET SOME PHYSICAL INTUITION

CONSIDER THE WEAK COUPLING LIMIT

\[
\frac{4\pi^2}{(g^2_{YM})_\alpha} = \left( x_\alpha^6 - x_{\alpha-1}^6 \right) / g_s
\]

\[
\sim \text{Re} \left( S_\alpha - S_{\alpha-1} \right)
\]

VALID FOR WEAK \( g^2_{YM}(SU(n)_\alpha) \)

HOLOMORPHY \( \Rightarrow \)

\[
-\pi i \tau_\alpha = -\pi i \left( \frac{\Theta_\alpha}{2\pi} + \frac{4m^2}{g^2_\alpha} \right) = S_\alpha - S_{\alpha-1}
\]
WEAK COUPLING LIMIT

1. \( \left| \frac{t_\alpha}{t_{\alpha-1}} \right| = \varepsilon_\alpha \to 0 \quad 1 \leq \alpha \leq n \)

2. \( \tilde{q}_\alpha(v) = \pm_0 \cdots t_{\alpha-1} \tilde{q}_\alpha(v) \)
   HOLDING \( \tilde{q}_\alpha(v) \) ORDER 1.

EXERCISE: SHOW THAT IN THIS LIMIT, FOR \( |t_\beta| \ll |t| \ll |t_{\beta-1}| \)
THE ROOTS ARE APPROX. \( \tilde{q}_\beta(v) = 0 \).

\[ q_\alpha(v) = C_\alpha \left( V^k - \mu^{(\alpha)} V^{k-1} U_2^{(\alpha)} V^{k-2} - \cdots - U_K^{(\alpha)} \right) \]

\[ \text{gauge couplings} \quad \text{bifund. masses} \quad \text{adjoint reps of} \quad \text{SU(K)}_\alpha \]

as well see
Claim: \[ \sum = \{ (t, v) \mid F(t, v) = 0 \} \]

is the SW curve of the effective 4D gauge theory.

A computation that makes this plausible is the K.E. of the scalars describing fluctuations of the scalars in the \( v \)-direction:

\[ \mathbb{R}^{1,3} \times \mathbb{C}^* \times \mathbb{C}_{6,10} \times \mathbb{C}_{4,5} \]

\[ (x^\mu, t, v(t, \xi_i(x^\mu))) \]

\( \xi_i \sim \) parameters of \( \Sigma \)

slowly varying

\[ \xi_2 = 0 : \text{some susy basepoint} \]
Use the DBI action for the single M5:

\[ ds^2 = dx^\mu dx_{\mu} + l^2 |dv|^2 + R_{\mu\nu} \left| \frac{dt}{t} \right|^2 + \ldots \]

\[ \frac{2\pi}{\ell^6} \int_{M^5} \text{vol} - \frac{2\pi}{\ell^6} \int_{M^5} \text{vol} \left| \frac{\delta}{\delta \xi^0} \right|_{\xi^0 = 0} \]

\[ = \frac{R_{11}^2}{\ell^2} \int_{\mathbb{R}^{11,3}} dx^{0123} \int \frac{dt dt^*}{(1 + t^2)} \sum \frac{\delta v}{\delta \xi^j} \frac{\delta v}{\delta \xi^i} \frac{\delta v}{\delta \xi^0} \frac{\delta v}{\delta \xi^0} + O(\delta^3) \]

\[ = \frac{R_{11}^2}{\ell^2} \int_{\mathbb{R}^{11,3}} dx^{0123} \partial_\mu \xi^i \partial_\nu \xi^j \int \sum \frac{\delta v}{\delta \xi^0} \frac{\delta v}{\delta \xi^0} + \frac{\delta v}{\delta \xi^0} \frac{\delta v}{\delta \xi^0} \]
• **Definition:** Normalizable Def's $\xi_i$:

$$\sum \int \frac{dv}{d\xi_i : t} \times \frac{dv}{d\xi_i : t} < \infty$$

$\Rightarrow \xi_i \text{ varies } u_i^{(x)}$, but not the positions $t_\alpha$ or residues of pole of $v_i(t)$ at $t = t_\alpha$.

