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Three Lectures on Fukaya-Seidel Categories and Web-Based Formalism

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ABSTRACT: Lecture notes for three lectures in Florida, Jan. 26-29, 2015. These lectures describe some aspects of a forthcoming paper with D. Gaiotto and E. Witten, Version: January 29, 2015

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1. Lecture 1: Background And Motivation

1.1 Introduction

These lectures attempt to survey some work I have done over the past few years with Davide Gaiotto and Edward Witten. There is a rather long (about 370 pp.) paper "Algebra of the Infrared: String Field Theoretic Structures in Massive $\mathcal{N} = (2,2)$ Field Theory in Two Dimensions," that is thankfully nearing completion. It will be cited as AlgebraicStructures [8] and is the basis for everything said here. There is also a "short" (c. 53pp.) summary of the web-based formalism "Concise Summary of Web-Based Formalism," which has been completed, but not yet posted on the arxiv. Finally, these notes are available at

http://www.physics.rutgers.edu/∼gmoore/FloridaLectures2.pdf

Three closely related references:

1. K. Hori et. al. "Mirror Symmetry...." Clay volume. Hori:2003ic [11]
2. P. Seidel, *Fukaya categories and Picard-Lefschetz theory*, SeidelBook [15]
3. Recent paper of Kapranov, Kontsevich, and Soibelman: Kapranov:2014uwa [12]

1.2 Goals

Let X be a Kähler manifold, and $W : X \to \mathbb{C}$ a holomorphic Morse function. To this data physicists associate a "Landau-Ginzburg model." It is closely related to the Fukaya-Seidel (FS) category.

My goals will be:

1. To construct an A_∞- category of branes in this model, using only data "visible at long distances" - that is, only data about BPS solitons and their interactions. This is the "web-based formalism."

2. To explain how the "web-based" construction of an A_∞-category of branes is related to the FS category.

3. To construct an A_∞ 2-category of theories, interfaces, and boundary operators.

4. To show how these interfaces categorify the wall-crossing formula for BPS solitons as well as the wall-crossing formulae for so-called framed BPS states.

5. To sketch how the formalism might be useful in formulating a theory of knot homology.

1.3 A review of Landau-Ginzburg models

To warm up, let us review some well-known facts about the physicist's Landau-Ginzburg theory. We want to understand the groundstates of the model in various geometries with various boundary conditions. We approach the subject from the viewpoint of Morse theory.

1.3.1 Supersymmetric Quantum Mechanics And Morse Theory

From a physicist's point of view Morse theory is the theory of the computation of groundstates in supersymmetric quantum mechanics (SQM) Witten:1982im [16]. Recall that in SQM we have a particle moving on a Riemannian manifold $q : \mathbb{R} \to M$ together with a real Morse function $h : M \to \mathbb{R}$ and we consider the (Euclidean) action

$$S_{SQM} = \int dt \left(\frac{1}{2}|\dot{q}|^2 + \frac{1}{2}|dh|^2 + \cdots \right) \tag{1.1}$$

There is a uniquely determined perturbative vacuum $\Psi(p_i)$ associated to each critical point p_i of h. True vacua are linear combinations of the $\Psi(p_i)$. How do we find them?

To find the true vacua we introduce the MSW ("Morse-Smale-Witten") complex generated by the perturbative ground states

$$\mathbb{M} = \oplus_{p_i : dh(p_i)=0} \mathbb{Z} \cdot \Psi(p_i) \tag{1.2}$$

The complex is graded by the Fermion number operator $\mathcal{F}$, whose value on $\Psi(p_i)$ is:

$$f = \frac{1}{2}(n_- - n_+) \tag{1.3}$$

where $n_\pm$ is the number of $\pm$ eigenvalues of the Hessian.

The matrix elements of the differential Q are obtained by counting the number of solutions to the instanton equation:

$$\frac{dq}{d\tau} = \nabla h \tag{1.4}$$

which have no reduced moduli and interpolate between two critical points.

By "counting" we always mean "counting with signs determined by certain orientations." Getting the signs right is a highly technical business and we will avoid it altogether in these lectures.

The space of true ground states is the cohomology $H^*(\mathbb{M}, Q)$ of the MSW complex.

1.3.2 Landau-Ginzburg Models From Supersymmetric Quantum Mechanics

Now, to formulate LG models, we apply the above picture of Morse theory to the case where the target manifold of the SQM is a space of maps $D \to X$ where D is a one-dimensional manifold, possibly with boundary:

$$M = \mathrm{Map}(D \to X) \tag{1.5}$$

The Morse function is

$$h = -\int_D \left(\phi^*(\lambda) - \frac{1}{2}\mathrm{Re}(\zeta^{-1}W)dx \right) \tag{1.6}$$

eq:MorseFun

Here ζ is a phase. For simplicity we assume that the Kahler manifold is exact and choose a trivialization of the symplectic form $\omega = d\lambda$.

We apply this story to our case with target space $M = Map(D, X)$ and superpotential eq:MorseFun (1.6). If we work out the SQM action we get a $1+1$ dimensional field theory. The bosonic terms in the action are

$$\int_{D\times\mathbb{R}} \frac{1}{2}|d\phi|^2 + \frac{1}{2}|\nabla W|^2 + \cdots \tag{1.7}$$

Now, this theory has massive vacua on $D = \mathbb{R}$ at the critical points $\phi_i \in X$ of W:

$$\phi(x,t) = \phi_i \in \mathbb{V} \tag{1.8}$$

Sometimes, boundary conditions do not admit solutions with ϕ a constant vacuum. In this case groundstates are given by solitons - solutions of $\delta h = 0$.

The stationary points of the Morse function h are solutions of the ζ-*soliton equation*

$$\frac{\mathrm{d}}{\mathrm{d}x}\phi^I = g^{I\bar{J}}\frac{\mathrm{i}\zeta}{2}\frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}} \tag{1.9}$$

eq:LG-flow

Later we will find it useful to note that the ζ-soliton equation is equivalent to

1. Upwards gradient flow with potential $\mathrm{Im}(\zeta^{-1}W)$.

2. Hamiltonian flow with Hamiltonian $\mathrm{Re}(\zeta^{-1}W)$.

1.3.3 Solitons On The Real Line

Now suppose $D = \mathbb{R}$. We choose boundary conditions of finite energy:

$$\lim_{x\to-\infty}\phi = \phi_i \tag{1.10}$$

eq:left-infty-bc

where $\phi_i \in \mathbb{V}$. Similarly, if D extends to $+\infty$ then we require

$$\lim_{x\to+\infty}\phi = \phi_j \tag{1.11}$$

eq:right-infty-bc

with $\phi_i \neq \phi_j$. What is the MSW complex in this case?

Recall that solutions to eq:LG-flow (1.9) project to straight lines of slope $\mathrm{i}\zeta$ in the complex W-plane. Therefore, there is no solution for generic ζ. There can only be a solution for

$$\mathrm{i}\zeta = \mathrm{i}\zeta_{ji} := \frac{W_j - W_i}{|W_j - W_i|} \tag{1.12}$$

hopeful

in which case a solution projects in the W-plane to a line segment from W_i to W_j.

We assume that the left and right Lefshetz thimbles intersect transversally in the fiber over a regular value of W on the line segment $[W_i, W_j]$. In this case, there will be a finite number of classical solitons, one for each intersection point $p \in \mathcal{S}_{ij}$. The MSW complex is then:

$$\mathbb{M}_{ij} = \oplus_{p\in\mathcal{S}_{ij}}\left(\mathbb{Z}\Psi^{f}_{ij}(p) \oplus \mathbb{Z}\Psi^{f+1}_{ij}(p)\right) \tag{1.13}$$

eq:MorseComplexR

The grading of the complex is

$$f = -\frac{\eta(\mathcal{D}+\varepsilon)}{2}. \tag{1.14}$$

where $\mathcal{D}$ is the Dirac operator obtained by linearizing the ζ-soliton equation (1.9) and ε small and positive. For fixed ij the complex is graded by a $\mathbb{Z}$-torsor. [1] We can now introduce the *BPS soliton degeneracies* [2]:

$$\mu_{ij} := -\mathrm{Tr}_{\mathbb{M}_{ij}}\mathcal{F}(-1)^{\mathcal{F}}. \tag{1.15}$$

These will show up in Lecture 2 (Section §2.4) and again in Lecture 3 when we discuss wall-crossing. We can already note that, in some sense, $\mathbb{M}_{ij}$ has "categorified the 2d BPS degeneracies."

The differential on $\mathbb{M}_{ij}$ is given by counting solutions to the ζ-*instanton equation*:

$$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial \tau}\right)\phi^I = \frac{\mathrm{i}\zeta}{2} g^{I\bar{J}}\frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}, \tag{1.16}$$

with boundary conditions illustrated in Figure 1:

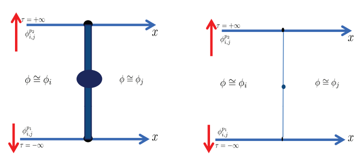


Figure 1: Left: An instanton configuration contributing to the differential on the MSW complex. The black regions indicate the locus where the field $\phi(x,\tau)$ varies vary significantly from the vacuum configurations ϕ_i or ϕ_j. The length scale here is ℓ_W, set by the superpotential. Right: Viewed from a large distance compared to the length scale ℓ_W the instanton looks like a straight line $x = x_0$, where the vacuum changes discontinuously from vacuum ϕ_i to ϕ_j. The nontrivial τ-dependence of the instanton configuration, interpolating from a soliton p_1 to another soliton p_2 has been contracted to a single vertex located at $\tau = \tau_0$. This illustrates the origin of the 2-valent vertices of extended webs in the context of LG theory.

Written out this is:

$$\lim_{x\to-\infty}\phi(x,\tau) = \phi_i \qquad \lim_{x\to+\infty}\phi(x,\tau) = \phi_j \tag{1.17}$$

$$\lim_{\tau\to-\infty}\phi(x,\tau) = \phi_{ij}^{p_1}(x) \qquad \lim_{\tau\to+\infty}\phi(x,\tau) = \phi_{ij}^{p_2}(x). \tag{1.18}$$

[1] There is a tricky point here. In the algebraic manipulations below it is important to use the Koszul rule. But that only makes sense when there is an integral grading. One needs to write $f = f_i - f_j + n_{ij}$, where n_{ij} is integral, and remove the f_i, which turn out to be the phase of the determinant of the Hessian.

Following the rules of SQM, the matrix elements of the differential are obtained by counting the solutions with no reduced moduli, (i.e. the solutions with two moduli).

Remarks:

1. The complex (eq:MorseComplexR 1.13) is not a standard mathematical Morse theory complex: h is degenerate because of translation invariance. The critical set is $\mathbb{R}$, parametrizing the "center" of the soliton. But we take neither the cohomology nor the compactly supported cohomology of of this critical set. Rather, we attach a certain Clifford module to each critical locus. ("Quantization of the collective coordinate.")

2. Supersymmetric quantum mechanics has two supersymmetries satsifying $\{Q, \bar{Q}\} = 2H$. When the spatial domain is $D = \mathbb{R}$ there are more symmetries in the problem not manifest from the SQM viewpoint. Namely the LG model has (2,2) supersymmetry:

$$\begin{aligned} \{Q_+, \bar{Q}_+\} &= H + P & \{Q_+, Q_-\} &= \bar{Z} \\ \{Q_-, \bar{Q}_-\} &= H - P & \{\bar{Q}_+, \bar{Q}_-\} &= Z. \end{aligned} \tag{1.19}$$ eq:22susy

   The supersymmetries of the SQM are

$$\mathcal{Q}_\zeta := Q_- - \zeta^{-1}\bar{Q}_+, \quad \bar{\mathcal{Q}}_\zeta := \bar{Q}_- - \zeta Q_+. \tag{1.20}$$ manifest

   The ζ-soliton and -instanton equations are the $\mathcal{Q}_\zeta$-fixed point equations for the classical field configurations. When D is a half-line or an interval, with suitable boundary conditions only the two-dimensional supersymmetry algebra will be preserved.

3. Now comes an important physics point: The theory is *massive* with a length scale ℓ_W corresponding to the inverse of the lightest soliton. Physical correlations should decay exponentially beyond that scale. We can picture the solitons and instantons as in Figure fig:INSTANTON-ON-R 1.

1.4 LG Models On A Half-Plane And The Strip

1.4.1 Boundary Conditions

If D has a left-boundary $x_\ell \le x$ or a right boundary $x \le x_r$ at finite distance then we need to put boundary conditions to get a good Morse theory, or QFT.

1. At $x = x_\ell, x_r$, the boundary value ϕ^∂ must be valued in a maximal Lagrangian submanifold $\mathcal{L}_\ell, \mathcal{L}_r$ of X in order to have elliptic boundary conditions for the Dirac equation on the fermions.

2. The theory is simplest when the Lagrangian submanifolds are exact: $\iota^*(\lambda) = dk$, for a single-valued k, and we will make that assumption here. Indeed, the Morse function (eq:MorseFun 1.6) is replaced by $h \to h \pm k(\phi^\partial)$, where the sign is for the negative/positive half-plane, respectively.

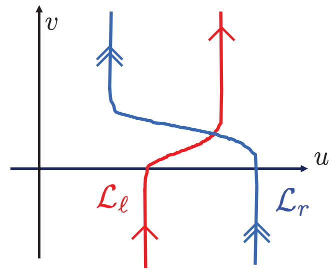


Figure 2: A pair of Lagrangian submanifolds Ł$_\ell$, Ł$_r$ embedded in the $u - v$ plane. Ł$_\ell$ and Ł$_r$ intersect at the one point indicated. u is plotted horizontally and we assume that Ł$_\ell$, Ł$_r$ are embedded in the half-plane $u > 0$. fig:Lagrangians

We are certainly interested in X which is noncompact (since we want W to be nontrivial) and we are typically interested in noncompact Lagrangians. Now, we want to have well-defined spaces of quantum states on an interval $\mathcal{H}_{\mathcal{L}_\ell,\mathcal{L}_r}$, invariant under separate Hamiltonian symplectomorphisms of the left and right branes.

The generators of the MSW complex in this case can be identified with the intersection points

$$\mathcal{L}_\ell^{(\Delta x)} \cap \mathcal{L}_r \tag{1.21}$$

where we regard the ζ-soliton equation (1.9) as a flow in x and $\mathcal{L}^{(\Delta x)}$ means the flow has been applied for a range (Δx).

But now there is a problem: Intersection points can go to infinity as the length of the interval is changed (or if independent Hamiltonian symplectomorphisms are applied to left and right branes).

Example: Consider $\zeta^{-1}W = \mathrm{i}\phi^2$ and consider the candidate left and right branes shown in Figure 2. We regard the ζ-soliton equation as a flow in x, and if $\phi = u + \mathrm{i}v$ is the decomposition into real and imaginary parts then

$$\partial_x u = u \qquad \partial_x v = -v \tag{1.22}$$

Therefore, the flow in x of $\mathcal{L}_\ell$ will not intersect $\mathcal{L}_r$ for sufficiently large x. Therefore there will be supersymmetric states for small width of $[x_\ell, x_r]$ but none for large width of $[x_\ell, x_r]$. This is a problem for the kind of "partially topological field theory" we are studying.

In [8] we find that there are *two* distinct criteria we could impose on the allowed Lagrangians to avoid the above problem. In these lectures we focus on just one namely, we will restrict the Lagrangians to be *Branes of class* T_κ: Choose a phase $\kappa \neq \pm\zeta$, and constants c, c'. The precise choices don't matter too much, although which component of the circle κ sits in is significant. Branes of class T_κ are based on Lagrangians which project

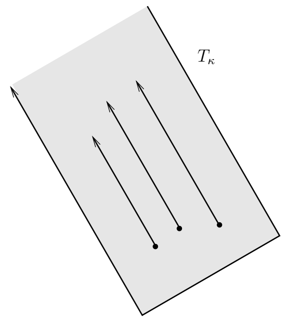


Figure 3: The rays in the complex W-plane that start at critical points and all run in the $\imath\kappa$ direction fit into the semi-infinite strip T_κ, which is shown as a shaded region. manyrays

under W to a semi-infinite rectangle in the W-plane:

$$\begin{aligned} |\mathrm{Re}\,(\kappa^{-1}W)| &\le c \\ \mathrm{Im}\,(\kappa^{-1}W) &\ge c', \end{aligned} \tag{1.23}$$ zelbo

as in Figure 3 (manyrays). The for this is that, under the x-flow of the ζ-soliton equation we have

$$\frac{d}{dx}\mathrm{Re}\,(\kappa^{-1}W) = -\frac{1}{2}\{\mathrm{Re}(\zeta^{-1}W), \mathrm{Re}(\kappa^{-1}W)\} = \frac{1}{4}\mathrm{Im}(\frac{\zeta}{\kappa})|dW|^2 \tag{1.24}$$ eq:x-flow

Then, points at infinity flow very fast out of the rectangle and hence intersection points $\mathcal{L}_\ell^{(\Delta x)} \cap \mathcal{L}_r$ always sit in a bounded region and cannot escape to infinity.

