WALL-CROSSING FORMULA FOR
BPS STATES & SOME APPLICATIONS

CLAY WORKSHOP ON K3 & MODULAR FORMS
MARCH 20, 2008

BASED ON WORK DONE WITH
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1. Introduction

The "space of BPS states" has been a central concept in SUSY gauge theory & string theory for almost 30 years.

Today I'll focus on recent progress in understanding phenomena associated to marginal stability.

1. Introduction for Mathematicians

2. Wall-crossing formulae:
   - Primitive, semi-primitive, Kontsevich-Soibelman

3. Physical derivation

4. D6D2D0 system: Ideal sheaves & DT

5. D4D2D0 system: Modular gen. functions

6. OSV

7. Problems at weak coupling

8. Conclude
A. DEFINING THE "SPACE OF BPS STATES"

FOR DEFINITENESS, WE FOCUS ON THEORIES WITH $d=4$, $\mathcal{N}=2$ SUSY IN (ASYMPTOTIC) MINKOWSKIAN SPACE $M_4$

HILBERT SPACE OF ONE-PARTICLE STATES, $\mathcal{H}$, IS A REP. OF THE $d=4$, $\mathcal{N}=2$ ALGEBRA.

$\hat{\mathcal{Z}}$: CENTRAL CHARGE OPERATOR

$$\{\hat{Q}^{i\alpha}, \hat{Q}^{j\beta}\} = \delta_{ij} (C_{\mu})_{\alpha\beta} \hat{P}^\mu + \epsilon_{ij} C_{\alpha\beta} \hat{\mathcal{Z}}$$

DECOMPOSE $\mathcal{H} = \bigoplus_{z \in \mathbb{C}} \mathcal{H}^\mathcal{Z}_{z} = \mathcal{Z}$

**LEMMA**: $E \geq |z|$ ON $\mathcal{H}_{z}$

**DEF'_N**: $\mathcal{H}_{\text{BPS}}$ IS THE SUBSPACE OF $\mathcal{H}$ WHERE $E = |z|$. 
NOW—SPECIALIZE TO TYPE II STRING THEORY ON $M_4 \times X$.

- $M_4$ is noncompact $\Rightarrow$ to define the Hilbert space as a rep. of $W=2$ we must specify boundary cond's for the massless fields:

$$\lim_{x \to \infty} (g_{\mu\nu}, \phi, B_{\mu\nu}, RR) := \mathfrak{M}_\infty \in \tilde{M}$$

$\mathcal{H}_{\mathfrak{M}_\infty} : 1$-Particle Hilbert space depends on $\mathfrak{M}_\infty$

- Generalized Maxwell theory $\Rightarrow$ $\mathcal{H}_{\mathfrak{M}_\infty}$ is graded by electric/magnetic charge sectors:

$$\mathcal{H}_{\mathfrak{M}_\infty} = \bigoplus_{\Gamma} \mathcal{H}_{\mathfrak{M}_\infty}^{\Gamma}$$

$\Gamma \in (twisted) K$-theory($X$) mod torsion: $\Gamma \in \Lambda$, a symplectic lattice
Now we put these things together:

Consider IIA strings with

1. $X = \text{static, compact, CY 3-fold}$
2. Flat B-field: $B \in H^2(X, \mathbb{R})$
3. Flat RR fields

$\Rightarrow \quad W = 2, \quad d = 4 \quad \text{SUGRA}$

- Each $H^\Gamma_{\Phi_\infty}$ is a rep of $W = 2$
- Central charge $Z = Z(\Gamma; \Phi_\infty)$

So, we study the BPS spectrum

$H_{\text{BPS}} = \bigoplus_{\Gamma \in K^0(X)} H^\Gamma_{\Phi_\infty, \text{BRS}}$

Finite dimensional
B. Dependence on Moduli

The spaces $\overline{\mathcal{M}}_{\infty, \text{BPS}}$ are locally constant but not globally constant as functions of $\Phi_{\infty}$.

