Comments On Continuous Families Of Quantum Systems

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LIFE AFTER RCFT

- OR -

WHERE DO WE GO FROM HERE?

I. RCFT: GO THE DISTANCE
II. GENERALIZE
III. WHAT ABOUT EXPERIMENT?
**MODULAR TENSOR CATEGORY**

**DATA:**

1. $I = \text{LABELS, REPS, S-SEL, SECTORS}$
   
   $i \rightarrow i^\nu \rightarrow i; \; \Omega^\nu = 0$

2. $V^i_{jk} \chi^j = \mathcal{H}(i \begin{array}{c} j \end{array} k)$  
   "3-POINT COUPLINGS"

3. $\Omega^i_{jk} : \mathcal{H}(i \begin{array}{c} j \end{array} k) \cong \mathcal{H}( \begin{array}{c} i \end{array} k \begin{array}{c} j \end{array} )$

4. $F : \mathcal{H}( \begin{array}{c} j \end{array} k \begin{array}{c} l \end{array} ) \cong \mathcal{H}(i \begin{array}{c} j \end{array} k \begin{array}{c} l \end{array} )$

5. $S(p) : \mathcal{H}(p \begin{array}{c} l \end{array} \begin{array}{c} m \end{array} \begin{array}{c} n \end{array}) \cong \mathcal{H}(p \begin{array}{c} m \end{array} n \begin{array}{c} l \end{array})$

6. $e^{2\pi i c/\gamma}$
PART III: EXPERIMENT??

"REAL WORLD" APPLICATIONS IN
CONDENSED MATTER PHYSICS?

2D 2nd ORDER PHASE TRANS - OF COURSE

CSW ⇒ APPL’S TO F.Q.H.E. & ANYONS

DATA FOR MTC +

2+1 QFT ⇒ g=0 AXIOMS

BASIC PRINCIPLES OF (NON)RELATIVISTIC

NONABELIAN ANYONS NOT RULED OUT

F.Q.H. SYSTEM ⇒ L.G. THEORY

\[ A(z) \sim \int \frac{\langle 1 \psi^+ \psi(z') \rangle}{\overline{z-z'}} \, d^2 z' \]

W/ CS TERM

LOW ENERGY, LONG RANGE: PURE CSW

⇒ FULL MTC!?
NEW STATES

EXAMPLE: \[ \Psi_{PF} = \left[ \frac{1}{z_{ij}} \prod_{i<j} (z_i - z_j)^q \right] \]

\[ \Psi_{PF} = \left< \psi(z_1), \ldots, \psi(z_{2N}) \right> \frac{1}{2^N e^{i\varphi(z)}} \]

1. \( \nu = \frac{1}{q} \) \( q \) even!

2. \exists HAMILTONIAN SUCH THAT \( \Psi_{PF} \) is NONDEGENERATE, INCOMPRESSIBLE, GROUNDSTATE

\[ \mathcal{H} = \sum (-i \nabla_i - A)^2 + \nu \sum_{i<j} P^{2i} + \text{MANY BODY INT's} \]

3. ORDER PARAMETER: \[ U^q(z) \left[ \frac{1}{z-w} \right] \frac{1}{U(w) \Psi(z)} \]

N. Read
Doubly degenerate:

... \psi \psi \psi \psi \psi \psi 1 1 \psi 1 \times e^{2\pi i/8}

Analytic continuation:

... \psi \psi \psi \psi \psi \psi 1 1 \psi 1 \times e^{2\pi i/8}

"Physical" realization of nonabelian anyons
Part II

A Comment On Berry Connections
Philosophy

If a physical result is not mathematically natural, there might well be an underlying important physical issue.

We will illustrate this with continuous families of quantum systems

i.e. quantum systems parametrized by a space $X$ of control parameters.

In this context one naturally encounters Berry connections – an enormously successful idea.
Given a continuous family of Hamiltonians with a gap in the spectrum there is, in general, not one Berry connection, but rather a family of Berry connections.