1.4.2 LG Ground States On A Half-Line

Now, we consider the theory on the positive half-plane. We choose ζ so that it does *not* coincide with any of the ζ_{ij} defining the solitons for $D = \mathbb{R}$. What are the groundstates preserving $\mathcal{Q}_\zeta$ supersymmetry?

The MSW complex $\mathbb{M}_{\mathcal{L}_\ell, j}$ is generated by the ζ-solitons on the half-plane satisfying the above boundary conditions.

The grading on the complex is a little nontrivial. We only know how to do it when X is Calabi-Yau. In this case case we define

$$e^{\imath\vartheta} = \frac{\mathbf{vol}}{\Omega|_{\mathcal{L}}} \tag{1.25}$$ dommy

(where Ω trivializes K_X and is normalized so that $\Omega\bar{\Omega}$ is the volume form on X) and we need to be able to define a single-valued logarithm ϑ. In this case we define the fermion number (on the interval) to be:

$$f = -\frac{1}{2}\eta(\mathcal{D}) - 2\frac{\varphi_r - \varphi_\ell}{2\pi}. \tag{1.26}$$ cgt

where $\varphi = \vartheta(\phi^\partial)$. On a half-line we drop φ_r or φ_ℓ as appropriate.

The differential on the complex is given by counting ζ-instantons. The picture of the instantons on the half-plane is shown in Figure 4 (fig:HALFPLANE-INSTANTON-1)

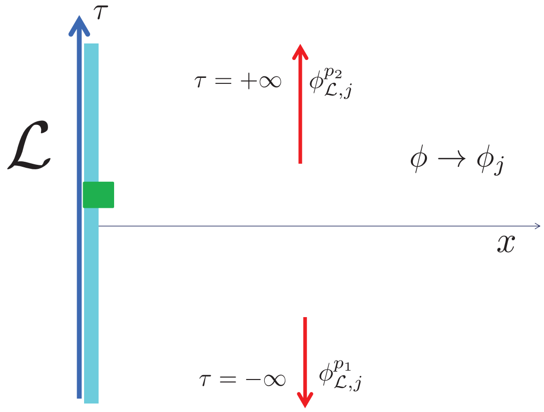


Figure 4: An instanton in the complex $\mathbb{M}_{\mathcal{L},j}$. The solitons corresponding to $p_1, p_2 \in \mathcal{L} \cap R^{\zeta}_j$ are exponentially close to the vacuum ϕ_j except for a small region, shown in turquoise, of width ℓ_W. In addition, the instanton transitions from one soliton to another in a time interval of length ℓ_W, indicated by the green square. At large distances the green square becomes the 0-valent vertex used in extended half-plane webs. fig:HALFPLANE-INS

1.4.3 LG Ground States On The Strip

The story on the strip is very similar to the half-plane, but there is an interesting wrinkle that provides a nice example where naive categorification of formulae for BPS degeneracies fails.

We consider the LG theory on $\mathbb{R} \times [x_\ell, x_r]$. When $|x_r - x_\ell| \gg \ell_W$ the ζ-solitons must nearly "factorize" so there is a natural isomorphism:

$$\mathbb{M}_{\mathcal{L}_\ell,\mathcal{L}_r} \cong \oplus_{i\in\mathbb{V}} \mathbb{M}_{\mathcal{L}_\ell,i} \otimes \mathbb{M}_{i,\mathcal{L}_r}. \tag{1.27}$$ eq:appxt-complex-

So if we define the BPS degeneracy of the half-line solitons:

$$\mu_{\mathcal{L}_\ell,i} := \mathrm{Tr}_{\mathbb{M}_{\mathcal{L},i}} e^{i\pi\mathcal{F}} \tag{1.28}$$

then the Euler-Poincaré principle guarantees

$$\mu_{\mathcal{L}_\ell,\mathcal{L}_r} = \sum_{i\in\mathbb{V}} \mu_{\mathcal{L}_\ell,i}\mu_{i,\mathcal{L}_r} \tag{1.29}$$ eq:WittenIndexFac

Now, the naive categorification would state:

$$H^*(\mathbb{M}_{\mathcal{L}_\ell,\mathcal{L}_r}) \overset{?}{\cong} \oplus_{i\in\mathbb{V}} H^*(\mathbb{M}_{\mathcal{L}_\ell,i}) \otimes H^*(\mathbb{M}_{i,\mathcal{L}_r}). \tag{1.30}$$ wrong

Here we have used the natural differential on the tensor-product complex. It corresponds to the ζ-instantons of Figure fig:NaiveStripDifferential 5. As we will see, equation (wrong 1.30) is wrong. The reason is that there are other ζ-instantons which are also relevant. One example is a ζ-instanton that looks like Figure fig:StripInstanton 6:

We will interpret this figure more precisely in Lecture 2 at the end of Section §subsubsec:BoostSol 2.1.1.

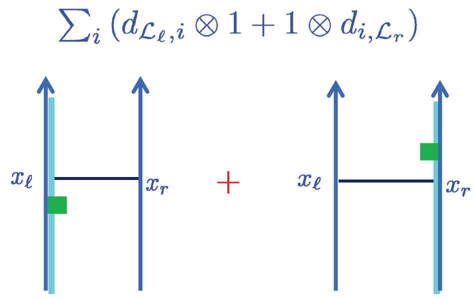


Figure 5: Naive differential on the strip. fig:NaiveStripDif

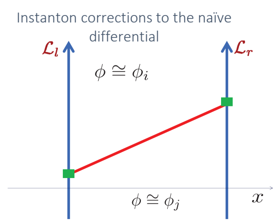


Figure 6: An instanton correction to the naive differential on the strip. fig:StripInstanto

1.5 The Fukaya-Seidel Category, d'après Les Physiciens

Finally, we sketch the Fukaya-Seidel (FS) category, at least the way a physicist would approach it. [2]

Fix ζ. Our objects will be branes based on Lagrangians in class T_κ, where κ is in one of the two components of $U(1) - \{\pm\zeta\}$. The morphism space is the MSW complex $\mathbb{M}_{\mathcal{L}_\ell,\mathcal{L}_r}$ generated by solutions of the ζ-soliton equation.

Then, to compute the differential M_1, we count ζ-instantons with one-dimensional moduli space. (That is, zero-dimensional reduced moduli space.)

To compute the higher A_∞-products we follow the example of open string field theory in light-cone gauge. We divide up the interval into equal length subintervals and consider the diagram in Figure fig:Worldsheets 7

We have to integrate over the moduli - the relative positions of the joining times. When the fermion numbers of the incomming and outgoing states are such that the amplitude is not trivially zero the expected dimension of the moduli space will be zero, and in fact the

[2]We thank Nick Sheridan for many useful discussions about the mathematical approaches to the FS category.

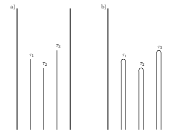


Figure 7: (a) n open strings all of width w come in from the past ($\tau = -\infty$) and a single one of width nw goes out to the future ($\tau = +\infty$). There are $n-1$ values of τ at which two open strings combine to one. The linearly independent differences between these critical values of τ are the $n-2$ real moduli of this worldsheet. (b) The picture in (a) can be slightly modified in this fashion so that H becomes smooth. The moduli are still the differences between the critical values of τ. fig:Worldsheets

solutions will only exist for a finite set of critical values $\tau_i - \tau_{i+1}$ where the strings join. The amplitude is obtained by counting over the finite set of solutions to the ζ-instanton equation.

Remark: The A_∞-category we have sketched above is not precisely what we one finds in the papers in the literature. (See, for example SeidelBook [15].) Our understanding from experts in the subject is that what we have described is well-known to be conceptually the correct definition, but it does not appear in papers because there are some technical difficulties in handling the PDE's. We fully expect it to be A_∞-equivalent to the standard mathematical construction.

1.6 Two Motivations For The Theory Of Interfaces

Now we sketch in some detail two of the motivations for this work, and in particular the contents of Lecture 3 on interfaces and categorified wall-crossing.

1.6.1 Motivation 1: Knot Homology

c:KnotHomology

We follow the approach of Witten WItten:2011pz [18] as further developed by Gaiotto and Witten Gaiotto:2011nm [6].

Let $L \subset M_3$ be a link in an oriented 3-fold. We would like to construct a (doubly graded) chain complex $\widehat{\mathcal{K}(L)}$ whose Euler character gives interesting knot polynomials such that there are chain maps associated with knot bordisms obeying natural topological conditions.

The complex is associated with a certain list of data, and the first piece of data is a choice of a compact simple Lie group G together with an irreducible representation R_a of G associated to each connected component L_a of L.

Witten's approach starts with the famous 6d (2,0) theory for $\mathfrak{g} = Lie(G)$ on a six-manifold

$$\mathbb{R} \times M_3 \times D \tag{1.31}$$

where D is a cigar. Since nobody knows what the $(2,0)$ theory is we then KK reduce with respect to the $U(1)$ symmetry of the cigar to get a description in terms of 5d SYM on

$$\underbrace{\mathbb{R}}_{x_0} \times \underbrace{M_3}_{x^1,x^2,x^3} \times \underbrace{\mathbb{R}_+}_{y} \tag{1.32}$$

where underneath the factors we have written typical coordinates on these spaces.

Witten's basic idea is that $\widehat{\mathcal{K}(L)}$ is a Morse-Smale-Witten complex generated by solutions of the Kapustin-Witten (KW) equations on $M_4 = M_3 \times \mathbb{R}_+$, where the differential on the complex is obtained by counting solutions to a certain 5d equation (written independently by A. Haydys and E. Witten). The boundary conditions on the 4d and 5d equations at $y = 0$ are slightly subtle. At generic points of M_3 they are Nahm pole boundary conditions and near L_a they encode the data of the link, in particular R_a. See [18, 14, 17, 19] for detailed discussion. It is also convenient to put boundary conditions at $y = \infty$ to reduce the structure group from G to an abelian subgroup.

WItten:2011pz,Mazzeo:2013zga,Witten:201

The connection to our story starts to emerge when we consider the special case

$$M_3 = \underbrace{\mathbb{R}}_{x^1} \times \underbrace{C}_{z=x^2+\mathrm{i}x^3} \tag{1.33}$$

(Here we deliberately use the same notation for the Riemann surface C that is often used for the ultraviolet curve in theories of class S.) In this case, the KW and HW equations are identical to those of a Landau-Ginzburg model on a spacetime $\mathbb{R}^2$ that should be thought of as the $x^0 - x^1$ plane:

$$\underbrace{\mathbb{R}}_{x^0} \times \underbrace{\mathbb{R}}_{x^1} \tag{1.34}$$

The data (X, W) of the model is then given by:

1. The target space of the model is a space of complexified gauge fields on a principal G^c-bundle $E^c \to \tilde{M}_3$ where
$$\tilde{M}_3 = C \times \mathbb{R}_+ \tag{1.35}$$
with complex gauge field $\mathcal{A} = A + \mathrm{i}\phi$. (Here A is a unitary connection on a principal G bundle E.) The boundary conditions at $y \to 0, \infty$ that encode the link and its data $\{R_a\}$ are used in the precise definition of the allowed gauge potentials $\mathcal{U}^c$. Then
$$X = \mathcal{U}^c / \mathcal{G}^c \tag{1.36}$$
for a suitable group of complexified gauge transformations $\mathcal{G}^c$.

2. The Kahler metric and symplectic structure are:
$$ds^2 = \int_{\tilde{M}_3} \mathrm{Tr}(\delta\mathcal{A} * \delta\overline{\mathcal{A}}) \tag{1.37}$$
eq:Uc-metric
$$\omega = \int_{\tilde{M}_3} \mathrm{Tr}(\delta A * \delta\phi). \tag{1.38}$$
eq:Uc-Symp

3. The holomorphic superpotential is the Chern-Simons form
$$W^{cs}(\mathcal{A}) = \int_{\tilde{M}_3} \mathrm{Tr}(\mathcal{A} d\mathcal{A} + \frac{2}{3}\mathcal{A}^3) \tag{1.39}$$
eq:CS-kappa

This formulation, while exact, is not easy to work with because the target space of the LG model is infinite-dimensional. In some situations the problem can be reduced to questions about a *finite-dimensional* LG model. This is one of the main results of Gaiotto and Witten [6].

The basic idea is to describe the link in $M_3 = \mathbb{R} \times C$ as an evolving set of points in C, $z_a(x^1)$, $a = 1, \dots n$, as in Figure 8:

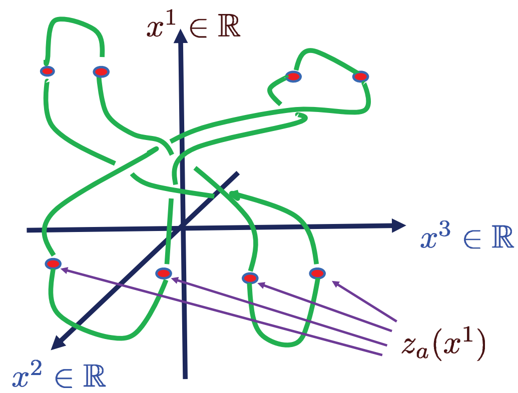


Figure 8: This figure depicts the link L in the boundary at $y = 0$ at a fixed value of x^0. It is presented as a tangle evolving in the x^1 direction and therefore can be characterized as a trajectory of points $z_a(x^1)$ in the complex $z = x^2 + ix^3$ plane. The tangle is closed by "creation" and "annihilation" of the points z_a in pairs (with identical values of k_a). fig:KNOT-HOM-4

We first consider the case where the $z_a(x^1)$ are *constant* and study the MSW complex in that case, finding equivalence to certain LG models with finite-dimensional target space.

Then, when we introduce "slow" x^1-dependence we have a family, parametrized by x^1, of LG models, a so-called "Janus" or "interface" of theories: See the discussion in Lecture 3, Section 3.1.

Gaiotto and Witten studied the case that $C = \mathbb{C}$ or $\mathbb{CP}^1$ and $G = SU(2)$ or $G = SO(3)$. In this case the representations on the link components define positive integers k_a at the points z_a (so the representations have dimension $k_a + 1$). They showed that, in the case that $z_a(x^1)$ are constant, the stationary points of the CS-LG theory correspond to "opers with monodromy-free singularities" on $\tilde{M}_3$. By this we mean the following:

We have a flat gauge $SL(2, \mathbb{C})$ gauge field

$$\mathcal{D}_a = \partial_a + \mathcal{A}_a \qquad a = z, \bar{z}, y \tag{1.40}$$

The gauge fields obey boundary conditions:

$$\mathcal{A} \to \frac{1}{2y} \begin{pmatrix} dy & 2dz \\ 0 & -dy \end{pmatrix} + \cdots \qquad z \neq z_a, y \to 0 \tag{1.41}$$ eq:NP-2

For $y \to \infty$ at fixed z we have

$$\mathcal{A} \to -\frac{dz}{\xi}\begin{pmatrix} \mathbf{c} & 0 \\ 0 & -\mathbf{c} \end{pmatrix} + \xi d\bar{z}\begin{pmatrix} \bar{\mathbf{c}} & 0 \\ 0 & -\bar{\mathbf{c}} \end{pmatrix} + dy\begin{pmatrix} c_1 & 0 \\ 0 & -c_1 \end{pmatrix} \quad (1.42) \quad \text{eq:y-infty-bc}$$

where $\mathbf{c}$ is complex, c_1 is real, and ξ is a phase, with a flat section:

$$\begin{aligned} \mathcal{D}_{\bar{z}} s &= 0 \\ \mathcal{D}_y s &= 0 \end{aligned} \quad (1.43) \quad \text{eq:Flt-Sec}$$

and moreover has the properties that for $y \to 0$ at fixed z, ♣also at $z = z_a$? ♣

$$s \wedge \mathcal{D}_z s = K(z) := \prod_{a=1}^{n} (z - z_a)^{k_a}, \qquad y \to 0 \quad (1.44) \quad \text{eq:Flt-Sec-bc-1}$$

Now, opers with monodromy-free singularities have known connections to the Gaudin model, WZW models, and free-field representations of conformal blocks. Using this, the problem is simplified to a study of the following Landau-Ginzburg theory, which we call the *Yang-Yang-Landau-Ginzburg model*:

1. Fix n distinct points $z_a \in \mathbb{C}$ labeled with positive integers k_a and let $q := \frac{1}{2}\sum_a k_a$. The target space of the model is a covering space of the configuration space $\mathcal{C}(q; \{z_a\})$ of q distinct, but indistinguishable points w_i, $i = 1, \dots, q$ on $C := \mathbb{C} - \{z_1, \dots, z_n\}$.