Moduli space $\tilde{\mathcal{M}}$ is a product:

\begin{align*}
\text{Hypermultiplets} \times \text{Vectormultiplets} \times \left[ \text{Cplx Str., } \phi, \text{ RR Fields} \right] \times \left[ \text{Complexified Kähler} \right]
\end{align*}

We work at a generic hypermultiplet.

Recent progress has been concerned with the dependence on vectormultiplets, in this talk,

\[ t = B + i J \]

- The jumping locus is real codimension one
Define an index

\[ \Omega(\Gamma, t_\infty) = -\frac{1}{2} \text{Tr}_{\mathcal{H}_{\infty, \text{BPS}}} (2J_3)^2 (-1)^{2J_3} \]

- Technical point:

\[ \mathcal{H}_{\infty, \text{BPS}} = \mathcal{H}_{\frac{1}{2} \text{HM}} \otimes \mathcal{H}(\Gamma, t_\infty) \]

½ hyper 2(0) + (½) as spin rep

\[ \Omega(\Gamma, t_\infty) = \text{Tr}_{\mathcal{H}(\Gamma, t_\infty)} (-1)^F \]

Henceforth focus on \( \mathcal{H}(\Gamma, t_\infty) \)

- Can also study "Hodge polynomial"

\[ \text{Tr}_{\mathcal{H}(\Gamma, t)} (-x)^{J_3+R} (-y)^{J_3-R} \]

- Key point: \( \Omega \) changes across walls of marginal stability
C. WHY DO WE CARE?

PHYSICS MOTIVATION

1. THE MAIN MOTIVATION FOR RECENT WORK IS THE PROGRAM, INITIATED BY STROMINGER–VAFA (1995) OF ACCOUNTING FOR BH ENTROPY VIA MICROSTATE COUNTING. THAT GOAL IS STILL NOT FULLY ACCOMPLISHED.

WE DON'T KNOW BPS DEGENERACRY FOR CERTAIN NATURAL CHARGE REGIMES, FOR EXAMPLE:

$$\Gamma \rightarrow 2\Gamma \quad \lambda \rightarrow \infty$$

2. OSV CONJECTURE:

RELATION BETWEEN

$$\Omega(\Gamma) \in GW/DT/GV INVARIANTS$$

$$\Rightarrow$$ NONPTVE TOPOLOGICAL STRING?
1. Physical stability of BPS states is related to math. Stability in the bounded derived category of coherent sheaves on a C.Y.: Kontsevich, Douglas, Bridgeland, Thomas, Pandharipande ...

Physics $\Rightarrow$ Predictions/Constraints on what we expect should be true.

2. Many interesting connections to automorphic forms and analytic number theory; some relations to arithmetic C.Y's.

3. There are several other more speculative applications, e.g. BPS algebras: Generalizing Nakajima's work and suggested by Type II/Het duality should be closely related.
2. **Wall-Crossing Formulae: Statement**

\( N = 2, \ d = 4 \) Algebra \( \Rightarrow \)

- **Moduli of vacua** \( \tilde{\mathcal{M}} \)
- **Lattice of electric/magnetic charges** \( \Lambda \)
- **Central Charge** \( Z : \Lambda \times \tilde{\mathcal{M}} \rightarrow \mathbb{C} \)

**Walls where \( \mathcal{H}_{BPS} \) might jump**

\( MS(\Gamma_1, \Gamma_2) := \{ t \mid Z(\Gamma_1, t) = \lambda Z(\Gamma_2, t), \lambda \in \mathbb{R}_+^* \} \)

**Cecotti, Intriligator, Nafa; Seiberg Witten:**

A boundstate of particles with charges \( \Gamma_1, \Gamma_2 \) can decay

We want to say how many states decay.
PRIMITIVE WALL-CROSSING FORMULA:

∧ HAS SYMPLECTIC FORM \( \langle \cdot, \cdot \rangle \)

LET \( J_{12} := \langle \pi, \pi \rangle \)

\[ I_{12} \text{Im}(z_1 z_2^*) > 0 \]

\[ I_{12} \text{Im}(z_1 z_2^*) < 0 \]