Example: For band insulators there is a family of natural Berry connections, whose gauge equivalence classes are parametrized by a real-space torus.

This has consequences for topological contributions to electric polarization and magneto-electric polarizability: a 3D Chern ``insulator”’’ has a bulk QHE.
THE ORIGIN OF THE PROBLEM IS THE PROBLEM OF THE ORIGIN.
Affine Space

Like a vector space – but no natural choice of origin.

Definition: There is a transitive and free action of a vector space.
Non-symmorphic crystals

E(n): The Euclidean group of length-preserving transformations of affine n-dimensional space.

There is a natural subgroup $\mathbb{R}^n$ of translations.

But there is no natural subgroup isomorphic to O(n):

One must CHOOSE an origin to define such a group.

That’s why there are non-symmmorphic crystal structures.
Hilbert Bundles

Hilbert bundle over a space $X$ of control parameters

Diagram:

$$\mathcal{H}_x \xrightarrow{\pi} \mathcal{H} \xrightarrow{x} X$$
Sections Of A Hilbert Bundle

Space of sections: $\Gamma[\mathcal{H} \to X]$

$\Psi : x \mapsto \psi(x) \in \mathcal{H}_x$
Projected Bundles

Given a continuous family of projection operators: \( P(x) : \mathcal{H}_x \rightarrow \mathcal{H}_x \)

Projected bundle \( \mathcal{V} : \text{Subbundle with sections:} \)

\[ \Gamma(\mathcal{V}) := \{ \psi(x) | P(x)\psi(x) = \psi(x) \} \subset \Gamma(\mathcal{H}) \]

\[ \mathcal{H} = S^2 \times \mathbb{C}^2 \quad P(\hat{x}) = \frac{1}{2} (1 + \hat{x} \cdot \vec{\sigma}) \]

Definition: A vector bundle \( \mathcal{V} \) is a projected bundle.
Projected Connection

Connection:

\[ \nabla : \Gamma(\mathcal{V}) \rightarrow \Omega^1(\mathcal{V}) \]

\[ \nabla(f \Psi) = df \otimes \Psi + f \nabla \Psi \]

Remark: The space of connections on a vector bundle is an affine space modeled on the vector space: \( \Omega^1(\text{End}(\mathcal{V})) \)

If the vector bundle \( \mathcal{V} \) is defined using a family of projection operators \( P(x) \) and we choose a connection \( \nabla^\mathcal{H} \) on \( \mathcal{H} \):

we get a connection on the bundle \( \mathcal{V} : \)

\[ \nabla^P := P \circ \nabla^\mathcal{H} \circ \iota \quad \iota : \Gamma(\mathcal{V}) \hookrightarrow \Gamma(\mathcal{H}) \]
Berry Connection

Given a continuous family of Hamiltonians $H_x$ on $\mathcal{H}_x$, if there is a gap:

$$E_{\text{gap}} \notin \cup_{x \in X} \text{Spec}(H_x)$$

we have a continuous family of projection operators:

$$P(x) = \Theta(E_{\text{gap}} - H_x)$$

$$\nabla^B := P \circ \nabla^{\mathcal{H}} \circ \iota$$

[M. Berry (1983); B. Simon 1983]]

Note that it requires a CHOICE of $\nabla^{\mathcal{H}}$
Commonly assumed: $\mathcal{H}$ has been trivialized:

$$\mathcal{H} = X \times \mathcal{H}_0$$

Natural choice of $\nabla^\mathcal{H}$:

The trivial connection.

$$\nabla^\mathcal{H} \psi(x) = dx^\mu \frac{\partial}{\partial x^\mu} \psi(x)$$

$$\vec{A}_{\text{Berry}} = \langle \psi | \vec{\nabla}_R | \psi \rangle$$

But in general there is no natural trivialization of $\mathcal{H}$!
Hilbert Bundle Over Brillouin Torus

Crystal in n-dimensional affine space: \( C \subset \mathbb{A}^n \)

Invariant under a lattice of translations: \( L \subset \mathbb{R}^n \)

Brillouin torus: \( = \{ \text{unitary irreps of } L \} \).