2. To define the covering space X we introduce the function:

$$W = \sum_{i,a} k_a \log(w_i - z_a) - \sum_{i<j} \log(w_i - w_j)^2 - c \sum_i w_i \quad (1.45) \quad \text{eq:YangYangW-1}$$

   where c is a nonzero complex number related to the boundary conditions of the oper at $y = \infty$. The target space X of the YYLG model should be the smallest cover on which W is single-valued as a function of the w_i.

3. We simply take the obvious Euclidean metric on X induced from $\sum_i |dw_i|^2$.

Now, as the $z_a(x^1)$ evolve we have a family of theories, and when the z_a undergo braiding or creation/fusion then we have interfaces between the theories. In a way we will describe in Lecture 3 we can construct a complex out of these interfaces, and this will be the knot homology complex.

1.6.2 Motivation 2: Spectral Networks

Consider a Hitchin system, say for $SU(N)$, on a (punctured) Riemann surface C. The spectral curve is

$$\Sigma := \{\det(\lambda - \varphi) = 0\} \subset T^*C \quad (1.46)$$

and is an $N : 1$ branched cover

$$\pi : \Sigma \to C \quad (1.47)$$

One of the mathematical outcomes of the paper Gaiotto:2012rg [7] was the definition of a "nonabelianization" map, a kind of inverse of the standard abelianization map of Higgs bundle theory. The statement is that, given a flat $\mathbb{C}^*$-connection on a complex line bundle

$$(L, \nabla^{ab}) \to \Sigma \tag{1.48}$$

we can push it forward to a flat rank N bundle

$$(E, \nabla) \to C \tag{1.49}$$

To do this, fix a phase ζ. One constructs a network of paths on C where the edges are integral curves

$$\int^z (\lambda_i - \lambda_j) = \zeta \tag{1.50}$$ eq:edge-equation

and where i, j label two sheets of the cover.

How to construct the actual network is a long story. One begins from branch points, (assumed simple) where three edges emerge as in Figure fig:SpectralNetworkBranchpoint2 9:

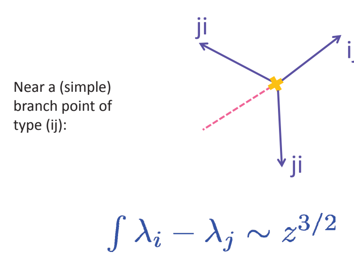


Figure 9: WKB paths in the neighborhood of a simple branchpoint exchanging sheets ij. fig:SpectralNetwo

The edges are then evolved using the equation eq:edge-equation (1.50), supplemented by some local rules at intersections. Call the spectral network $\mathcal{W}_\zeta$. Away from $\mathcal{W}_\zeta$ we have an isomorphism

$$E \cong \pi_*(L) \tag{1.51}$$

To define (E, ∇) everywhere we say what its parallel transport is along a path

$$\wp : z_1 \to z_2 \tag{1.52}$$

We say the parallel transport is

$$F(\wp) = P \exp \int_\wp \nabla = \sum_{\gamma_{ij'}} \underline{\overline{\Omega}}(\wp, \vartheta, \gamma_{ij'}) \mathcal{Y}_{\gamma_{ij'}} \tag{1.53}$$

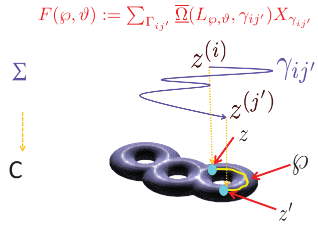


Figure 10: The lift of a path $\wp$ from z to z' is associated with a framed BPS state. Its "charge" is the relative homology class $\gamma_{ij'}$ of this path. fig:GammaIJ-Prime

where the sum is over homology paths $\gamma_{ij'}$ on Σ beginning and ending at preimages $z_1^{(i)}$ and $z_2^{(j')}$ as in Figure fig:GammaIJ-Prime 10

Moreover,

$$\mathcal{Y}_{\gamma_{ij'}} = \exp \int_{\gamma_{ij'}} \nabla^{ab} \tag{1.54}$$

and the $\overline{\underline{\Omega}}(\wp,\vartheta,\gamma_{ij'})$ are integers, known as "framed BPS degeneracies" . The crucial aspect of this definition is that if we consider a family of small paths where the endpoint crosses one of the edges of a spectral network, as in Figure fig:DetourRule1 11

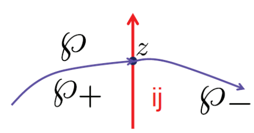


Figure 11: A small path $\wp$ crosses a single S-wall of type ij. fig:DetourRule1

then

$$\begin{aligned} F(\wp) = & \sum_i \exp \int_{\wp^{(i)}} \nabla^{ab} \\ & + \sum_{\gamma_{ij}\in\Gamma_{ij}(z,z)} \mu(\gamma_{ij}) \left(\exp \int_{\wp_+^{(i)}} \nabla^{ab}\right)\left(\exp \int_{\gamma_{ij}} \nabla^{ab}\right)\left(\exp \int_{\wp_-^{(i)}} \nabla^{ab}\right) \end{aligned} \tag{1.55}$$

eq:FRWC

where $\mu(\gamma_{ij})$ is another set of integers, known as "2d BPS soliton degeneracies."

Now, in this Hitchin situation physicists associate a canonical 1+1 dimensional quantum field theory $\mathbb{S}_z$ to each point $z \in C$. The vacua of $\mathbb{S}_z$ are in 1-1 correspondence with the preimages $z^{(i)}$ and in the above construction there are physical interpretations:

1. $\wp$ corresponds to an *interface* between theories $\mathbb{S}_{z_1}$ and $\mathbb{S}_{z_2}$.
2. $\overline{\underline{\Omega}}(\wp,\vartheta,\gamma_{ij'})$ correspond to degeneracies of BPS states of particles bound to the interface.

3. $\mu(\gamma_{ij})$ correspond to degneracies of 2d soliton states within the theory $\mathbb{S}_z$.

4. The rule (eq:FRWC 1.55) corresponds to a wall-crossing formula for the framed BPS states.

Remark: The quantum field theory $\mathbb{S}_z$ is in general not a Landau-Ginzburg theory, but it has many of the same features: It has massive vacua and $(2,2)$ supersymmetry. Pairs of critical points of the LG superpotential W correspond to relative homology classes γ_{ij} and differences of critical values of W, i.e., central charges in solitons sectors, correspond to $\int_{\gamma_{ij}} \lambda$.

One of the goals of AlgebraicStructures [8] is to categorify this story and replace the BPS degeneracies by chain complexes and the wall-crossing formula by a statement about suitable categories. We achieve this goal in Lecture 3 in Section §subsubsec:Cat-S-Wall 3.4.1.

2. Lecture 2: The Web Formalism

2.1 Boosted Solitons And ζ-Webs

Now we would like to interpret more precisely the meaning of Figure fig:StripInstanton 6.

2.1.1 Boosted Solitons

subsec:BoostSol

Recall that ζ-instantons satisfy

$$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial \tau}\right)\phi^I = \frac{\mathrm{i}\zeta}{2} g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}, \tag{2.1}$$

eq:LG-INST-p

and we are interested in solutions for arbitrary phase ζ.

Recall too that ζ-solitons on $D = \mathbb{R}$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}x}\phi^I = g^{I\bar{J}} \frac{\mathrm{i}\zeta}{2} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}} \tag{2.2}$$

eq:LG-flow-p

and with boundary conditions (ϕ_i, ϕ_j) at $x = -\infty, +\infty$ only exist for special phases $\mathrm{i}\zeta_{ji}$ given by the phase of the difference of critical values $W_j - W_i$.

We can use solitons of type ij to produce solutions of the ζ-instanton equation on the Euclidean plane by taking the ansatz:

$$\phi_{ij}^{\mathrm{boosted}}(x,\tau) := \phi_{ij}^{\mathrm{soliton}}(\cos \mu x + \sin \mu \tau), \tag{2.3}$$

obeys

$$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial \tau}\right)\phi_{ij}^{\mathrm{boosted},I}(x,\tau) = \frac{\mathrm{i}e^{\mathrm{i}\mu}\zeta_{ji}}{2} g^{I\bar{J}} \partial_{\bar{J}} \bar{W}(\phi_{ij}(x)) \tag{2.4}$$

eq:boosted-solito

so we choose the rotation μ so that

$$e^{\mathrm{i}\mu}\zeta_{ji} = \zeta \tag{2.5}$$

eq:xi-to-zeta

We call such solutions to the ζ-instanton equation *ζ-boosted solitons*. A short computation show that the "worldline" (i.e. the region where the solution is not exponential close to

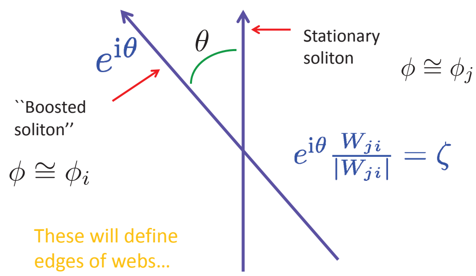


Figure 12: The boosted soliton. A short computation show that the "worldline" is parallel to the complex number $z_{ij} := z_i - z_j$ where $z_i = \zeta \bar{W}_i$. fig:BoostedSolito

one of the vacua ϕ_i or ϕ_j) is parallel to the complex number $z_{ij} := z_i - z_j$ where $z_i = \zeta \bar{W}_i$. See Figure fig:BoostedSolitonFlorida 12.

Now we can start to interpret the "extra" ζ-instanton illustrated in Figure fig:StripInstanton 6. The idea is that if the width of the interval is much larger than ℓ_W then the ζ-instanton is well-approximated, away from the boundaries, by a boosted soliton. There is some kind of "emission amplitude" and "absorption amplitude" associated with the region where the boosted soliton joins the boundaries. In order to discuss these we first consider the ζ-instanton equation on the plane, but with some unusual boundary conditions.

2.1.2 Fan Boundary Conditions

We would like to make a solution to the ζ-instanton equation that looks like several boosted solitons at infinity. Thus suppose we have a *cyclic fan of solitons*:

$$\mathcal{F} = \{\phi^{p_1}_{i_1,i_2}, \ldots, \phi^{p_n}_{i_n,i_1}\}. \tag{2.6}$$ eq:solseq

We would like to have a solution which looks like the corresponding boosted solitons as z moves clockwise around a circle at infinity, as in Figure fig:WEDGES 13.

Note this only makes sense when the phases of the successive differences $z_{i_k,i_{k+1}}$ are clockwise ordered. We call such a sequence of vacua a *cyclic fan of vacua*.

If the index of a certain Dirac operator is positive then we expect, from index theory, that there will be ζ-instantons which approach such a cyclic fan of solitons at infinity. In fact, physicists studying domain wall junctions have in fact established the existence of such solutions in some special cases Carroll:1999wr,Gibbons:1999np [1, 9]. We will assume that a moduli space of such solutions $\mathcal{M}(\mathcal{F})$ exists. Based on physical intuition we expect these moduli spaces to satisfy two crucial properties:

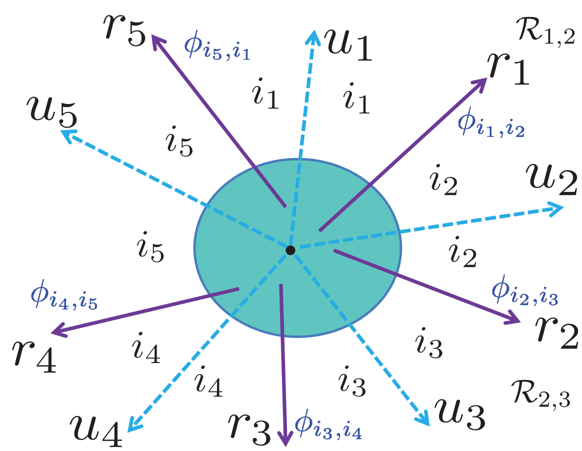


Figure 13: Boundary conditions on the ζ-instanton equation defined by a cyclic fan of solitons. fig:WEDGES

1. *Gluing*: Under favorable conditions, two solutions which only differ significantly from fan solutions inside a bounded region can be glued together as in Figure fig:GluedSolution 14 This process can be iterated to produce what we call ζ-*webs*, shown in Figure fig:ZetaWeb 15.

2. *Ends*: The moduli space $\mathcal{M}(\mathcal{F})$ can have several connected components. Some of these components will be noncompact, and the "ends," or "boundaries at infinity," of the moduli space will be described by ζ-webs.

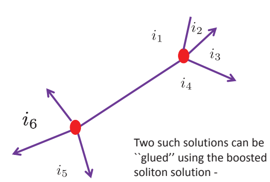


Figure 14: Gluing two solutions with fan boundary conditions to produce a new solution with fan boundary conditions. The red regions indicate where the solution deviates significantly from the boosted solitons and the vacua. When the "centers" of the two ζ-instantons are far separated the approximate, glued, field configuration can be corrected to a true solution. fig:GluedSolution

The compact connected components of ζ-webs are called ζ-*vertices*. We are most interested in the ζ-vertices of dimension zero: These will contribute to the path integral of the LG model with fan boundary conditions provided the fermion number of the outgoing states sums to 2. We claim that counting such points for fixed fans of solitons produces interesting integers that satisfy L_∞ identities. We will state that a bit more precisely later.

This picture is the inspiration for the web-formalism, to which we turn next. It will give us the language to state the above claim in more precise terms.

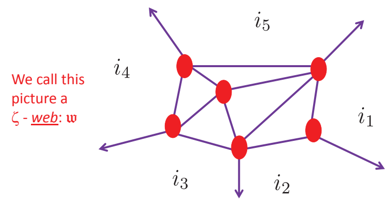


Figure 15: Several solutions can be glued together to produce a ζ-web solution fig:ZetaWeb

2.2 The Web Formalism On The Plane

Now switch to a mathematical formalism that we call the *web-based formalism* for describing the above physics.

2.2.1 Planar Webs And Their Convolution Identity

Definition: The *vacuum data* is the pair $(\mathbb{V}, z)$ where $\mathbb{V}$ is a finite set called the *set of vacua* and $z : \mathbb{V} \to \mathbb{C}$ defines the *vacuum weights*.

Remarks:

1. Vacua are denoted $i, j, \dots \in \mathbb{V}$. The vacuum weight associated to i is denoted z_i.
2. The vacuum weights $\{z_i\}$ are assumed to be in general position. This means

$$\{z_1, \dots, z_N\} \in \mathcal{V} := \mathbb{C}^N - \mathfrak{E} \tag{2.7}$$

eq:VacWtSpace

where $\mathfrak{E}$ is the exceptional set. Thus, $z_{ij} \neq 0$ for $i \neq j$. Moreover, no three vacuum weights are colinear and finally there are no exceptional webs. [3]

Definition: A *plane web* is a graph in $\mathbb{R}^2$, together with a coloring of the *faces* by vacua such that the labels across each edge are different and moreover, when oriented with i on the left and j on the right the edge is straight and parallel to the complex number $z_{ij} := z_i - z_j$. We take plane webs to have all vertices of valence at least two.