\( \pi_1, \pi_2 \ \text{PRIMITIVE}, t_{m5} \ \text{GENERIC} \Rightarrow \)

\[ \mathcal{H}_+ - \mathcal{H}_- = (J_{12}) \otimes \mathcal{H}(\pi_1; t_{m5}) \otimes \mathcal{H}(\pi_2; t_{m5}) \]

\[ J_{12} = \frac{1}{2}(|I_{12}| - 1) \]

\[ \Delta \Omega = (-1)^{|I_{12}|} \text{Im} \Omega(\pi_1; t_{m5}) \Omega(\pi_2; t_{m5}) \]
**Semi-Primitive Wall-Crossing Formula**

In addition to $\Gamma_1 + \Gamma_2$ boundstates, we can also form $N_1 \Gamma_1 + N_2 \Gamma_2$ boundstates:

$$MS(\Gamma_1, \Gamma_2) = MS(N_1 \Gamma_1, N_2 \Gamma_2) \quad N_1, N_2 \in \mathbb{Z}_+$$

Consider $N_1 = 1$, $N_2 \geq 1$:

$$\bigoplus_{N_2} U^{N_2} \Delta \mathcal{H} / \Gamma \rightarrow \Gamma_1 + N_2 \Gamma_2$$

**Claim:** This is a $\mathbb{Z}_2$-graded Fock space

$$\mathcal{H}(\Gamma_1; t_{m_5}) \bigotimes_{k=1}^{\infty} F\left( k^{\Gamma_1, k\Gamma_2} \otimes \mathcal{H}(k\Gamma_2; t_{m_5}) \right)$$

**Graded space of oscillators**

In particular:

$$\Omega_1 + \sum_{N > 0} U^N \Delta \Sigma_2(\Gamma_1 + N \Gamma_2) =$$

$$= \Sigma(\Gamma_1) \prod_{k > 0} \left( 1 - (-1)^{\langle \Gamma_1, k\Gamma_2 \rangle} U^k \right) \langle \Gamma_1, k\Gamma_2 \rangle \Omega(k\Gamma_2)$$
The Kontsevich–Soibelman Formula

For the lattice $\Lambda$ of charges introduce a Lie algebra $\mathbb{Z}[\Lambda]$ with one generator for each $y \in \Lambda$:

$$[e_y, e_{y_2}] = (-1)^{\langle y, y_2 \rangle} \langle y, y_2 \rangle e_{y+y_2}$$

At fixed $t$, $\mathbb{Z} : \Lambda \rightarrow \mathbb{C}$, choose any convex angular sector $V$.

INCREASING SLOPE

$$\prod_{y \in \mathbb{Z}^1(V) \cap \Lambda} \left( \exp \sum_{n=1}^{\infty} \frac{e_{ny}}{n^2} \right)^{\Omega^-(y)}$$

DECREASING SLOPE

$$\prod_{y \in \mathbb{Z}^1(V) \cap \Lambda} \left( \exp \sum_{n=1}^{\infty} \frac{e_{ny}}{n^2} \right)^{\Omega^+(y)}$$
3. PHYSICAL DERIVATION OF WCF

A. SUPERGRAVITY TOOLS

D-BRANES ARE OBJECTS IN A CATEGORY IN TYPE II A/CY, THE SUBCATEGORY OF SUSY BRANES IS PROBABLY THE BOUNDED DERIVED CATEGORY OF COHERENT SHEAVES. BUT WE WANT TO DESCRIBE THE (PHYSICALLY) STABLE OBJECTS.

AT WEAK STRING COUPLING, AND \( J \to \infty \) \( J \) A BEAUTIFUL DESCRIPTION OF STABLE BPS STATES USING SUGRA.