LET: $\chi_\alpha = \text{basis for } H_1(\Sigma, \mathbb{Z})$

$\mathcal{I}^{ab} = \text{intersection form}$

ACTION = \frac{R_{11}^2}{l^2} \int dx^{0123} \mathcal{I}^{ab} \partial_\mu \overline{\Pi}_a \partial_\mu \overline{\Pi}_b$

WITH $\overline{\Pi}_a = \oint_{\gamma_a} v \frac{dt}{t}$
Choosing a duality frame

\[ \frac{R_{11}^2}{\ell^2} \int dx^{0123} \ \text{Im} \tau_{ij} \ d_\mu a^i \ d^{\mu} \overline{a}^j \]

with S-W Diff'�

\[ \lambda = \nu \frac{dt}{t} \]

Generalizations ...

Exercise: Compute the residue of \( \lambda \) at \( t = t_0 \), in the weak coupling limit, and show it is \( \mu_0 - \mu_{0-1} \). Thereby make a connection of \( \mu_0 \) to bifund. masses.
E: Adding $\frac{1}{\ell}$ decoupling flavors

D6-brane at point in $x^4, x^5, x^6$

$\text{NS5} \otimes \text{D4} \otimes \text{NS5}$

$k_0, k_1, k_2, \ldots, k_n \ldots, k_{n+1}$

leads to linear quiver

$\text{K}_1 \rightarrow \text{K}_2 \rightarrow \text{K}_3 \rightarrow \ldots \rightarrow \text{K}_n$

$\text{d}_1 \rightarrow \text{d}_2 \rightarrow \text{d}_3 \rightarrow \ldots \rightarrow \text{d}_n$
ASYMPTOTIC FREEDOM \Rightarrow

-2k_\alpha + k_{\alpha-1} + k_{\alpha+1} + d_\alpha \leq 0

\Rightarrow

D6 LIFT TO MULTI-TAUB-NUT. \\
\Sigma \subset TN IS HOLOMORPHIC CURVE.

MAIN NEW FEATURE: SOME ROOTS \nu_i(t) OF F(t, v) = 0 \\
GO TO \infty FOR t \to 0, \infty.
F. THE GENERAL STORY

• M. THEORY ON $\mathbb{R}^{1,3} \times Q \times \mathbb{R}^{7,8,9}$

$Q = H.K. \ 4$-FOLD

PREVIOUS $Q = T^* C$, $C = C^*$

• K-WRAPPED M5 ON HOLOMORPHIC CURVE $C \subset Q$, INTERSECTS
  SINGLY WRAPPED $C_\alpha$ TRANSVERSALLY.

• DECOUPLE GRAVITY $\Rightarrow$ $SU(k)$ $(2,0)$

THEORY ON $\mathbb{R}^{1,3} \times C$ WITH DEFECTS

AT $C \cap C_\alpha$. (TWISTED TO PRESERVE $d=4, N=2$ SUSY.)

• SMALL FLUCTUATIONS ONLY

PROBE INF. NBD. OF $C \subset Q \Rightarrow T^* C$
• SW CURVE OF THE D=4, N=2 THEORY IS A HOLOMORPHIC CURVE \( \Sigma \subset T^*C \).

• SW DIFFERENTIAL IS RESTRICTION OF CANONICAL \( \omega = dx \wedge dz = d(xdz) \)

\[ \lambda = xdz \]

• LOW ENERGY EFFECTIVE THEORY IS COMPACTIFICATION OF SINGLE M5 ON \( \Sigma \).
3. MAPPING TO HITCHIN SYSTEMS

- We have now seen that we get a $D=4$, $N=2$ theory by compactifying a $D=6$ $(2,0)$ theory on $C$.

- To get our hyperkähler $0$-model we then compactify on $S^1_R$.