Reciprocal lattice: \( L^\vee \subset \mathcal{K} \cong (\mathbb{R}^n)^\vee \cong \mathbb{R}^n \)

\( \bar{k} \in T^\vee = \mathcal{K}/L^\vee \quad \chi_{\bar{k}}(R) = e^{2\pi i \bar{k} \cdot R} \quad R \in L \)

Bloch states define a Hilbert bundle \( \mathcal{H} \) over the Brillouin torus:

\( \mathcal{H}_{\bar{k}} := \{ \psi_{\bar{k}} \mid \psi_{\bar{k}}(x + R) = e^{2\pi i \bar{k} \cdot R} \psi_{\bar{k}}(x) \} \)
Trivializations Of $\mathcal{H}$

$\mathcal{H}$ can be trivialized by choosing Bloch functions

$$\psi_{n, \bar{k}}(x + R) = e^{2\pi i k \cdot R} \psi_{n, \bar{k}}(x) \quad n \in \mathbb{N}$$

smooth $\quad \forall \bar{k} \in T^\vee$

$\{\psi_{n, \bar{k}}\}$ A basis for Hilbert space $\mathcal{H}_{\bar{k}}$

But in general there is no natural trivialization of $\mathcal{H}$!
A Family Of Connections on $\mathcal{H}$

So: There is no such thing as "THE" Berry connection in the context of band structure.

But, there is a natural family of connections on $\mathcal{H}$:

$$\nabla_{\mathcal{H}, x_0}$$

They depend on a choice of origin $x_0$ modulo L:

$$\nabla_{\mathcal{H}, x_0} - \nabla_{\mathcal{H}, x_0'} = \alpha$$

$$\alpha = 2\pi i \, dk \cdot (x_0 - x_0') \otimes 1_{\mathcal{H}}$$

[Freed & Moore, 2012]
Berry Connections For Insulators

Insulator: Projected bundle \( \mathcal{F} \) of filled bands:

\[
\mathcal{F}_{\mathbf{k}} = \Theta (E_f - H_{\mathbf{k}}) \cdot \mathcal{H}_{\mathbf{k}} \subset \mathcal{H}_{\mathbf{k}}
\]

\[
\nabla^B, x_0 - \nabla^B, x'_0 = \alpha
\]

\[
\alpha = 2\pi i \, dk \cdot (x_0 - x'_0) \otimes 1_{\mathcal{F}}
\]

So what?

\[
F(\nabla^B, x_0) = F(\nabla^B, x'_0)
\]

All Chern numbers unchanged....
Electric polarization:

\[
\langle K, P/e \rangle = \int_{T_K^\perp} \text{Im} \log \det \text{Hol}(\nabla^B, x_0, \gamma_K) \mod 2\pi
\]

[ King-Smith & Vanderbilt (1993); Resta (1994) ]

Magnetoelectric Polarizability

\[
\mathcal{L}_{\text{Maxwell eff.}} \supset \int_{\mathbb{R}^4} \alpha^{ij} E_i B_j
\]

``Axion angle''

\[
\theta(x_0) = \frac{1}{3} \alpha^i_i = \int_{T^\vee} CS(\nabla^B, x_0)
\]

[ Qi, Hughes, Zhang; Essin, Joel Moore, Vanderbilt ]
Dependence Of Axion Angle On $x_0$

$$CS(\nabla + \alpha) - CS(\nabla) = \text{Tr}(2\alpha F + \alpha D_A \alpha + \frac{2}{3} \alpha^3)$$

$$\tilde{c} := \int_{T^\lor} c_1(\mathcal{F}) \in L^\lor$$

$$\theta(x_0) - \theta(x'_0) = 2\pi \tilde{c} \cdot (x_0 - x'_0)$$

$$\mathcal{L}_{\text{Maxwell}}^{\text{Maxwell}} \supset \frac{1}{4\pi} \int_{\mathbb{R}^4} \langle \tilde{c}, d\tilde{x} \rangle \wedge CS(A^{\text{Maxwell}})$$