IN THE SEMICLASSICAL LIMIT \( \Psi \in H_{\text{BPS}} \) \( \Psi \to \) BPS SOLUTION OF SUGRA EQUATIONS

* SUPERGRAVITY ALLOWS ONE TO IDENTIFY MANY "STABLE OBJECTS" THANKS TO THE ATTRACTOR MECHANISM.
ATTRACTOR MECHANISM: (F.K.S.; STROMINGER)

$\Gamma, \mathcal{L}_\mathcal{E}$ Spherical Symmetry

$\implies \exists$ AT MOST ONE BPS SOLUTION OF SUGRA.

IF IT EXISTS...

SCALAR FIELDS $t = t(\Gamma)$, AND EVOLUTION FROM $r = \infty$ TO $r = 0$ APPROACHES AN ATTRACTIVE FIXED POINT $t_*(\Gamma)$:

$\mu_{VM}$

RADIAL MOTION TO HORIZON
Attractor Flow = Gradient Flow for

$$\log |\mathcal{Z}(\Gamma; t)|^2 \Rightarrow$$

**Basic Trichotomy**

1. $\mathcal{T}_*(\Gamma) \in \text{Interior}(\tilde{\mathcal{M}})$
   and $\mathcal{Z}(\Gamma; t^*(\Gamma)) \neq 0$

"Regular Attractor Point"

2. $\exists$ Nonempty Subvariety $\subset \tilde{\mathcal{M}}$
   $\mathcal{Z}(\Gamma; t) = 0$

3. $\mathcal{T}_*(\Gamma) \in \partial \tilde{\mathcal{M}}$

(1.) $\exists$ Spherically Symmetric BPS
   Black Holes in $\mathcal{H}_{BPS}(\Gamma; t)$ for all $t$

(2.) $\mathcal{H}_{BPS}(\Gamma; t) = \emptyset$ in an open region of the zero locus.
   $\mathcal{H}_{BPS}$ might be nonempty further away

(3.) Cannot use SUGRA to establish existence; must use microscopic arguments.
**SPLIT ATTRACTOR FLOWS**

If \( \mathcal{Z}(\Gamma; t) = 0 \) has solutions in the interior of moduli space then use:

**Denef's Rule:** \( \mathcal{Z}(\Gamma; t) \neq 0 \iff \exists \text{ a split attractor flow (S.A.F.)} \)

**S.A.F.:** A piecewise attractor flow, joined along walls of M.S., conserving charge at the vertices, terminating on R.A.P.'s:

\[ \Gamma = \Gamma_1 + \Gamma_2 \]

![Diagram](image)
- IF SUCH ATTRACTOR FLOW TREES EXIST WE CAN CONSTRUCT A CORRESPONDING SOLUTION OF SUGRA.

- SPACETIME PICTURE:

  \[ \mathbb{R}^3 \rightarrow (x_i, \Gamma_i) \]
  \[ (x_j, \Gamma_j) \]
  \[ (x_n, \Gamma_n) \]

- NEAR EACH POINT \( x_i \): THE SOLUTION LOOKS LIKE THE SINGLE-CENTERED SOLUTION: "BLACK-HOLE MOLECULES"

"INTEGRABILITY CONDITION"

\[ \forall i \sum_{j \neq i} \frac{\langle \Gamma_i, \Gamma_j \rangle}{|x_i - x_j|^2} = 2 \operatorname{Im} \left( \frac{z_i \overline{z}}{|z|^2} \right)_{t = 0} \]
B. DERIVATION OF PRIMITIVE WCF:

**CONSIDER BOUND STATE OF TWO PRIMITIVE CHARGES:**

\[ R = \frac{1}{2} \langle \Gamma_1, \Gamma_2 \rangle \frac{|Z_1 + Z_2|_\infty}{\text{Im} (Z_1 \overline{Z_2})_\infty} \]

- **NOTE:** \( \langle \Gamma_1, \Gamma_2 \rangle \text{Im} (Z_1 \overline{Z_2})_\infty > 0 \)

- **NOTE THAT BY CHANGING \( t_\infty \) WE CAN MAKE** \( \text{Im} (Z_1 \overline{Z_2})_t \rightarrow 0 \)