- But, by a kind of "Fubini theorem," we could get the low energy theory compactifying in the other order....
6D $\mathbb{A}_{K-1} / \mathbb{R}^{1,2} \times S^1_R \times C + \text{DEFECTS}$

$l_C \ll l_{S^1}, l_{\mathbb{R}^{1,2}}$

$l_{S^1} \ll l_C, l_{\mathbb{R}^{1,2}}$

$5D \frac{U(K) \text{ SYM}}{\mathbb{R}^{1,2} \times C}$

$4D \frac{W=2 \text{ GAUGE THEORY}}{\mathbb{R}^{1,2} \times S^1_R}$

$l_{S^1} \ll l_{\mathbb{R}^{1,2}}$

$l_C \ll l_{\mathbb{R}^{1,2}}$

$\sigma-$ MODEL: $\mathbb{R}^{1,2} \rightarrow \mathcal{M}$
ON THE OTHER HAND... REDUCTION

OF THE 5D $U(K)(2,0)$ THEORY ON $\mathcal{C} \Rightarrow$

HITCHIN Eqs. \[
\begin{cases}
F + R^2[\varphi, \overline{\varphi}] = 0 \\
\overline{\partial}_{\overline{A}} \varphi = 0 \\
\partial_{A} \overline{\varphi} = 0
\end{cases}
\]

$\varphi = (\overline{\Phi}_4 + i \overline{\Phi}_5) : (1,0) \text{ form on } \mathcal{C}$

+ SINGULAR BC'S AT $S_\infty$

\[
\Rightarrow \mathcal{M} = \text{MODULE SPACE OF HITCHIN SYSTEM.}
\]

(Remark: The topological twisting on $\mathcal{C}$ makes $\varphi$ into a $(1,0)$ form on $\mathcal{C}$.)
THE SPECTRAL CURVE

RETURN TO \[ C \approx C^*_{6+i\omega} \Rightarrow t = e^{-s} \]

\[ \varphi = \varphi_s ds \]

* EIGENVALUES OF \( \varphi_s \) = POSITIONS OF D4 BRANES

THESE MUST BE THE ROOTS OF \( v \) IN \( F(t, v) = 0 \)!

indeed, given a soln of the hitchin system, an easy computation shows that

\[ \frac{\partial}{\partial t} \left( \det (v + \varphi_s) \right) = 0 \]

(Note \( \frac{\partial}{\partial t} \varphi_s \neq 0 \)!)
MORE INVARIENTLY:

\[ \det (v ds + q) = 0 \]

DEFINES THE SPECTRAL CURVE IN \( T^*C \).

WORKING IN \( t = e^{-s} \)

\[ \det \left( v \frac{dt}{t} - q_t dt \right) = 0. \]

OR

\[ \det (\lambda - \varphi) = 0 \]

THIS SHOULD DEFINE \( \Sigma \) SO,

COMPARING TO:
\[ F(t,v) = \sum_{\alpha=0}^{n+1} t^{n+1-\alpha} q_\alpha(v) \]

\[ = p_0(t) V^k + p_1(t) V^{k-1} + \ldots + p_k(t) \]

(MAKING IT MONIC WE DIVIDE BY \( p_0(t) \) AND IDENTIFY:

\[ V^k + R_1(t) V^{k-1} + \ldots + R_k(t) = \det (v - t \varphi_t) \]

PREVIOUSLY WE SAW THAT THE ROOTS BEHAVE AS:

1.) \( V_i(t) \sim \text{FINITE} \quad t \to t^\alpha \)

EXCEPT FOR ONE ROOT \( V_\ast(t) \sim \frac{p_\alpha}{t-t^\alpha} \)

2.) \( V_i(t) \rightarrow \begin{cases} V_i^{(s)} & t \to 0 \\ V_i^{(\infty)} & t \to \infty \end{cases} \)
THEREFORE, THE S-W DIFF'L BEHAVES LIKE