QHE in the \textit{bulk} of the ``insulator'' in the plane orthogonal to $\tilde{c}$.
Part III

Born Rule For Families Of Quantum Systems Parametrized By A Noncommutative Manifold
Quantum Systems

Set of physical "states" \( S \)

Set of physical "observables" \( O \)

Born Rule: \( BR : S \times O \rightarrow \mathcal{P} \)

\( \mathcal{P} \) Probability measures on \( \mathbb{R} \).

\( m \in \mathcal{M}(\mathbb{R}) \quad \Rightarrow \quad 0 \leq \varphi(m) \leq 1 \)

\( m = [r_1, r_2] \subset \mathbb{R} \quad BR(s, O)([r_1, r_2]) \)

is the probability that a measurement of the observable \( O \) in the state \( s \) has value between \( r_1 \) and \( r_2 \).
Standard Dirac-von Neumann Axioms

Density matrices $\rho$: Positive trace class operators on Hilbert space of trace = 1

Self-adjoint operators $T$ on Hilbert space

Spectral Theorem: There is a one-one correspondence of self-adjoint operators $T$ and projection valued measures:

$$m \in \mathcal{M}(\mathbb{R}) \rightarrow P_T(m)$$

Example: $T = \sum \lambda P_\lambda$  

$$P_T([r_1, r_2]) = \sum_{r_1 \leq \lambda \leq r_2} P_\lambda$$

$$m \in \mathcal{M}(\mathbb{R}) \quad BR(\rho, T)(m) = \text{Tr}_\mathcal{H}(\rho P_T(m))$$
Continuous Families Of Quantum Systems

Hilbert bundle over space $X$ of control parameters.

For each $x$ get a probability measure $\varrho_x$:

$$m \in \mathcal{M}(\mathbb{R}) \mapsto \varrho_x(m) := \text{Tr}_{\mathcal{H}_x}(\rho_x P_{T_x}(m))$$

$$BR : S \times \mathcal{O} \times X \to \mathcal{P}$$

$$BR(\rho, T, x) = \varrho_x$$
Noncommutative Families?

What happens when the space $X$ of control parameters is replaced by a noncommutative space?

How does the Born rule change?

Why ask this question?
Curiosity.

With irrational magnetic flux the Brillouin torus is replaced by a noncommutative manifold. (Bellisard, Connes, Gruber,...)

NC $tt^*$ geometry (S. Cecotti, D. Gaiotto, C. Vafa)

Boundaries of Narain moduli spaces of toroidal heterotic string compactifications are NC

The “early universe” might be NC
**C* Algebras**

A C* algebra is a (normed) algebra $\mathcal{A}$ over the complex numbers with an involution:

$$a \in \mathcal{A} \rightarrow a^* \in \mathcal{A} \quad (ab)^* = b^* a^*$$

such that ....

**Example 1:** $\mathcal{A} = C(X) := \{f : X \rightarrow \mathbb{C}\}$

**Example 2:** $\mathcal{A} = Mat_n(\mathbb{C})$

**Self-adjoint:**

$$a^* = a$$

**Positive:**

$$a = b^* b$$
Gelfand’s Theorem

The topology of a (Hausdorff) space $X$ is completely captured by the C*-algebra of continuous functions on $X$:

$$C(X) := \{ f : X \to \mathbb{C} \}$$

$(f_1 + f_2)(x) = f_1(x) + f_2(x)$  $(f_1 \cdot f_2)(x) := f_1(x)f_2(x)$

``Points” become 1D representations:

$\text{ev}_{x_0} : f \in C(X) \mapsto f(x_0) \in \mathbb{C}$

Commutative C* algebra:

$A \rightarrow \text{Irrep}(A)$  A topological space

$A \cong C(\text{Irrep}(A))$
Noncommutative Geometry

Statements about the topology/geometry of \( X \) are equivalent to algebraic statements about \( C(X) \)

Replace \( C(X) \) by a noncommutative \( C^* \) algebra \( \mathcal{A} \)

Interpret \( \mathcal{A} \) as the "algebra of functions on a noncommutative space" ... ... even though there are no points.