Definition The *deformation type* of a web is the equivalence class under stretching of internal edges and overall translation. There is a moduli space of deformation types and it can be oriented. We denote an oriented deformation type by $\mathfrak{w}$.

Example: An example of two different deformation types of web is shown in Figure fig:DIFFERENT-DEFORMATION- 16.

A key construction we can make with webs is known as *convolution*. To define it we introduce some terminology:

[3] *Exceptional webs* are defined to be webs whose deformation space has a dimension larger than the expected dimension $2V - E$. For details see AlgebraicStructures [8].

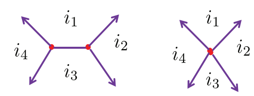


Figure 16: The two webs shown here are considered to be different deformation types, even though the web on the left can clearly degenerate to the web on the right.

1. *The local fan at a vertex* $v \in \mathfrak{w}$*:* is denoted $I_v(\mathfrak{w})$.
2. *The fan of vacua at infinity:* is denoted $I_\infty(\mathfrak{w})$.

For example see Figure 17:

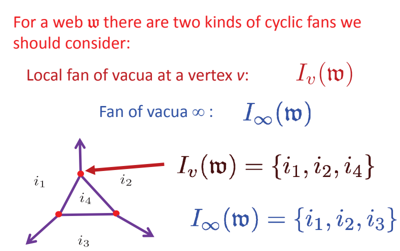


Figure 17: Illustrating the local fan of vacua and the fan of vacua at infinity for a web $\mathfrak{w}$.

Now, suppose we have two webs $\mathfrak{w}$ and $\mathfrak{w}'$ such that there is a vertex v of $\mathfrak{w}$ we have

$$I_v(\mathfrak{w}) = I_\infty(\mathfrak{w}'). \tag{2.8}$$

Then define $\mathfrak{w} *_v \mathfrak{w}'$ to be the deformation type of a web obtained by cutting out a small disk around v and gluing in a suitably scaled and translated copy of the deformation type of $\mathfrak{w}'$. The procedure is illustrated in Figure 18.

The upshot is that if $\mathcal{W}$ is the free abelian group generated by oriented deformation types of webs then convolution defines a product

$$\mathcal{W} \times \mathcal{W} \to \mathcal{W} \tag{2.9}$$

(making it a "pre-Lie algebra" in the sense of [4]).

Now, we consider the *taut webs*. These are, by definition, those with only one internal degree of freedom. That is, the moduli space of the taut webs is three-dimensional. See Figure 19:

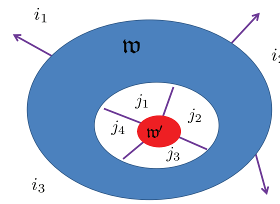


Figure 18: Illustrating the convolution of a web $\mathfrak{w}$ with internal vertex v having a fan $I_v(\mathfrak{w}) = \{j_1, j_2, j_3, j_4\}$ with a web $\mathfrak{w}'$ having an external fan $I_\infty(\mathfrak{w}') = \{j_1, j_2, j_3, j_4\}$. fig:CONVOLUTION

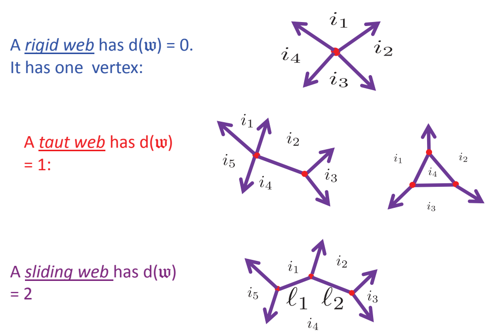


Figure 19: Illustrating rigid, taut, and sliding webs with 0, 1, and 2 internal degrees of freedom. fig:RigidTautSlid

We define the *taut element* to be the sum over all the taut webs:

$$\mathfrak{t} := \sum_{d(\mathfrak{w})=3} \mathfrak{w}. \tag{2.10}$$

eq:taut-planar

we can coherently orient all the taut webs in, say, the direction of getting larger.

Now the key theorem is that

$$\mathfrak{t} * \mathfrak{t} = 0. \tag{2.11}$$

The proof is that if we expand this out then we can group products in pairs which cancel. The pairs correspond to opposite ends of a moduli space of "sliding" webs, with two internal degrees of freedom. The idea is illustrated in Figure 20: fig:TAUT-SQUARE

2.2.2 Representation Of Webs

Definition: A *representation of webs* is a pair $\mathcal{R} = (\{R_{ij}\}, \{K_{ij}\})$ where R_{ij} are $\mathbb{Z}$-graded $\mathbb{Z}$-modules defined for all ordered pairs ij of distinct vacua and K_{ij} is a degree -1 symmetric perfect pairing

$$K_{ij} : R_{ij} \otimes R_{ji} \to \mathbb{Z}. \tag{2.12}$$

whichone

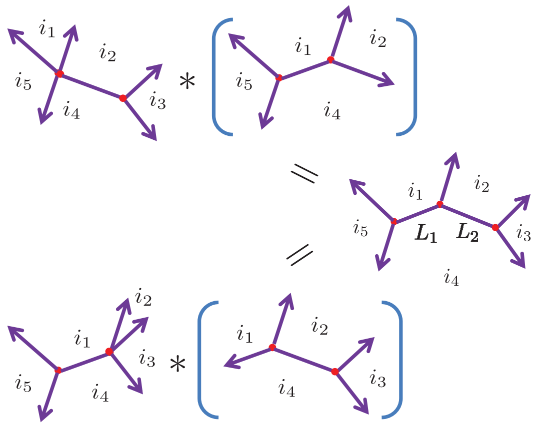


Figure 20: The two boundaries of the deformation type of the sliding web shown on the right correspond to different convolutions shown above and below. If we use the lengths L_1, L_2 of the edges as coordinates then the orientation from the top convolution is $dL_2 \wedge dL_1$. On the other hand the orientation from the bottom convolution is $dL_1 \wedge dL_2$ and hence the sum of these two convolutions is zero. This is the key idea in the demonstration that $\mathfrak{t} * \mathfrak{t} = 0$. fig:TAUT-SQUARE

Given a representation of webs, we define a representation of a cyclic fan of vacua $I = \{i_1, i_2, \ldots, i_n\}$ to be

$$R_I := R_{i_1,i_2} \otimes R_{i_2,i_3} \otimes \cdots \otimes R_{i_n,i_1} \tag{2.13}$$

when I is the cyclic fan at a vertex of a web we refer to $R_{I_v(\mathfrak{w})}$ to as the *representation of the vertex*. Elements of this representation are called *interior vectors*.

Next we collect the representations of all the vertices by forming

$$R^{\rm int} := \oplus_I R_I \tag{2.14}$$ eq:Rint-def

where the sum is over all cyclic fans of vacua. We want to define a map

$$\rho(\mathfrak{w}) : TR^{\rm int} \to R^{\rm int} \tag{2.15}$$

where for any $\mathbb{Z}$-module M we define the tensor algebra to be

$$TM := M \oplus M^{\otimes 2} \oplus M^{\otimes 3} \oplus \cdots \tag{2.16}$$

In fact, the operation will be graded-symmetric so it descends to a map from the symmetric algebra $SR^{\rm int} \to R^{\rm int}$.

We now define the *contraction operation*:

We take $\rho(\mathfrak{w})[r_1, \ldots, r_n]$ to be zero unless $n = V(\mathfrak{w})$ and there exists an order $\{v_1, \ldots, v_n\}$ for the vertices of $\mathfrak{w}$ such that $r_a \in R_{I_{v_a}(\mathfrak{w})}$. If such an order exists, we will define our map

$$\rho(\mathfrak{w}) : \otimes_{v \in \mathcal{V}(\mathfrak{w})} R_{I_v(\mathfrak{w})} \to R_{I_\infty(\mathfrak{w})} \tag{2.17}$$ eq:web-rep-1

as the application of the contraction map K to all internal edges of the web. Indeed, if an edge joins two vertices $v_1, v_2 \in \mathcal{V}(\mathfrak{w})$ then if $R_{I_{v_1}(\mathfrak{w})}$ contains a tensor factor R_{ij} it follows that $R_{I_{v_2}(\mathfrak{w})}$ contains a tensor factor R_{ji} and these two factors can be paired by K as shown in Figure fig:WEBEDGE 21:

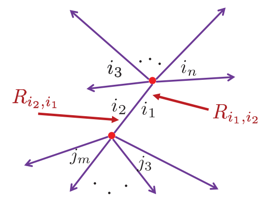


Figure 21: The internal lines of a web naturally pair spaces R_{i_1,i_2} with R_{i_2,i_1} in a web representation, as shown here. fig:WEBEDGE

It is not difficult to see that the convolution identity $\mathfrak{t} * \mathfrak{t} = 0$ implies that $\rho(\mathfrak{t})$ satisfies the axioms of an L_∞ algebra $\rho(\mathfrak{t}) : TR^{\rm int} \to R^{\rm int}$:

$$\sum_{\mathrm{Sh}_2(S)} \epsilon_{S_1,S_2}\ \rho(\mathfrak{t})[\rho(\mathfrak{t})[S_1], S_2] = 0 \tag{2.18}$$

eq:L-infty-rho

where we sum over 2-shuffles of the ordered set $S = \{r_1, \dots, r_n\}$ and ϵ_{S_1,S_2} is a sign factor discussed at length in AlgebraicStructures [8].

Definition: An *interior amplitude* is an element $\beta \in R^{\rm int}$ of degree $+2$ so that if we define $e^\beta \in TR^{\rm int} \otimes \mathbb{Q}$ by the exponential series then

$$\rho(\mathfrak{t})(e^\beta) = 0. \tag{2.19}$$

eq:bulk-amp

Definition: A *Theory* $\mathcal{T}$ consists of a set of vacuum data $(\mathbb{V}, z)$, a representation of webs $\mathcal{R} = (\{R_{ij}\}, \{K_{ij}\})$ and an interior amplitude β.

Remark: If β is an interior amplitude and we define $\rho_\beta(\mathfrak{w})[r_1, \dots, r_\ell] := \rho(\mathfrak{w})[r_1, \dots, r_\ell, e^\beta]$ then $\rho_\beta(\mathfrak{t}) : TR^{\rm int} \to R^{\rm int}$ satisfies the L_∞ Maurer-Cartan equation.

2.2.3 Realization Via LG Models

1. *Vacua*: $\mathbb{V}$ is the set of critical points of W.

2. *Vacuum weights*: $z_i = \zeta \bar{W}_i$

3. *Web representation*:
$$R_{ij} := \oplus_{p \in \mathcal{S}_{ij}} \mathbb{Z} \Psi^{f+1}(p) \tag{2.20}$$
and the contraction K is defined by the path integral.

4. *Interior amplitude*: Suitably interpreted, the path integral leads to a counting of ζ-instantons with fan boundary conditions and defines an element in R^{int} which is an interior amplitude β. This follows from localization of the path integral on the moduli space of ζ-instantons and the fact that the path integral must create a $\mathcal{Q}_\zeta$-closed state AlgebraicStructures [8].

2.2.4 Examples: Theories With Cyclic Weights

Two useful examples have $\mathbb{V} = \mathbb{Z}/N\mathbb{Z}$. We break the cyclic symmetry and label vacua by $i \in \{0, \dots, N-1\}$ with weights:

$$\mathbb{V}_\vartheta^N : z_k = e^{-\mathrm{i}\vartheta - \frac{2\pi \mathrm{i}}{N} k} \qquad k = 0, \cdots N-1 \tag{2.21}$$ eq:CyclicWt

The first example is $\mathcal{T}_\vartheta^N$ with a single chiral superfield and superpotential

$$W = \zeta \frac{N+1}{N} \left(\phi - e^{-\mathrm{i}N\vartheta} \frac{\phi^{N+1}}{N+1} \right). \tag{2.22}$$ eq:TN-Superpot

The web-representation is

$$\begin{aligned} R_{ij} &= \mathbb{Z}^{[1]} \qquad i < j \\ R_{ij} &= \mathbb{Z} \qquad i > j \end{aligned} \tag{2.23}$$ eq:ExpleTN-webrep

At a vertex of valence n we have $\deg R_I = n-1$ and hence only 3-valent vertices contribute to the MC equations, so the only nonzero amplitudes are $\beta_{ijk} \in R_{ijk}$ for $0 \le i < j < k \le N-1$. The L_∞ equations come from the two taut webs of Figure fig:TNEXAMPLE-1 22:

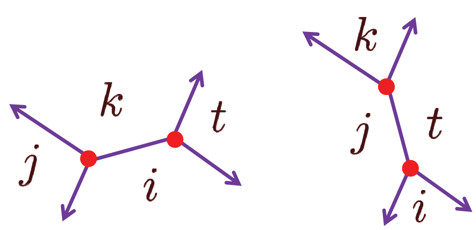


Figure 22: The two terms in the component of the L_∞ equations for $i < j < k < t$. fig:TNEXAMPLE-1

and are just:

$$b_{ijk} b_{ikt} - b_{ijt} b_{jkt} = 0 \qquad i < j < k < t \tag{2.24}$$ eq:ExpleMC-1

A more elaborate set of examples is provided by the mirror dual to the B-model on $\mathbb{CP}^{N-1}$ with $SU(N)$ symmetry. This again has vacuum weights (eq:CyclicWt 2.21) but now we take

$$R_{ij} = A^{[1]}_{j-i} \qquad i < j$$
$$R_{ij} = A_{N+j-i} \qquad i > j \tag{2.25}$$ eq:SUN-Rij

where A_ℓ is the ℓ-th antisymmetric power of a fundamental representation of $SU(N)$ and

$$K_{ij}(v_1 \otimes v_2) = \kappa_{ij} \frac{v_1 \wedge v_2}{\mathrm{vol}} \tag{2.26}$$ eq:Kij-TSUN

with $\kappa_{ij} \in \{\pm 1\}$. An $SU(N)$-invariant ansatz for the interior amplitude reduces the L_∞ MC equations to (2.24) above.

2.3 The Web Formalism On The Half-Plane

Fix a half-plane $\mathcal{H} \subset \mathbb{R}^2$ in the (x, τ) plane. Most of our pictures will take the positive or negative half-plane, $x \geq x_\ell$ or $x \leq x_r$, but it could be any half-plane.

Definition: Suppose $\partial\mathcal{H}$ is not parallel to any of the z_{ij}. A *half-plane web* in $\mathcal{H}$ is a graph in the half-plane, which allows some vertices to be subsets of the boundary. We apply the same rule as for plane webs: Label connected components of the complement of the graph by vacua so that if the edges are oriented with i on the left and j on the right then they are parallel to z_{ij}.

We can again speak of a deformation type of a half-plane web $\mathfrak{u}$. Now translations parallel to the boundary of $\mathcal{H}$ act freely on the moduli space. Once again we define half-plane webs to be *rigid, taut,* and *sliding* if $d(\mathfrak{u}) = 1, 2, 3$, respectively. Similarly, we can define oriented deformation type in an obvious way and consider the free abelian group $\mathcal{W}_{\mathcal{H}}$ of oriented deformation types of half-plane webs in the half-plane $\mathcal{H}$. Some examples where $\mathcal{H} = \mathcal{H}_L$ is the positive half-plane are shown in Figures 23 and 24.