**WHILE** \( |Z_1 + Z_2|_{t_\infty} \neq 0 \)

**ILLUSTRATES THE KEY POINT OF MARGINAL STABILITY:**
\[ \text{MS}(\Gamma_1, \Gamma_2) := \left\{ \pm \epsilon \in \mathbb{U} \left| \frac{Z_1}{Z_2} \in \mathbb{R}^+ \right. \right\} \]

- NO \( \Gamma_1 + \Gamma_2 \) BOUND STATE EXISTS HERE

\[ \text{CHANGE BC'S @ } r = \infty \Rightarrow R_{12} \rightarrow \infty \]

**If** \( Z(\Gamma; t) \) **has a zero then there is no bound state of type** \( \Gamma_1 + \Gamma_2 \) **in the blue region.**

\[ Z(\Gamma; t_0) = 0 \]
Macroscopic Argument for WCF:

\[ R_{12} = \frac{1}{2} \langle \Gamma_1, \Gamma_2 \rangle \frac{|Z_1 + Z_2|_\infty}{\text{Im}(Z_1 \overline{Z_2})_\infty} \]

Electromagnetic field of two dyons has spin:

\[ \mathcal{J}_{12} = \frac{1}{2} \left( K \langle \Gamma_1, \Gamma_2 \rangle |1 - 1| \right) \text{, quantum correction} \]

Locality \implies \text{For } \Gamma_1, \Gamma_2 \text{ primitive:}

States lost from \( \mathcal{H}(\Gamma; t\infty) \) are

\( (\mathcal{J}_{12}) \otimes \mathcal{H}(\Gamma_1; t_{\text{ms}}) \otimes \mathcal{H}(\Gamma_2; t_{\text{ms}}) \)
MICROSCOPIC ARGUMENT FOR WCF: WHEN $\Theta = \arg \frac{Z_2}{Z_1} \to 0$, MODEL LIGHT D.O.F. BY A QUIVER GAUGE THRT:

$$
\begin{array}{c}
\Gamma_1 \\
\bullet \\
\longrightarrow \\
\begin{array}{c}
\Gamma_1 \\
\bullet \\
\longleftrightarrow \\
\n_-
\end{array} \\
\longrightarrow \\
\begin{array}{c}
\Gamma_2 \\
\bullet \\
\end{array}
\end{array}
$$

TRANSLATION TO SUPERGRAVITY:

STABILITY DATA: $(\tau, -\tau)$

$n_+ - n_- = \mathbb{Z}_{12}$

GENERICALLY $n_+ = 0$ or $n_- = 0$.

SUPPOSE $n_- = 0$:

$\tau > 0$ \hspace{1cm} $\mathcal{M} = \mathbb{CP}^{n_+ - 1}$

$\tau < 0$ \hspace{1cm} $\mathcal{M} = \emptyset$

$\Delta \mathcal{L} = H^*(\mathbb{CP}^{n_+ - 1})$

$\text{spin}(3) \cong \text{Lefschetz}$
Quiver Quantum Mechanics

$O+1$ SUSY QED with

- $1$ VM $(A_0, \vec{x}, \lambda)$
- $n_{\pm}$ CM's $(\Phi_\pm, \overline{\Phi}_\pm)$ charge $\pm 1$

Small $|\vec{x}| \Rightarrow$ Higgs Branch $= \text{Moduli of Stable Quiver Rep's}$

Large $|\vec{x}| \Rightarrow$ Integrate Out $\Phi_\pm$

$\left(\begin{array}{c}
\text{Denef} \\
\text{QQHHH}
\end{array}\right)$

$V_{\text{eff}} = \frac{1}{2\mu} \left( \vartheta + \frac{n_+ - n_-}{2r} \right)^2$

$(n_+ - n_-)$ BPS States of Spin $\frac{1}{2}(n_+ - n_- - 1)$

$\vartheta < 0$

$n_+ \text{ Higgs Br.}$

BPS States

$\vartheta = 0$

$(n_+ - n_-)$ Coulomb Br. $\rightarrow \infty$

$\vartheta > 0$

$n_+ \text{ Higgs Br.}$

BPS States
C. DERIVATION OF SEMI-PRIMITIVE WGF

HALO STATES

Suppose \( \langle \Gamma_1, \Gamma_2 \rangle \neq 0 \),

\[
\Gamma_j = \lambda_j \Gamma_2 \quad \lambda_j > 0, j = 2, \ldots, N
\]

are all mutually local.