``pointless geometry''

Example: Noncommutative torus:

\[
U_i U_i^* = U_i^* U_i = 1 \quad U_i U_j = e^{2\pi i \phi_{ij}} U_j U_i
\]
Noncommutative Control Parameters

We would like to define a family of quantum systems parametrized by a NC manifold whose ``algebra of functions'' is a general C* algebra \( \mathcal{A} \).

What are observables?

What are states?

What is the Born rule?

What replaces the Hilbert bundle?
Noncommutative Hilbert Bundles

Definition: Hilbert C* module $\mathcal{E}$ over C*-algebra $\mathcal{A}$.

Complex vector space $\mathcal{E}$ with a right-action of $\mathcal{A}$ and an "inner product" valued in $\mathcal{A}$

$\Psi_1, \Psi_2 \in \mathcal{E}$ \quad $(\Psi_1, \Psi_2)_{\mathcal{A}} \in \mathcal{A}$

$(\Psi_1, \Psi_2)^*_{\mathcal{A}} = (\Psi_2, \Psi_1)_{\mathcal{A}}$

$(\Psi, \Psi)_{\mathcal{A}} \geq 0$ \quad (Positive element of the C* algebra.)

such that .....
Quantum Mechanics With Noncommutative Amplitudes

Basic idea: Replace the Hilbert space by a Hilbert C* module

\[ \mathcal{H} \rightarrow \mathcal{E} \]

\[ \Psi_1, \Psi_2 \in \mathcal{E} \quad (\Psi_1, \Psi_2)_A \in A \]

Overlaps are valued in a possibly noncommutative algebra.

QM:
\[ 0 \leq \varphi(\lambda) = (\psi_\lambda, \psi)(\psi_\lambda, \psi)^* \leq 1 \]

QMNA:
\[ (\Psi_\lambda, \Psi)(\Psi_\lambda, \Psi)^* \in A \]
Example 1: Hilbert Bundle Over A Commutative Manifold

\[ \mathcal{E} = \Gamma[\mathcal{H} \to X] \quad \mathfrak{A} = C(X) \]

\[ \Psi : x \mapsto \psi(x) \in \mathcal{H}_x \]

\[ (\Psi_1, \Psi_2)_{\mathfrak{A}} \in \mathfrak{A} := C(X) \]

\[ (\Psi_1, \Psi_2)_{\mathfrak{A}}(x) := (\psi_1(x), \psi_2(x))_{\mathcal{H}_x} \in \mathbb{C} \]
Example 2: Hilbert Bundle Over A Fuzzy Point

Def: "fuzzy point" has \( \mathfrak{A} \cong \text{Mat}_{a \times a}(\mathbb{C}) \)

\[ \mathcal{E} = \text{Mat}_{b \times a}(\mathbb{C}) \]

\[ (\Psi_1, \Psi_2)_{\mathfrak{A}} = \Psi_1^\dagger \Psi_2 \]
Observables In QMNA

Consider ``adjointable operators'' \( T : \mathcal{E} \to \mathcal{E} \)

\[
(\Psi_1, T\Psi_2)_{\mathcal{A}} = (T^*\Psi_1, \Psi_2)_{\mathcal{A}}
\]

The adjointable operators \( \mathcal{B} \) are another C* algebra.