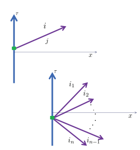


Figure 23: Two examples of rigid positive-half-plane webs. fig:BNDRY-WEB-1

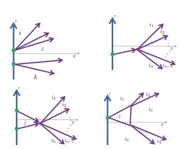


Figure 24: Four examples of taut positive-half-plane webs fig:HALFPLANE-TAU

There are now two kinds of convolutions:

1. Convolution at a boundary vertex defines

$$* : \mathcal{W}_{\mathcal{H}} \times \mathcal{W}_{\mathcal{H}} \to \mathcal{W}_{\mathcal{H}} \tag{2.27}$$ eq:bbstar

2. Convolution at an interior vertex defines:

$$*: \mathcal{W}_{\mathcal{H}} \times \mathcal{W} \to \mathcal{W}_{\mathcal{H}} \tag{2.28}$$ eq:bistar

We now define the half-space taut element (oriented in the direction in which the web gets bigger):

$$\mathfrak{t}_{\mathcal{H}} := \sum_{d(\mathfrak{u})=2} \mathfrak{u}. \tag{2.29}$$

The convolution identity is

$$\mathfrak{t}_{\mathcal{H}} * \mathfrak{t}_{\mathcal{H}} + \mathfrak{t}_{\mathcal{H}} * \mathfrak{t}_p = 0. \tag{2.30}$$

The idea of the proof is the same as in the planar case. An example is shown in Figure fig:BLK-BDRY-WEBIDENT 25:

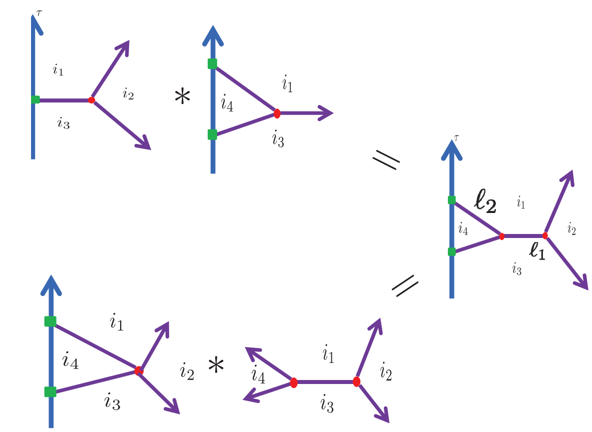


Figure 25: An example of the identity on plane and half-plane taut elements. On the right is a sliding half-plane web. Above is a convolution of two taut half-plane webs with orientation $dy \wedge d\ell_1 \wedge d\ell_2$. Below is a convolution of a taut half-plane web with a taut plane web. The orientation is $dy \wedge d\ell_2 \wedge d\ell_1$. The two convolutions determine the same deformation type but have opposite orientation, and hence cancel. fig:BLK-BDRY-WEBI

2.4 Categorification Of The 2D Spectrum Generator

ubsec:Cat-Muij

Given a half-plane and a representation of webs we can introduce a collection of chain complexes $\widehat{R}_{ij}$ that will play and important role in what follows.

One way to motivate the $\widehat{R}_{ij}$ is to recall the Cecotti-Vafa-Kontsevich-Soibelman wall-crossing formula Cecotti:1992rm,Kontsevich:2008fj [3, 13] for the Witten indices/BPS degeneracies $\mu_{ij} = \mathrm{Tr}_{R_{ij}}(-1)^{\mathcal{F}}$ of 2d solitons. The μ_{ij} were extensively studied in Fendley:1992dm,Cecotti:1992qh,Cecotti:1992rm [5, 2, 3] where the wall-crossing phenomenon was first discussed. One way to state the formula uses the matrix of BPS degeneracies

$$\mathbf{1} + \oplus_{z_{ij}\in\mathcal{H}} \widehat{\mu}_{ij} e_{ij} = \bigotimes_{z_{ij}\in\mathcal{H}} (\mathbf{1} + \mu_{ij} e_{ij}) \tag{2.31}$$ eq:2d-CVKS-prod

where we assume there are N vacua so we can identify $\mathbb{V} = \{1, \dots, N\}$, e_{ij} are elementary $N \times N$ matrices, $\mathbf{1}$ is the $N \times N$ unit matrix, and in the tensor product we order the factors left to right by the clockwise order of the phase of z_{ij}. Continuous deformations of the Kähler metric $g_{I\bar{J}}$ or the superpotential will lead to jumps of the μ_{ij} when z_{ij} become parallel. The wall-crossing formula states that nevertheless, the matrix (2.31) remains constant as long as no ray enters of leaves $\mathcal{H}$.

The matrix (2.31) is sometimes called the "spectrum generator." We now "categorify" the spectrum generator, and define $\widehat{R}_{ij}$ from the formal product

$$\widehat{R} := \oplus_{i,j=1}^{N} \widehat{R}_{ij} e_{ij} := \bigotimes_{z_{ij} \in \mathcal{H}} (\mathbb{Z} \cdot \mathbf{1} + R_{ij} e_{ij}) \tag{2.32}$$

eq:Cat-KS-prod

Note that $\widehat{R}_{ii} = \mathbb{Z}$ is concentrated in degree zero and $\widehat{R}_{ij} = 0$ if z_{ij} points in the opposite half-plane $-\mathcal{H}$. If $J = \{j_1, \dots, j_n\}$ is a half-plane fan in $\mathcal{H}$ then we define

$$R_J := R_{j_1,j_2} \otimes \cdots \otimes R_{j_{n-1},j_n} \tag{2.33}$$

eq:RJ

and $\widehat{R}_{ij}$ is just the direct sum over all R_J for half-plane fans J that begin with i and end with j.

Remarks:

1. We can "enhance" the (categorified) spectrum generator $\widehat{R}$ with "Chan-Paton factors." By definition, *Chan-Paton data* is an assignment $i \to \mathcal{E}_i$ of a $\mathbb{Z}$-graded module to each vacuum $i \in \mathbb{V}$. The modules $\mathcal{E}_i$ will be referred to as *Chan-Paton factors.* The enhanced spectrum generator is defined to be

$$\widehat{R}(\mathcal{E}) := \oplus_{i,j \in \mathbb{V}} \widehat{R}_{ij}(\mathcal{E}) e_{ij} := \left(\oplus_{i \in \mathbb{V}} \mathcal{E}_i e_{ii}\right) \widehat{R} \left(\oplus_{j \in \mathbb{V}} \mathcal{E}_j e_{jj}\right)^* \tag{2.34}$$

eq:Add-CP-Hop

2. Phase ordered products such as (2.31) have also appeared in many previous works on Stokes data, so the R_{ij} can also be considered to be "categorified Stokes factors" and $\widehat{R}$ is an "categorified Stokes matrix."

3. If we consider a family of theories where the rays z_{ij} and z_{jk} pass through each other then the categorified spectrum generator $\widehat{R}$ is in general *not* invariant. In Lecture 3 will discuss the categorified version of the above wall-crossing formula.

2.5 A_∞-Categories Of Thimbles And Branes

2.5.1 The A_∞-Category Of Thimbles

We now want to define the *A_∞-category of Thimbles*, denoted $\mathfrak{Vac}$: Suppose we are given the data of a Theory $\mathcal{T}$ and a half-plane $\mathcal{H}$. Then $\mathfrak{Vac}$ has as objects the vacua $i, j, \dots \in \mathbb{V}$ (as we will see, they are better thought of as Thimble branes $\mathfrak{T}_i, \mathfrak{T}_j, \dots$). The space of morphisms $\mathrm{Hom}(j,i)$, (which we also denote as $\mathrm{Hop}(i,j) := \mathrm{Hom}(j,i)$ since many formulae in A_∞-theory look much nicer when written in terms of Hop) is simply

$$\mathrm{Hop}(i,j) := \widehat{R}_{ij} \tag{2.35}$$

We can enhance the category with Chan-Paton factors. The morphism spaces are simply

$$\mathrm{Hop}^{\mathcal{E}}(i,j) := \widehat{R}_{ij}(\mathcal{E}) = \mathcal{E}_i \widehat{R}_{ij} \mathcal{E}_j^* . \tag{2.36}$$

The corresponding category is denoted $\mathfrak{Vac}(\mathcal{E})$.

Now we need to define the A_∞-multiplication in $\mathfrak{Vac}(\mathcal{E})$ of an n-tuple of composable morphisms. As a first step, for any half-plane web $\mathfrak{u}$ we define a map

$$\rho(\mathfrak{u}) : T\widehat{R}(\mathcal{E}) \otimes TR^{\mathrm{int}} \to \widehat{R}(\mathcal{E}) \tag{2.37}$$

It will be graded symmetric on the second tensor factor. As usual, we define the element

$$\rho(\mathfrak{u})[r_1^\partial, \dots, r_m^\partial; r_1, \dots, r_n] \tag{2.38}$$

eq:Rho-rp-r

by contraction. We will abbreviate this to $\rho(\mathfrak{u})[P;S]$ where $P = \{r_1^\partial, \dots, r_m^\partial\}$ and $S = \{r_1, \dots, r_n\}$. We define $\rho(\mathfrak{u})[P;S]$ to be zero unless the following conditions hold:

- The numbers of interior and boundary vertices of $\mathfrak{u}$ match the number of arguments of either type: $V_\partial(\mathfrak{u}) = m$ and $V_i(\mathfrak{u}) = n$.
- The boundary arguments match in order and type those of the boundary vertices: $r_a^\partial \in R_{J_{v_a^\partial}(\mathfrak{u})}(\mathcal{E})$.
- We can find an order of the interior vertices $\mathcal{V}_i(\mathfrak{u}) = \{v_1, \dots, v_n\}$ of $\mathfrak{u}$ such that they match the order and type of the interior arguments: $r_a \in R_{I_{v_a}(\mathfrak{u})}$.

If the above conditions hold, we will simply contract all internal lines with K and contract the Chan Paton elements of consecutive pairs of r_a^∂ by the natural pairing $\mathcal{E}_i \otimes \mathcal{E}_j^* \to \delta_{ij}\mathbb{Z}$. With this definition in hand, we can check that the convolution identity for taut elements implies a corresponding identity for $\rho[\mathfrak{t}_{\mathcal{H}}]$:

$$\sum_{\mathrm{Sh}_2(S), \mathrm{Pa}_3(P)} \epsilon\, \rho(\mathfrak{t}_{\mathcal{H}})[P_1, \rho(\mathfrak{t}_{\mathcal{H}})[P_2; S_1], P_3; S_2] + \sum_{\mathrm{Sh}_2(S)} \epsilon\, \rho(\mathfrak{t}_{\mathcal{H}})[P; \rho(\mathfrak{t}_p)[S_1], S_2] = 0. \tag{2.39}$$

eq:big-rel-rho

where $\mathrm{Pa}_3(P)$ is the set of partitions of the ordered set P into an ordered set of three disjoint ordered sets, all inheriting the ordering of P. We call (2.39) the LA_∞ relations.

The most important consequence of these identities is that if we are given an interior amplitude β, we can immediately produce an A_∞ category where the multiplication

$$\rho_\beta(\mathfrak{t}_{\mathcal{H}}) : T\widehat{R}(\mathcal{E}) \to \widehat{R}(\mathcal{E}) \tag{2.40}$$

eq:R-AFTYALG

is defined by saturating all the interior vertices with the interior amplitude:

$$\rho_\beta(\mathfrak{t}_{\mathcal{H}})[r_1^\partial, \dots, r_m^\partial] := \rho(\mathfrak{t}_{\mathcal{H}})[r_1^\partial, \dots, r_m^\partial; e^\beta]. \tag{2.41}$$

This has the effect of killing the second term in (2.39) and combining the first summand into the usual defining relations for an A_∞-category. The product is illustrated in Figure 26.

Remark: The conceptual meaning of (2.39) is that there is an L_∞ morphism from the L_∞ algebra R^{int} to the L_∞ algebra of the Hochschild cochain complex of the A_∞ category $\mathfrak{Vac}(\mathcal{E})$. The paper [12] shows that in the present context the map is in fact an L_∞ isomorphism.

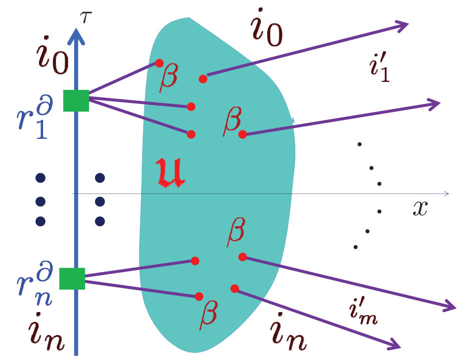


Figure 26: Illustrating the A_∞-product on time-ordered boundary vectors $r_1^\partial, \dots, r_n^\partial$. We sum over taut half-plane webs $\mathfrak{u}$, indicated by the green blob, and saturate all interior vertices with the interior amplitude β. fig:AINFTY-PRODUC

ec:VacCategory

2.5.2 The A_∞-Category Of Branes

We define a *Brane*, denoted $\mathfrak{B} = (\mathcal{E}, \mathcal{B})$ to be a choice of Chan-Paton data $\mathcal{E}$ together with a *boundary amplitude*, that is, a degree +1 element

$$\mathcal{B} \in \widehat{R}(\mathcal{E}) \tag{2.42}$$

which solves the Maurer-Cartan equations

$$\sum_{n=1}^{\infty} \rho_\beta(\mathfrak{t}_{\mathcal{H}})[\mathcal{B}^{\otimes n}] = \rho_\beta(\mathfrak{t}_{\mathcal{H}})[\frac{\mathcal{B}}{1-\mathcal{B}}] = 0. \tag{2.43}$$

eq:boundary-amp

The category of Branes is denoted $\mathfrak{Br}$. It depends on the Theory $\mathcal{T}$ and the half-plane $\mathcal{H}$. Its objects are Branes $\mathfrak{B} = (\mathcal{E}, \mathcal{B})$ where $\mathcal{E}$ is *any* choice of Chan Paton data $\mathcal{E}$ and $\mathcal{B}$ is a compatible boundary amplitude. The space of morphisms from $\mathfrak{B}_2$ to $\mathfrak{B}_1$ is defined by simply modifying the enhanced spectrum generator to

$$\mathrm{Hop}(\mathfrak{B}_1, \mathfrak{B}_2) := \left(\oplus_i \mathcal{E}_i^1 e_{ii}\right) \otimes \widehat{R} \otimes \left(\oplus_i \mathcal{E}_i^2 e_{ii}\right)^* . \tag{2.44}$$

eq:BHOM

In order to define the composition of morphisms

$$\delta_1 \in \mathrm{Hop}(\mathfrak{B}_0, \mathfrak{B}_1), \quad \delta_2 \in \mathrm{Hop}(\mathfrak{B}_1, \mathfrak{B}_2), \dots, \delta_n \in \mathrm{Hop}(\mathfrak{B}_{n-1}, \mathfrak{B}_n) \tag{2.45}$$

we use the formula

$$M_n(\delta_1, \dots, \delta_n) := \rho_\beta(\mathfrak{t}_{\mathcal{H}}) \left(\frac{1}{1-\mathcal{B}_0}, \delta_1, \frac{1}{1-\mathcal{B}_1}, \delta_2, \dots, \delta_n, \frac{1}{1-\mathcal{B}_n} \right). \tag{2.46}$$

eq:BraneMultiplic

Note that $M_n(\delta_1, \dots, \delta_n) \in \mathrm{Hop}(\mathfrak{B}_0, \mathfrak{B}_n)$. After some work (making repeated use of the fact that the $\mathcal{B}_a$ solve the A_∞-Maurer-Cartan equation) one can show that the M_n satisfy the A_∞-relations and hence $\mathfrak{Br}$ is an A_∞-category.

Remarks:

1. The multiplication (eq:BraneMultiplications 2.46) can be illustrated much as in Figure fig:AINFTY-PRODUCT 26. The only difference is that now the boundary vectors r_s^{∂} don't have to saturate all boundary vertices. Rather, boundary vertices between r_k^{∂} and r_{k+1}^{∂} can be saturated by the boundary amplitude $\mathcal{B}_k$.

2. For each vacuum i we define the Thimble Brane $\mathfrak{T}_i$ to be the brane with CP data $\mathcal{E}(\mathfrak{T}_i)_j = \delta_{i,j}\mathbb{Z}$ with boundary amplitude $\mathcal{B}(\mathfrak{T}_i) = 0$. Then the category of Thimbles $\mathfrak{Vac}$ is a full subcategory of $\mathfrak{Br}$. ♣Say it also for $\mathfrak{Vac}(\mathcal{E})$. ♣

2.5.3 Realization In The LG Model

sec:BraneAmpLG

Choose $\mathcal{H}$ to be the positive half-plane with boundary conditions set by a Lagrangian $\mathcal{L} \subset X$. The Chan-Paton data is given by the MSW complex:

$$\mathcal{E}_i = \mathbb{M}_{\mathcal{L},i} \tag{2.47}$$

We consider amplitudes with boundary conditions shown in Figure fig:HALFPLANEBC 27. Counting the number of ζ-instantons satisfying these boundary conditions can be used to define an element in $\mathcal{B}_J \in \mathcal{E} \otimes R_J \otimes \mathcal{E}^*$. As with the case of the interior amplitude, localization of the path integral to the moduli space of ζ-instantons together with $\mathcal{Q}_\zeta$-closure of the state produced by the path integral implies that $\mathcal{B}$ is a boundary amplitude in the above sense.