1.) \( \lambda: \infty \) \text{FINITE ON ALL BRANCHES FOR } t \to t_\alpha \text{ AND}

\[
\lambda \sim \frac{m_\alpha}{t-t_\alpha} \ dt \quad t \to t_\alpha
\]

ON ONE BRANCH

2.) \( \text{FOR } t \to 0, \infty \text{ THE SW DIFF'L HAS POLE } \lambda \rightarrow \left\{ \begin{array}{ll}
V_i^{(0)} \frac{dt}{t} \\
V_i^{(\infty)} \frac{dt}{t}
\end{array} \right. \text{ ON EACH BRANCH}
\]

AS \( t \to 0, \infty \) RESPECTIVELY

\[
\text{BUT } \det (\lambda - \varphi) = 0
\]

SO THE HIGGS FIELD \( \varphi \) HAS SINGULARITIES
\( \varphi(t) \sim \frac{dt}{t-t_\alpha} \quad t \to t_\alpha \)

\( \varphi(t) \sim \frac{dt}{t} \quad t \to 0 \)

\( \varphi(t) \sim \frac{dt}{t} \quad t \to \infty \)

(When we add flavors or consider other generalizations we have irregular singular points.)
AN IMPORTANT SPECIAL CASE: \( k = 2 \)

\[ \sum 2:1 \to C \]

\[ \varphi \sim \begin{pmatrix} \lambda & \\
-\lambda & \end{pmatrix} \in \text{su}(2) \]

(AFTER SUBTRACTING OVERALL CENTER OF MASS OF BRANE SYSTEM)

\[ \varphi \sim \frac{dt}{t-t_\alpha} \begin{pmatrix} m_\alpha & \\
& -m_\alpha \end{pmatrix} \]

\[ A \sim \cdots \]

\[ \chi^2 = \sum_\alpha \left( \frac{m_\alpha^2}{(t-t_\alpha)^2} + \frac{c_\alpha}{t-t_\alpha} \right) (dt)^2 \]

\[ c_\alpha \sim \text{PARAMETRIZE } B. \]
M5 ORIGIN OF THE BPS STATES IS FROM OPEN M2 BRANES BETWEEN II M5 (STROMINGER)

\[ \Rightarrow \text{BPS STRINGS IN THE} (2,0) \text{ THEORY} \]
BPS STATES IN THE HITCHIN FRAMEWORK

THE PHYSICAL DERIVATION LEADS TO THE FOLLOWING RULES FOR DESCRIBING BPS STATES IN THE HITCHIN FRAMEWORK (KLEMM, LERCHE, VAFA, WARNER)

\[ \Omega(\gamma, u) = 0 \quad \text{UNLESS THERE EXISTS A CURVE} \quad \tilde{c} \in \Sigma \quad \text{IN HOMOLOGY CLASS} \quad \gamma \quad \text{SO THAT} \quad \pi_x(\tilde{c}) = c \]

IS A CURVE IN C S.T.

\[ \langle \lambda, d_t \rangle \in e^{i \theta x} = \text{CONSTANT} \]

AND

ONE OF TWO THINGS HAPPENS:
1. **EITHER,**

   *C begins and ends on a branch point*

   $\Rightarrow$ **HM with $\Omega = +1$**

   **M-theory picture:** **open M2:**

2. **OR,** *C = closed curve* $\Rightarrow$

   **VM with $\Omega = -2$.**
N.B. FOR SUCH CURVES $\langle \tau, d_\tau \rangle = e^{i \nu_*}$

$\nu_* = \text{arg}(Z)$

(REMARK: FOR $\text{SU}(k)$, $k > 2$ THE BPS STATES ARE MORE COMPLICATED AND INVOLVE "STRING WEBS." )
5. Twistor Coordinates
For $k = 2$

We have constructed the $X_8$
and verified the properties 1-6 above

The $K$ transformations
and $\xi \to 0, \infty$ asymptotics
emerge very naturally...

Conjecturally a generalized
construction applies to $k > 2$.

This is the construction
we turn to next