Definition: **QMNA observables** are self-adjoint elements of \( \mathcal{B} \)

(Technical problem: There is no spectral theorem for self-adjoint elements of an abstract C* algebra.)
Definition: A $\textit{C*-algebra state}$ $\omega \in \mathcal{S}(\mathcal{A})$ is a positive linear functional

$$\omega : \mathcal{A} \rightarrow \mathbb{C} \quad \omega(1) = 1$$

$$\mathcal{A} = C(X) \quad \omega \in \mathcal{S}(\mathcal{A})$$

$$\omega(f) = \int_X f d\mu \quad d\mu = \text{a positive measure on } X:$$

$$\mathcal{A} \cong \text{Mat}_{a \times a}(\mathbb{C}) \quad \omega \in \mathcal{S}(\mathcal{A})$$

$$\omega(T) = \text{Tr}_\mathcal{H}(\rho T) \quad \rho = \text{a density matrix}$$
QMNA States

Definition: A **QMNA state** is a completely positive unital map

\[ \varphi : \mathcal{B} \rightarrow \mathcal{A} \]

``Completely positive” comes up naturally both in math and in quantum information theory.

Positive: \[ \varphi : \mathcal{B}_{\geq 0} \rightarrow \mathcal{A}_{\geq 0} \]

Unital: \[ \varphi(1_{\mathcal{B}}) = 1_{\mathcal{A}} \]

Completely positive

\[ \varphi \otimes 1 : (\mathcal{B} \otimes \text{Mat}_n(\mathbb{C}))_{\geq 0} \rightarrow (\mathcal{A} \otimes \text{Mat}_n(\mathbb{C}))_{\geq 0} \]
QMNA Born Rule

Main insight is that we should regard the Born Rule as a map

\[ BR : S^{QMNA} \times O^{QMNA} \times S(A) \rightarrow P \]

For general \( A \) the datum \( \omega \in S(A) \) together with complete positivity of \( \varphi \) give just the right information to state a Born rule in general:

\[ BR(\varphi, T, \omega) \in P \]
Family Of Quantum Systems Over A Fuzzy Point

\[ \mathcal{E} = \text{Mat}_{b \times a}(\mathbb{C}) = \mathbb{C}^b \otimes \mathbb{C}^a = \mathcal{H}_{\text{Bob}} \otimes \mathcal{H}_{\text{Alice}} \]

\[ \mathfrak{A} = \text{Mat}_{a}(\mathbb{C}) = \text{End}(\mathcal{H}_{\text{Alice}}) \]

\[ \mathfrak{B} = \text{Mat}_{b}(\mathbb{C}) = \text{End}(\mathcal{H}_{\text{Bob}}) \]

\[ BR(\varphi, T, \omega)(m) = \text{Tr}_{\mathcal{H}_A} \rho_A \varphi(P_T(m)) \]

``A NC measure \( \omega \in S(\mathfrak{A})'''' is equivalent to a density matrix \( \rho_A \) on \( \mathcal{H}_A \)

QMNA state:

\[ \varphi(T) = \sum_{\alpha} E_{\alpha}^\dagger T E_{\alpha} \quad \sum_{\alpha} E_{\alpha}^\dagger E_{\alpha} = 1 \]
Quantum Information Theory & Noncommutative Geometry

\[ BR(\varphi, T, \omega)(m) = \text{Tr}_{\mathcal{H}_A} \rho_A \varphi(P_T(m)) \]

\[ = \sum_\alpha \text{Tr}_{\mathcal{H}_A} \rho_A E_\alpha^\dagger(P_T(m)) E_\alpha \]

\[ = \sum_\alpha \text{Tr}_{\mathcal{H}_B} E_\alpha \rho_A E_\alpha^\dagger P_T(m) \]

\[ = \text{Tr}_{\mathcal{H}_B} \mathcal{E}(\rho_A) P_T(m) \]

Last expression is the measurement by Bob of \( T \) in the state \( \rho_A \) prepared by Alice and sent to Bob through quantum channel \( \mathcal{E} \).
Don’t be discouraged by negative response.

(Within reason)

Please say, "a Berry connection," not "the Berry connection."

(Unless you have specified $\mathcal{H}$)

QM can be generalized to QMNA.

(Is it really a generalization?)

(Is it useful?)