In general $\mathrm{Hop}(\mathfrak{B}_1, \mathfrak{B}_2)$ is a space of $\mathcal{Q}_\zeta$-closed local boundary operators and the physical interpretation of $M_n(\delta_1, \dots, \delta_n)$ is that we are taking a kind of "operator product." The $\mathcal{Q}_\zeta$ closure of the path integral implies that the M_n satisfy the A_∞-MC equation.

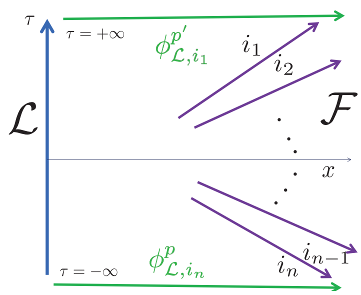


Figure 27: Boundary conditions for general half-plane instantons with fan boundary conditions at $x \to +\infty$ and solitons at $\tau \to \pm\infty$. fig:HALFPLANEBC

Remark: If we want good morphism spaces associated to the interval $[x_\ell, x_r]$ we need to restrict the class of Lagrangian submanifolds, as we have seen. In the web-based formalism we definitely do *not* want branes of class T_κ for $\kappa \in U(1) - \{\pm\zeta\}$!! One way to see this is that it is important to allow left-boundary branes which are left Lefshetz thimbles of phase ζ. These are in T_ζ, not T_κ. Moreover, the construction of the category should not depend on the particular position x_ℓ of the boundary. But again from equation (eq:x-flow 1.24) it is clear that it would depend on x_ℓ if we used branes of class T_κ. In AlgebraicStructures [8] it is argued that the suitable class of branes are *W-dominated branes* for which $\mathrm{Im}(\zeta^{-1}W) \to +\infty$ at infinity. (For right-branes on boundaries of the negative half-plane we require $\mathrm{Im}(\zeta^{-1}W) \to -\infty$.)

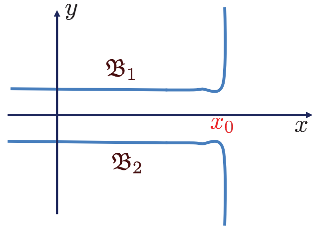


Figure 28: We count rigid ζ-instantons in the funnel geometry to define an A_∞-morphism between the FS category and the web-based category. The branes $\mathfrak{B}_1, \mathfrak{B}_2$ are in class T_ζ. fig:FUNNEL-STRIP

2.6 Relation Of The Web-Based Formalism To The FS Category

Now we would like the relate the A_∞-category constructed in the FS approach and in the web-based approach, say, for the positive half-plane. The web-based formalism applies to branes of class T_ζ and our description of the FS category applies to branes of class T_κ with $\kappa \neq \pm\zeta$.

To relate the two we strongly use the rotational non-invariance of the ζ-instanton equation and consider the FS category based on branes of class T_ζ but now the morphism spaces are defined by solving the equation on a horizontal strip, obtained from the vertical one by rotation by $\pi/2$. Thus, to define the MSW complex $\mathbb{M}_{\mathfrak{B}_1,\mathfrak{B}_2}$ the generators are given by solutions of the ζ-instanton equation which are invariant under translation in x, not in τ. Now we can use branes of class T_ζ on the upper and lower boundary.

To relate the FS and web-based categories we now consider the ζ-instanton equation on the funnel geometry of Figure 28:

A state in the far past at $x \to -\infty$ on the strip is an incoming soliton, in the above sense. A state in the morphisms in the web-based formalism gives half-plane fan boundary conditions at infinity for the positive half-plane. But these two states determine boundary conditions for the ζ-instanton equation on the space in Figure 28. We can therefore define a map

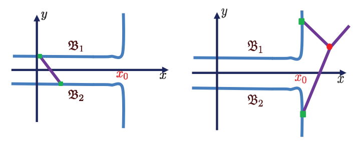
$$\mathcal{U} : \mathbb{M}_{\mathfrak{B}_1,\mathfrak{B}_2} \to \mathrm{Hop}(\mathfrak{B}_1, \mathfrak{B}_2) \tag{2.48}$$

The matrix elements of $\mathcal{U}$ are defined by counting ζ-instantons in the funnel geometry. When we consider states of the same fermion number the expected dimension of the moduli space is dimension zero and the moduli space is expected to be a finite set of points.

To prove that $\mathcal{U}$ is a chain map we consider the one-dimensional moduli spaces of solutions to the ζ-instanton equation between states whose fermion number differs by 1. The two ends correspond to ζ-instantons far down the strip - giving the differential on $\mathbb{M}_{\mathfrak{B}_1,\mathfrak{B}_2}$ and taut webs far out on the positive half-plane, giving the differential on $\mathrm{Hop}(\mathfrak{B}_1, \mathfrak{B}_2)$, so

$$\mathcal{U} \circ M_1^{\mathrm{FS}} - M_1^{\mathrm{web}} \circ \mathcal{U} = 0 \tag{2.49}$$

where M_1 denotes the differential on the morphisms in the A_∞-category. $\mathcal{U}$ can be extended to an A_∞-equivalence between the categories [8].



(a) (b)

Figure 29: When the difference of fermion numbers of ingoing and outgoing states is +1 there will be a one-dimensional moduli space of ζ-instantons. The two typical boundaries are indicated in (a) and (b). They lead to the two terms in the equation assuring that $\mathcal{U}$ is a chain map. fig:FUNNEL-CHAINM

3. Lecture 3: Interfaces And Categorified Wall-Crossing

3.1 Motivation: Interfaces In Landau-Ginzburg Models

c:LG-Interface

Suppose we have a family of superpotentials $W(\phi; c)$, parametrized by a point c in a topological space C. [4] Suppose $\wp : [x_\ell, x_r] \to C$ is a continuous path. Then we can define a variant of LG theory based on an x-dependent superpotential:

$$W_x(\phi) := W(\phi; \wp(x)), \tag{3.1}$$

so that $W_x(\phi)$ is constant (in x) for $x \le x_\ell$ and for $x \ge x_r$. Clearly this $1+1$ dimensional theory no longer has translational invariance. It does, however, still have two out of the four supersymmetries of LG theory. This is demonstrated most easily if we take the approach via Morse theory/SQM using the Morse function on $\mathrm{Map}(\mathbb{R}, X)$:

$$h = -\int_{\mathbb{R}} \left[\phi^*(\lambda) - \frac{1}{2}\mathrm{Re}(\zeta^{-1} W(\phi; \wp(x)) dx \right] \tag{3.2}$$ eq:Interface-h

Clearly the resulting theory has a kind of "defect" or "domain wall" localized near $[x_\ell, x_r]$ interpolating between the left LG theory defined with superpotential $W_{x_\ell}(\phi)$ and the right LG theory defined with superpotential $W_{x_r}(\phi)$).

We will refer to this as a (LG, supersymmetric) *interface*. The term "Janus" is also often used in the literature.

Thus we have a continuous family of vacuum weights

$$z_i(x) = \zeta \bar{W}_x(\phi_{i,x}) \tag{3.3}$$

where the vacuum i is parallel transported from the vacua in the theory at x_ℓ and $\phi_{i,x}$ are the critical points of the superpotential $W_x(\phi)$. The ζ-instanton equation now becomes:

$$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial \tau} \right) \phi^I = \frac{\mathrm{i}\zeta}{2} g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}(\bar{\phi}; \wp(x)) \tag{3.4}$$ eq:LG-forced-flow

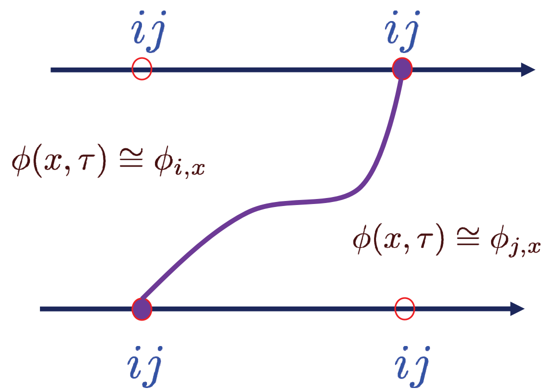


Figure 30: An analog of the boosted soliton for the case of a supersymmetric interface. fig:FORCEDFLOWBOO

and ζ-solitons are just τ-independent solutions. The analog of boosted solitons have curved worldlines, as in Figure 30

Now, we would like to define a relation of the branes in the left theory to the branes in the right theory by "parallel-transporting" across the interface.

3.2 Abstract Formulation: Flat Parallel Transport Of Brane Categories

Suppose we have a "continuous family of Theories." We use the term "Theory" in the sense of the web formalism. To make sense of this one must put a topology on the set of Theories. Note that the set of vacuum weights $\mathcal{V}$ of (2.7) carries a natural topology. Thus we can certainly speak of a continuous map

$$\wp : [x_\ell, x_r] \to \mathcal{V} = \mathbb{C}^N - \mathfrak{E} \tag{3.5}$$ eq:tame

We call this a *vacuum homotopy.*

More generally, one could also define a sense in which web representations and the interior amplitudes change continuously. So, in general, we have a continuous family of Theories $\mathcal{T}(x)$ on $[x_\ell, x_r]$. We would like to relate $\mathcal{T}^\ell = \mathcal{T}(x_\ell)$ to $\mathcal{T}^r = \mathcal{T}(x_r)$. More precisely, we want to define an A_∞-functor

$$\mathcal{F}(\wp) : \mathfrak{Br}(\mathcal{T}^\ell, \mathcal{H}) \to \mathfrak{Br}(\mathcal{T}^r, \mathcal{H}) \tag{3.6}$$

where $\mathcal{H}$ is, say, the positive half-plane.

The functor $\mathcal{F}(\wp)$ is meant to be a categorical version of parallel transport by a flat connection. Thus we want:

1. An A_∞-equivalence of functors:

$$\mathcal{F}(\wp_1) \circ \mathcal{F}(\wp_2) \cong \mathcal{F}(\wp_1 \circ \wp_2) \tag{3.7}$$

[4] C can be any space, but the notation is again chosen because one of the primary motivations is the theory of spectral networks and Hitchin systems.

for composable paths $\wp_1, \wp_2$.

2. An A_∞-equivalence of functors:

$$\mathcal{F}(\wp_1) \cong \mathcal{F}(\wp_2) \tag{3.8}$$

for paths $\wp_1, \wp_2$ homotopic in, say, $\mathcal{V}$.

We will show that one <u>can</u> construct such functors for "tame" vacuum homotopies, of the type (3.5). Flushed with success we then <u>want</u> to extend the construction to more general vacuum homotopies for paths of weights which cross the exceptional walls $\mathfrak{E}$. But you don't always get what you want:

> The existence of such a functor <u>forces</u> discontinuous changes of the web representation and the interior amplitude: This is the categorified version of wall-crossing.

The secret to constructing $\mathcal{F}(\wp)$ is the theory of *Interfaces* in the web-based formalism, to which we turn next.

3.3 Interface Webs And Composite Webs

3.3.1 The A_∞-Category Of Interfaces

In order to understand the parallel transport of Brane categories it will actually be very useful to consider *discontinuous* jumps between Theories.

Given a pair of vacuum data $(\mathbb{V}^-, z^-)$ and $(\mathbb{V}^+, z^+)$ we can define an interface web by using the data on the negative and positive half-planes, respectively. Examples are shown in Figures 31 and 33 below. We can define the taut element $\mathfrak{t}^{-,+}$ and write a convolution identity.

If we are given left and right Theories $(\mathcal{T}^-, \mathcal{T}^+)$ then we can define a representation of interface webs:

1. Chan-Paton factors now depend on a <u>pair</u> of vacua $\mathcal{E}_{j_-,j'_+}$.

2. At a boundary vertex we have the representation:

$$R_J(\mathcal{E}) := \mathcal{E}_{j_m,j'_1} \otimes R^+_{J'_+} \otimes \mathcal{E}^*_{j_1,j'_n} \otimes R^-_{J_-}. \tag{3.9}$$ eq:RJ-intfc

associated to the picture in Figure 31, where $J = (J_-, J'_+)$.

Now the categorified spectrum generator is given by the product

$$\widehat{R}(\mathcal{E}) = \left(\oplus_{i,i'}\mathcal{E}_{ii'}\ e_{ii}\otimes e_{i'i'}\right) \otimes \widehat{R}(\mathcal{T}^-,\mathcal{H}^-)^{\mathrm{tr}} \otimes \widehat{R}(\mathcal{T}^+,\mathcal{H}^+) \otimes \left(\oplus_{j,j'}\mathcal{E}_{jj'}\ e_{jj}\otimes e_{j'j'}\right)^* \tag{3.10}$$ eq:Inf-Vac-Homs2

See Figure 31 for a typical summand.

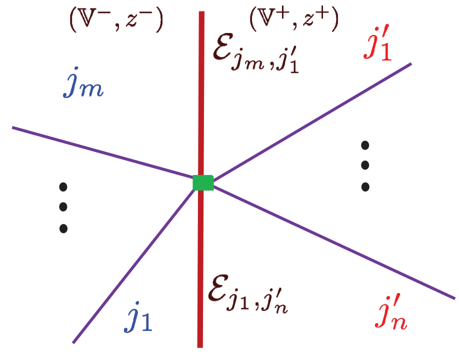


Figure 31: Conventions for Chan-Paton factors localized on interfaces. If representation spaces are attached to the rays then this figure would represent a typical summand in $\mathrm{Hom}(j_m j'_1, j_1 j'_n)$. We order such vertices from left to right using the conventions of positive half-plane webs. fig:DOMAINWALL-CH

Now an interface amplitude is a degree one element $\mathcal{B}^{-,+} \in \widehat{R}(\mathcal{E})$ satisfying the A_∞-MC equation:

$$\rho(\mathfrak{t}^{-,+}) \left(\frac{1}{1 - \mathcal{B}^{-,+}}; e^{\beta_-}; e^{\beta_+} \right) = 0 \tag{3.11}$$

We define an *Interface* to be a pair

$$\mathfrak{I}^{-,+} = (\mathcal{E}^{-,+}, \mathcal{B}^{-,+}) \tag{3.12}$$

and we can define an A_∞-category of Interfaces, denoted

$$\mathfrak{Br}(\mathcal{T}^-, \mathcal{T}^+). \tag{3.13}$$

The objects of $\mathfrak{Br}(\mathcal{T}^-, \mathcal{T}^+)$ are Interfaces, for some choice of CP data and the space of morphisms between $\mathfrak{I}_2^{-,+}$ and $\mathfrak{I}_1^{-,+}$ is the natural generalization of (eq:BHOM 2.44):

$$\mathrm{Hop}(\mathfrak{I}_1^{-,+}, \mathfrak{I}_2^{-,+}) := \left(\oplus_{i,i'} \mathcal{E}^1_{ii'}\ e_{ii} \otimes e_{i'i'}\right) \otimes \widehat{R}(\mathcal{T}^-, \mathcal{H}^-)^{\mathrm{tr}} \otimes \widehat{R}(\mathcal{T}^+, \mathcal{H}^+) \otimes \left(\oplus_{j,j'} \mathcal{E}^2_{jj'}\ e_{jj} \otimes e_{j'j'}\right)^* . \tag{3.14}$$

The A_∞-multiplications are given by the natural generalization of equation (eq:BraneMultiplications 2.46): we just contract with the taut element $\mathfrak{t}_{\mathcal{H}} \to \mathfrak{t}^{-,+}$ and saturate all interior vertices with the left or right interior amplitude β^-, β^+.

Remarks:

1. An Interface between the empty theory and itself is precisely the data of a Chain complex. See Figure fig:INTFCE-TRIV-TRIV 32 for the explanation.

2. *The identity Interface.* A very useful example of an Interface is the identity Interface $\mathfrak{Id} \in \mathfrak{Br}(\mathcal{T}, \mathcal{T})$. The CP spaces are

$$\mathcal{E}(\mathfrak{Id})_{ij} = \delta_{i,j} \mathbb{Z} \tag{3.15}$$

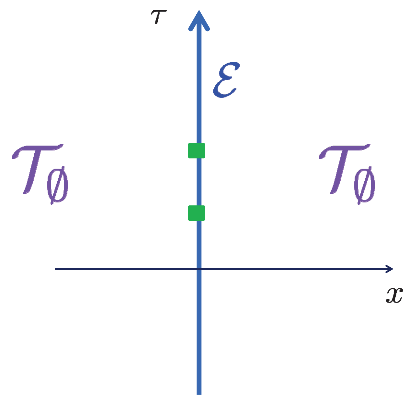


Figure 32: The only taut interface web when $\mathcal{T}^\ell, \mathcal{T}^r$ are the trivial theory has two boundary vertices. The boundary amplitude is associated to a single boundary vertex: $\mathcal{B} \in \mathcal{E} \otimes \mathcal{E}^*$ is a morphism of $\mathcal{E}$ of degree one. There is only one taut web, shown above. The MC therefore says that $\mathcal{B}^2 = 0$. Thus as Interface between the trivial theory and itself is the same thing as a chain complex. fig:INTFCE-TRIV-T

and

$$\widehat{R}(\mathcal{E}) = \oplus_{i,j} \widehat{R}^+_{ij} \otimes \widehat{R}^-_{ji} e_{ij} \otimes e_{ij} \tag{3.16}$$

where the superscripts $\pm$ indicate that $\widehat{R}$ is defined with respect to the positive, negative half-plane, respectively. To define the interface we take $\mathcal{B}_{\mathcal{I}}$ to have nonzero component only in summands of the form $R_{ij} \otimes R_{ji}$ corresponding to the fan $\{i, j; j, i\}$. The vertex looks like a straight line of a fixed slope running through the domain wall. The boundary amplitude is the element in $R_{ij} \otimes R_{ji}$ given by K_{ij}^{-1}. and the Maurer-Cartan equation is proved by Figure 33: fig:ID-INTERFACE

3. *Landau-Ginzburg interfaces and branes in the product theory*: In the context of Landau-Ginzburg models we can consider interfaces between a theory defined by (X_1, W_1) on the negative half-plane and (X_2, W_2) on the positive half-plane. By the doubling trick we would expect such interfaces to be related to branes for the positive half-plane of the theory based on $(\bar{X}_1 \times X_2, \bar{W}_1 + W_2)$. This is morally correct, but there are two closely related subtleties which should be pointed out. First, from the purely abstract formalism, if we try to related Interface amplitudes for a pair of Theories $\mathcal{T}^-, \mathcal{T}^+$ to boundary amplitudes for $\mathcal{T}^- \times \mathcal{T}^+$ we will, in general, fail: The vacua of the product theory are labeled by (j_-, j_+) but the slopes of the edges of the webs are the slopes of $z_{j^1_-, j^2_-} + z_{j^1_+, j^2_+}$. In general half-plane fans for the product theory will have nothing to do with pairs of half-plane fans in the left and right theories. The two concepts <u>will</u> be equivalent, however, in the special case that the web representations are of the form

$$R_{(j^1_-, j^1_+),(j^2_-, j^2_+)} = \delta_{j^1_-, j^2_-} R^+_{j^1_+, j^2_+} \oplus \delta_{j^1_+, j^2_+} R^-_{j^1_-, j^2_-}. \tag{3.17}$$ eq:SpecialRep

Second, on the Landau-Ginzburg side, if we literally take the product metric and the product superpotential then the Morse function h_1+h_2 is too degenerate: The critical manifolds are $\mathbb{R}\times\mathbb{R}$, corresponding to a center of mass collective coordinate for two separate solitons. We must perturb the theory by perturbing the superpotential with $\Delta W(\bar{\phi}_1, \phi_2)$. Generic such perturbations will in fact produce MSW complexes giving web representations of the form (eq:SpecialRep 3.17).

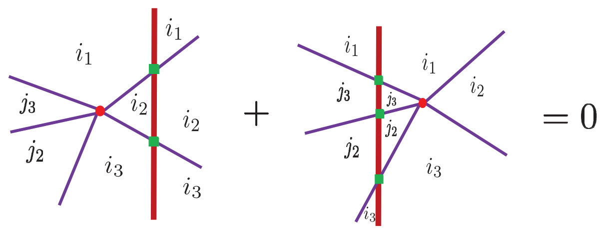


Figure 33: Examples of taut interface webs which contribute to the Maurer-Cartan equation for the identity interface $\mathfrak{I}\mathfrak{d}$ between a Theory and itself. fig:ID-INTERFACE

3.3.2 Composition Of Interfaces

A crucial new ingredient is that Interfaces can be composed. Suppose we have a situation as shown in Figure fig:CompInterface1 34 with a pair of Interfaces $\mathfrak{I}^{-,0}$ and $\mathfrak{I}^{0,+}$:

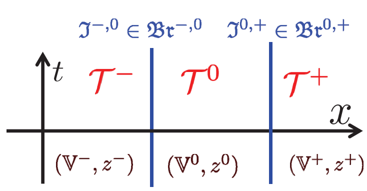


Figure 34: Two interfaces between a sequence of three Theories. fig:CompInterface

then we will produce a new Interface, denoted

$$\mathfrak{I}^{-,0} \boxtimes \mathfrak{I}^{0,+} \in \mathfrak{Br}(\mathcal{T}^-, \mathcal{T}^+) \tag{3.18}$$

as shown in Figure fig:CompInterface2 35:

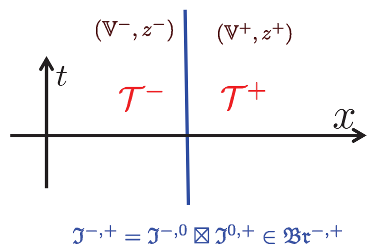


Figure 35: The Interface resulting from the "operator product" of the two Interfaces. fig:CompInterface

The key idea in the construction is to use "composite webs" $\mathfrak{c} = (\mathfrak{u}^{-}, \mathfrak{s}, \mathfrak{u}^{+})$. An example is shown in Figure fig:COMPOSITEWEB1 36:

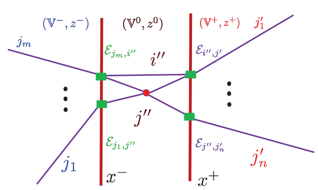


Figure 36: An example of a composite web, together with conventions for Chan-Paton factors. In this web the fan of vacua at infinity has $J_\infty(\mathfrak{c}) = \{j_1', \dots j_n'; j_1, \dots, j_m\}$ Reading from left to right the indices are in clockwise order. fig:COMPOSITEWEB1

Again one can develop the whole web theory, write taut elements and a convolution identity. (The convolution identity has some novel features. See AlgebraicStructures [8] for details.) The upshot is that the product Interface $\mathfrak{I}^{-,0} \boxtimes \mathfrak{I}^{0,+}$ has

1. *Chan-Paton data*:

$$\mathcal{E}(\mathfrak{I}^{-,0} \boxtimes \mathfrak{I}^{0,+})_{ii'} := \oplus_{i'' \in \mathbb{V}^0} \mathcal{E}^{-,0}_{i,i''} \otimes \mathcal{E}^{0,+}_{i'',i'} \tag{3.19}$$
eq:Comb-CP

2. *Interface amplitude*:

$$\mathcal{B}(\mathfrak{I}^{-,0} \boxtimes \mathfrak{I}^{0,+}) := \rho_\beta(\mathfrak{t}_c) \left[\frac{1}{1 - \mathcal{B}^{-,0}} ; \frac{1}{1 - \mathcal{B}^{0,+}} \right] \tag{3.20}$$
eq:InterfaceComp

where $\mathfrak{t}_c$ is the taut element for composite webs.

Using the convolution identity (omitted here) one can show that it indeed satisfies the Maurer Cartan equations for an interface amplitude between the theories $\mathcal{T}^{-}$ and $\mathcal{T}^{+}$ with Chan-Paton spaces (eq:Comb-CP 3.19).

Now one can show that we have an A_∞-bifunctor

$$\mathfrak{Br}(\mathcal{T}^-,\mathcal{T}^0) \times \mathfrak{Br}(\mathcal{T}^0,\mathcal{T}^+) \to \mathfrak{Br}(\mathcal{T}^-,\mathcal{T}^+) \tag{3.21}$$

This is illustrated in Figure fig:InterfaceBiFunctor 37

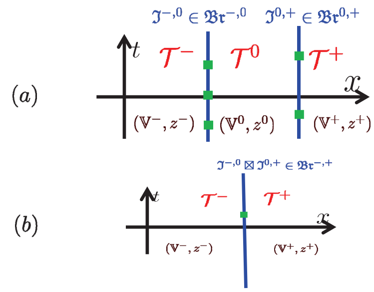


Figure 37: Illustrating the bi-functor property: We take the "OPE" of both local boundary operators on the interfaces, and of the interfaces, shown in (a), to produce a local operator on an interface, shown in (b). fig:InterfaceBiFu

An important special case is the case where $\mathcal{T}^-$ is the trivial Theory so that $\mathfrak{Br}(\mathcal{T}^-,\mathcal{T}^0) = \mathfrak{Br}(\mathcal{T}^0)$. Then we see that an Interface in $\mathfrak{Br}(\mathcal{T}^0,\mathcal{T}^+)$ gives an A_∞-functor on categories of Branes:

$$\mathfrak{Br}(\mathcal{T}^0) \times \mathfrak{Br}(\mathcal{T}^0,\mathcal{T}^+) \to \mathfrak{Br}(\mathcal{T}^+) \tag{3.22}$$

Physically: We are moving a $0,+$ interface into a boundary and mapping a boundary condition for Theory $\mathcal{T}^0$ to one for Theory $\mathcal{T}^+$.

Thus, our quest for parallel transport of Brane categories will be fulfilled if we can find suitable Interfaces $\mathfrak{I}[\wp]$ associated with paths between theories $\mathcal{T}^\ell$ and $\mathcal{T}^r$.

3.3.3 Homotopy of Branes and Interfaces

Part of the A_∞-structure of the category of Branes and Interfaces is that the Hop spaces have a differential: If $\delta \in \mathrm{Hop}(\mathfrak{B}_1,\mathfrak{B}_2)$ then

$$M_1(\delta) = \rho_\beta(\mathfrak{t}_{\mathcal{H}})\left(\frac{1}{1-\mathcal{B}_1}, \delta, \frac{1}{1-\mathcal{B}_2}\right) \tag{3.23}$$

and $M_1 \circ M_1 = 0$, when this makes sense. We can thus define a notion of homotopy equivalence of Branes (and entirely parallel definitions apply to Interfaces):

1. Two morphisms are homotopy equivalent if $\delta_1 - \delta_2 = M_1(\delta_3)$.

2. Two Branes are homotopy equivalent, denoted, $\mathcal{B} \sim \mathcal{B}'$, if there are two M_1-closed morphisms $\delta : \mathcal{B} \to \mathcal{B}'$ and $\delta' : \mathcal{B}' \to \mathcal{B}$ which are inverses up to homotopy. That is:

$$M_2(\delta,\delta') \sim \mathbf{Id} \qquad\qquad M_2(\delta',\delta) \sim \mathbf{Id}. \tag{3.24}$$

eq:hmtpy-Br

where $\mathbf{Id}$ is the natural identity in $\oplus_i \mathcal{E}_i \otimes \mathcal{E}_i^*$.

3.3.4 An A_∞ 2-Category Of Interfaces

A natural question to ask about the composition of Interfaces is whether it is associative. In fact, to define the composite webs we need to choose positions on the x-axis of the two domain walls and where the final interface should be located. These positions can influence the set of composite webs. So we should really denote the product of Interfaces by

$$\left(\mathfrak{I}^{-,0} \boxtimes \mathfrak{I}^{0,+}\right)_{x^{-,0},x^{0,+},x^{-,+}} \tag{3.25}$$

However, one can show that the product only depends on these positions up to homotopy equivalence. The proof, which is somewhat long involves developing a theory of webs which are time-dependent. Similarly, one can prove that the composition is associative, up to homotopy equivalence. All the details are in AlgebraicStructures [8].

The net result of this is that we have what might be called an "A_∞-2-category" structure:

1. The objects, or 0-morphisms are the Theories.

2. The 1-morphisms between two Theories are Interfaces $\mathfrak{I}^{-,+}$.

3. The 2-morphisms between two 1-morphisms are the boundary-changing operators on the Interface.

This is illustrated in Figure fig:TwoCategoryInterfaces 38:

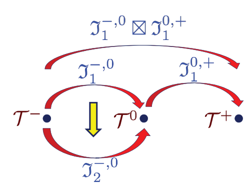


Figure 38: Illustrating the two category of Theories, Interfaces, and boundary operators. fig:TwoCategoryIr

3.4 An example of categorical transport

We will now sketch how one can actually construct a parallel transport interface for a tame vacuum homotopy:

$$\wp : x \mapsto \{z_i(x)\} \in \mathbb{C}^N - \mathfrak{E} \tag{3.26}$$

eq:SpinningWeight

which does not cross the exceptional walls $\mathfrak{E}$. We assume $\wp(x)$ only varies on a compact set $[x_\ell, x_r]$.

Our goal is to define an Interface

$$\mathfrak{I}[\wp] \in \mathfrak{Br}(\mathcal{T}^\ell, \mathcal{T}^r) \tag{3.27}$$

so that if $\wp^1(x) \sim \wp^2(x)$ give homotopic paths of vacuum weights with fixed endpoints then $\mathfrak{I}[\wp^1]$ and $\mathfrak{I}[\wp^2]$ are homotopy-equivalent Interfaces, and such that if we compose two paths then

$$\mathfrak{I}[\wp^1] \boxtimes \mathfrak{I}[\wp^2] \sim \mathfrak{I}[\wp^1 * \wp^2] \quad (3.28)$$

eq:trspt-1

where $\sim$ means homotopy equivalence.

The key is to construct an analogous theory of *curved webs* where the ij edges have tangents at (x, τ) parallel to $z_i(x) - z_j(x)$. One crucial new feature emerges for curved webs. Following the tangent vectors, sometimes the edges are forced to go to infinity at finite values of x. These special values of x are known as *binding points*. We can have "future stable" binding points as in Figure fig:FUTURESTABLE-1 39 or "past stable" binding points as in Figure fig:PASTSTABLE-1 40.

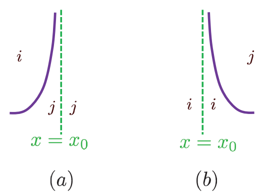


Figure 39: Near a future stable binding point x_0 of type ij the edges of type ij and of type ji asymptote to the dashed green line $x = x_0$. Figure (a) shows the behavior of edges of type ij and Figure (b) shows the behavior of edges of type ji. fig:FUTURESTABLE-

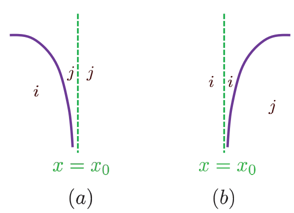


Figure 40: Near a past stable binding point x_0 of type ij the edges of type ij and of type ji asymptote to the dashed green line $x = x_0$. Figure (a) shows the behavior of edges of type ji and Figure (b) shows the behavior of edges of type ij. fig:PASTSTABLE-1

The binding points x_0 are characterized as the values of x for which

$$z_{ij}(x_0) \in \mathrm{i}\mathbb{R}_+ \quad (3.29)$$

eq:Sij-Ray-def

The future/past stability is determined by the sense in which $\mathrm{Re}\,(z_{ij}(x))$ passes through zero as x passes through x_0:

1. *Future stable binding point*: As x increases past x_0 $z_{ij}(x)$ goes through the positive imaginary axis in the counter-clockwise direction.

2. *Past stable binding point*: As x increases past x_0 $z_{ij}(x)$ goes through the positive imaginary axis in the clockwise direction.

Now we define Chan-Paton data of the desired Interface. For each binding point x_0 of type ij introduce a matrix with chain-complex entries. It depends on whether x_0 is future-stable or past stable:

$$S_{ij}(x_0) := \mathbb{Z} \cdot \mathbf{1} + R_{ij} e_{ij} \qquad \textit{future stable} \tag{3.30}$$ eq:SijFactor-def-

$$S_{ij}(x_0) := \mathbb{Z} \cdot \mathbf{1} + R^*_{ji} e_{ij} \qquad \textit{past stable.} \tag{3.31}$$ eq:SijFactor-def-

We will refer to $S_{ij}(x_0)$ as a *categorified S_{ij}-factor*, or just as an *S_{ij}-factor*, for short. Then we define the Chan-Paton factors of the Interface to be:

$$\oplus_{j,j' \in \mathbb{V}} \mathcal{E}_{j,j'} e_{j,j'} := \bigotimes_{i \neq j} \bigotimes_{x_0 \in \curlyvee_{ij} \cup \curlywedge_{ij}} S_{ij}(x_0) \tag{3.32}$$ eq:TautCurvedCP

where the tensor product on the RHS of (3.32) is ordered from left to right by increasing values of x_0. The amplitudes for the Interface are simply given by evaluating the taut curved web on the interior amplitude: $\rho(\mathfrak{t}_{\rm curved})(e^{\beta})$. (This formula needs some interpretation. See [8] for details.) In this way we get an Interface

$$\mathfrak{I}[\wp] \in \mathfrak{Br}(\mathcal{T}^{\ell}, \mathcal{T}^{r}). \tag{3.33}$$ eq:defIth

associated to the tame vacuum homotopy $\wp(x)$. It satisfies the desired properties for parallel transport.

In particular, thanks to the composition property (3.28) we can break up $\mathfrak{I}[\wp]$ as a product of Interfaces as in Figure 41:

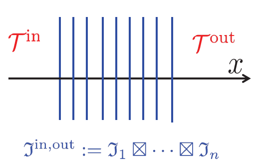


Figure 41: Breaking up the path $\wp$ into elementary paths we need only produce special interfaces for "trivial" transport, and for transport across S-walls. fig:ElementaryInt

We need only construct then the Interfaces for crossing the S_{ij} walls. These are denoted $\mathfrak{S}^{p,f}_{ij}$ for past and future stable crossings, respectively. The amplitudes can be described quite explicitly. See [8]. The functors $(\cdot) \boxtimes \mathfrak{S}^{p,f}_{ij}$ are closely related to mutations.

3.4.1 Categorified S-Wall-Crossing

sec:Cat-S-Wall

We now return to one of our motivations from Lecture 1.

Given an Interface $\mathfrak{I}^{-,+}$ associated with a path of theories the *framed BPS degeneracies* are, by definition:

$$\overline{\underline{\Omega}}(\mathfrak{I}^{-,+}, ij') := \mathrm{Tr}_{\mathcal{E}(\mathfrak{I}^{-,+})_{ij'}}(-1)^{\mathcal{F}} \tag{3.34}$$

eq:FramedBPS-def

If we consider a path $\wp_x$ whose endpoint terminates with $z(x)$, which crosses an ij binding point as x increases past x_0 (and hence $z(x)$ crosses an S_{ij}-wall) then the matrix of Witten indices

$$F[\wp_x] := \sum_{k,\ell} \overline{\underline{\Omega}}(\mathfrak{I}[\wp_x], k, \ell) e_{k,\ell}. \tag{3.35}$$

jumps by

$$F \mapsto \begin{cases} F \cdot (\mathbf{1} + \mu_{ij} e_{ij}) & x_{ij} \in \curlywedge_{ij} \\ F \cdot (\mathbf{1} - \mu_{ji} e_{ij}) & x_{ij} \in \curlyvee_{ij} \end{cases} \tag{3.36}$$

This is the framed wall-crossing. Now, since the Witten index of R_{ij} is μ_{ij} we recognize the formula for the change of the Interface

$$\mathfrak{I}[\wp_x] \to \mathfrak{I}[\wp_x] \boxtimes \mathfrak{S}_{ij}^{p,f} \tag{3.37}$$

as x crosses the binding point as a categorification of the S-wall crossing formula.

Example: Consider the Theory $\mathcal{T}^{N=2}$ above, that is $W \sim \phi^3 - z\phi$. The family is parametrized by $z \in C$ with $C = \mathbb{C}^*$. There are two massive vacua at $\phi_\pm = \pm z^{1/2}$. We choose a path $\wp$ defined by $z(x)$ in $\mathbb{C}^*$ where $x \in [\epsilon, 1-\epsilon]$ for ϵ infinitesimally small and positive with $z(x) = e^{\mathrm{i}(1-2x)\pi}$. There are two binding points of type $+-$ at $x = 1/3 - 0^+, 1 - 0^+$ and one binding point of type $-+$ at $x = 2/3 - 0^+$. They are all future stable. The wall-crossing formula for the framed BPS indices amounts to a simple matrix identity:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.38}$$

eq:Spl-Fr-WC

where the three factors on the LHS reflect the wall-crossing across the three S_{ij}-rays, and the matrix on the right accounts for the monodromy of the vacua. The categorification of the wall-crossing identity (3.38), at least at the level of Chan-Paton complexes, is obtained by generalizing the left-hand-side of (3.38) to:

$$\begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}[f_2] & \mathbb{Z} \end{pmatrix} \begin{pmatrix} \mathbb{Z} & \mathbb{Z}[f_1] \\ 0 & \mathbb{Z} \end{pmatrix} \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}[f_2] & \mathbb{Z} \end{pmatrix} = \begin{pmatrix} \mathcal{E}_{--} & \mathcal{E}_{-+} \\ \mathcal{E}_{+-} & \mathcal{E}_{++} \end{pmatrix} \tag{3.39}$$

eq:CP-prod

Here $\mathcal{E}_{-+} = \mathbb{Z}[f_1]$, while

$$\mathcal{E}_{--} = \mathcal{E}_{++} = \mathbb{Z} \oplus \mathbb{Z}[f_1 + f_2] \tag{3.40}$$

is a complex with a degree one differential (note that $f_1 + f_2 = 1$) and

$$\mathcal{E}_{+-} = \mathbb{Z}[f_2] \oplus \mathbb{Z}[f_2] \oplus \mathbb{Z}[f_2 + 1] \tag{3.41}$$

is another complex with a degree one differential. The matrix of complexes (3.39) is quasi-isomorphic to the categorified version of the monodromy:

$$\begin{pmatrix} 0 & \mathbb{Z}[1 - f_2] \\ \mathbb{Z}[f_2] & 0 \end{pmatrix}. \tag{3.42}$$

3.5 Categorified Wall-Crossing Ror 2d Solitons

The standard wall-crossing formula for BPS indices of 2d solitons was studied by Cecotti and Vafa in Cecotti:1992qh [2]. It is associated with a homotopy of vacuum weights so that the cyclic orders of the central charges gets reversed, as in Figure fig:CAT-CVWC-BPSRAYS 42:

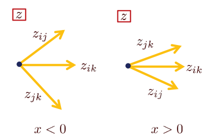


Figure 42: For the path of vacuum weights in Figure [fig:CAT-CVWC-1] we have BPS rays crossing as in the standard marginal stability analysis of the two-dimensional wall-crossing formula. fig:CAT-CVWC-BPSR

We can realize this by the explicit homotopy of vacuum weights shown in Figures fig:CAT-CVWC-1 43 and fig:CAT-CVWC-2 44:

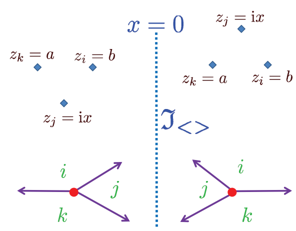


Figure 43: An example of a continuous path of vacuum weights crossing a wall of marginal stability. Here $z_k = a$ and $z_i = b$ with a, b real and $a < 0 < b$. They do not depend on x, while $z_j(x) = \mathrm{i}x$. We show typical vacuum weights for negative and positive x and the associated trivalent vertex. All other vacuum weights are assumed to be independent of x. As x passes through zero the vertex degenerates with $z_{jk}(x)$ and $z_{ij}(x)$ becoming real. Note that with this path of weights the $\{i, j, k\}$ form a *positive* half-plane fan in the negative half-plane, while $\{k, j, i\}$ form a *negative* half-plane fan in the positive half-plane. If we choose $x_\ell < 0 < x_r$ there is an associated interface $\mathfrak{I}_{<>}$. (We suppress the dependence on x_ℓ, x_r in the notation.) The only vertices are divalent vertices. These are all the standard amplitude K^{-1} familiar from the identity Interface $\mathfrak{I}\mathfrak{d}$, except for $\alpha_{<>}^{--} \in R_{ik}^{(2)} \otimes R_{ki}^{(1)}$. fig:CAT-CVWC-1

The wall-crossing of the BPS indices is a special case of the famous Kontsevich-Soibelman wall-crossing formula:

$$(1 + \mu_{ij}^{(1)} e_{ij})(1 + \mu_{ik}^{(1)} e_{ik})(1 + \mu_{jk}^{(1)} e_{jk}) = (1 + \mu_{jk}^{(2)} e_{ij})(1 + \mu_{ik}^{(2)} e_{ik})(1 + \mu_{ij}^{(2)} e_{jk}) \tag{3.43}$$ eq:CV-KS-WC

which gives:

$$\begin{aligned} \mu_{ij}^{(2)} &= \mu_{ij}^{(1)} \\ \mu_{jk}^{(2)} &= \mu_{jk}^{(1)} \\ \mu_{ik}^{(2)} &= \mu_{ik}^{(1)} + \mu_{ij}^{(1)} \mu_{jk}^{(1)}. \end{aligned} \tag{3.44}$$ eq:W-indx-wc

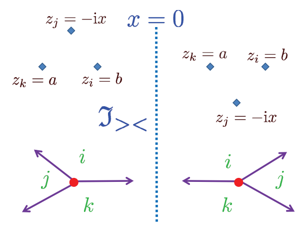


Figure 44: In this figure the path of weights shown in Figure [fig:CAT-CVWC-1] is reversed. Again, $z_k = a$ and $z_i = b$ with a, b real and $a < 0 < b$, but now $z_j(x) = -\mathrm{i}x$. We show typical vacuum weights for negative and positive x and the associated trivalent vertex. All other vacuum weights are assumed to be independent of x. Note that with this path of weights the $\{i, j, k\}$ form a *positive* half-plane fan in the positive half-plane, while $\{k, j, i\}$ form a *negative* half-plane fan in the negative half-plane. In order to define an interface we choose initial and final points for the path $-x_r < 0 < -x_\ell$ so that, after translation, it can be composed with the path defining $\mathfrak{I}_{<>}$. The interface $\mathfrak{I}_{><}$ has several nontrivial vertices. See Figure [fig:CAT-CVWC-8]. fig:CAT-CVWC-2

To categorify this we seek to define Interfaces:

$$\mathfrak{I}_{<>} \in \mathfrak{Br}(\mathcal{T}^\ell, \mathcal{T}^r) \qquad \& \qquad \mathfrak{I}_{><} \in \mathfrak{Br}(\mathcal{T}^r, \mathcal{T}^\ell) \tag{3.45}$$

(where the notation is meant to remind us how the half-plane fans are configured in the negative and positive half-planes). Now, the essential statement constraining these Interfaces is that the composition of the Interfaces should be homotopy equivalent to the identity Interface:

$$\mathfrak{I}_{<>} \boxtimes \mathfrak{I}_{><} \sim \mathfrak{Id}_{\mathcal{T}^\ell} \qquad \& \qquad \mathfrak{I}_{><} \boxtimes \mathfrak{I}_{<>} \sim \mathfrak{Id}_{\mathcal{T}^r}. \tag{3.46}$$ eq:Cat-WC-Form1

In [8] (AlgebraicStructures) we construct such Interfaces $\mathfrak{I}_{><}$ and $\mathfrak{I}_{<>}$ and show that the construction requires the relation:

$$\begin{aligned} R^{(2)}_{ij} &= R^{(1)}_{ij} \\ R^{(2)}_{jk} &= R^{(1)}_{jk} \\ R^{(2)}_{ik} - R^{(1)}_{ik} &= (R_{ij} \otimes R_{jk})^+ - (R_{ij} \otimes R_{jk})^- \\ &= \left(R^+_{ij} - R^-_{ij}\right) \otimes \left(R^+_{jk} - R^-_{jk}\right) \end{aligned} \tag{3.47}$$ eq:Cat-2dwc

where the superscript $\pm$ on the right hand side refers to the sign of $(-1)^F$. Although the categorified spectrum generator will jump, in general, the equation (3.47) (eq:Cat-2dwc) is clearly a categorification of the wall-crossing formulae (3.44) (eq:W-indx-wc).

3.6 Potential application to knot homology

To conclude, let us return to the motivation from knot homology. We consider the presentation of a tangle in $M_3 = \mathbb{R} \times C$, with $C = \mathbb{C}$, and $G = SU(2)$ or $G = SO(3)$ as in Figure 8 (fig:KNOT-HOM-4). As we explained, when the z_a are constant the Morse complex proposed by Witten is related to that of a Landau-Ginzburg theory called the Yang-Yang theory, described by superpotential (eq:YangYangW-1 1.45). As the $z_a(x^1)$ evolve we have, by the ideas of Section subsec:LG-Interface §3.1 an interface between LG theories.

In fact, the original theory at large negative x^1 corresponds to an oper with no singularities and is the trivial theory. Likewise, the final theory at large positive x^1 also corresponds to an oper with no singularities and is again the trivial theory. Recall that an Interface between the trivial theory and itself is nothing other than a chain complex. To construct this complex we can proceed by breaking up the path into elementary paths for which we compute elementary Interfaces as in Figure fig:ElementaryInterfaces 41.

It is clear that there will be three kinds of elementary Interfaces we must understand: Let us denote the YY theory appropriate to a fixed value of x^1 by $\mathcal{T}(\{z_a\},\{k_a\})$. (It turns out that boundary conditions at infinity forces the number q of chiral fields w_i to be given by $q = \frac{1}{2}\sum_a k_a$, so there is no need to indicate q.)

1. If the path $\wp^{\pm}_{a_1,a_2}$ braids two points $z_{a_1}(x)$ and $z_{a_2}(x)$ while all other points $z_a(x)$, for $a \neq a_1, a_2$ are fixed (on some small interval in x) then there will be *braiding Interfaces* $\mathfrak{I}^{\pm}(\wp^{\pm}_{a_1,a_2})$ between the theory $\mathcal{T}(\{z_a\},\{k_a\})$ and itself. The superscript indicates whether the braiding is clockwise or counterclockwise. These will be very similar to the S-wall interfaces discussed above.

2. If two points $z_{a_1}(x)$ and $z_{a_2}(x)$ annihilate (and then necessarily $k_{a_1} = k_{a_2}$) then there will be a *fusing Interface* between $\mathcal{T}(\{z_a\},\{k_a\})$ and the theory with z_{a_1} and z_{a_2} eliminated. Let us denote it by $\mathfrak{I}^{<}(a_1,a_2)$ These can probably be constructed using a theory of *cluster webs* described in AlgebraicStructures [8], but the full details have not been worked out yet.

3. The *creation Interface* $\mathfrak{I}^{>}(a_1,a_2)$ will just be the time reverse of the fusing Interface.

Now, a tangle such as shown in Figure fig:KNOT-HOM-4 8 is an x^1-ordered instruction of creation of pairs of points, braidings of points, and annihilations of pairs of points. Let us denote the corresponding ordered set of Interfaces for the tangle as $\mathfrak{I}_1,\dots,\mathfrak{I}_N$ for some N where each $\mathfrak{I}_s$ is one of the four types of interfaces described above. Then we can use the interface product $\boxtimes$ described above to construct

$$\mathfrak{I}(\text{Tangle}) := \mathfrak{I}_1 \boxtimes \cdots \boxtimes \mathfrak{I}_N. \tag{3.48}$$

This is an Interface between the trivial theory and itself. As just mentioned, it is therefore a chain complex. Let us call it $\widehat{\mathcal{K}}(L)$. We conjecture that the chain complexes so constructed define a knot homology theory. The required double-grading comes about as follows: The R_{ij} and Chan-Paton data have the usual grading by $(-1)^{\mathcal{F}}$. The second grading comes from integrating dW on cycles.

♣Need to add acknowledgements. ♣

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