Integrability conditions say

\[
j \geq 2: \quad \frac{\langle \Gamma_j, \Gamma_1 \rangle}{|\mathbf{x}_j - \mathbf{x}_1|} = 2 \frac{\text{Im} (Z(\Gamma_j) \overline{Z(\Gamma_1)})}{|Z(\Gamma_1)|}
\]

\[\Rightarrow \text{All } |\mathbf{x}_j - \mathbf{x}_1| \text{ are equal}
\]

Cross MS(\( \Gamma_1, \Gamma_2 \)): Halo radius \( \rightarrow \infty \)
THE PARTICLES IN THE HALO GENERATE A FOCK SPACE WITH

\((J_{\Gamma_1, k\Gamma_2}) \otimes \mathcal{H}(k\Gamma_2; itw)\) CREATION OPERATORS OF CHARGE \(k\Gamma_2\)

ALL WALLS \(W(\Gamma_1, N\Gamma_2)\) COINCIDE \(\Rightarrow\)
CROSSING A WALL WE LOSE ENTIRE FOCK SPACE:

\[ \Omega(\Gamma_1) + \sum_{N \geq 1} \Delta \Omega(\Gamma_2, \Gamma_1 + N\Gamma_2) u^N \]

\[ = \Omega(\Gamma_1) \prod_{k > 0} (1 - (-1)^k u^k)^{\langle \Gamma_1, k\Gamma_2 \rangle} |\Omega(k\Gamma_2)\rangle \]
Kontsevich–Soibelman Formula

- No physical derivation yet

Evidence that $K_i$'s $\Omega(\Gamma; t)$'s are the same as physical $\Sigma(\Gamma; t)$'s.

- Can recover primitive WCF
- Can recover semi-primitive WCF
- Nontrivial checks for $\text{SU}(2)$ Seiberg-Witten with $N_f = 0, 1, 2, 3$ hypermultiplets

(Last two are results w/ Wu-yan Chuang)
LIE ALGEBRA IS FILTERED \Rightarrow

CAN RESTRICT TO

\text{Heisenberg Algebra}\left\{ \begin{align*}
\left[ e_{0,1}, e_{1,0} \right] &= (-1)^{i} e_{1,2}, e_{1,1} \\
&= e_{1,1}, \text{ CENTRAL}
\end{align*} \right.

\mathcal{O}^{-}(\Gamma) \quad \mathcal{O}^{-}(\Gamma_{1} + \Gamma_{2}) \quad \mathcal{O}^{-}(2 \Gamma_{2})

\mathcal{O}^{+}(\Gamma_{2}) \quad \mathcal{O}^{+}(\Gamma_{1} + \Gamma_{2}) \quad \mathcal{O}^{+}(\Gamma_{1})

= \mathcal{O}^{-}(\Gamma_{2}) \quad \mathcal{O}^{-}(\Gamma_{1} + \Gamma_{2}) \quad \mathcal{O}^{-}(\Gamma_{1})

= \mathcal{O}^{+}(\Gamma_{2}) \quad \mathcal{O}^{+}(\Gamma_{1} + \Gamma_{2}) \quad \mathcal{O}^{+}(\Gamma_{1})

= \mathcal{U}_{1,1,0} \quad \mathcal{U}_{1,1} \quad \mathcal{U}_{0,0,1}

\mathcal{U}_{0,1} \mathcal{U}_{1,0} = \mathcal{U}_{1,1} \mathcal{U}_{0,0} \implies

\mathcal{O}^{+}(\Gamma_{1} + \Gamma_{2}) - \mathcal{O}^{-}(\Gamma_{1} + \Gamma_{2})

= \mathcal{O}^{+}(\Gamma_{1}) \mathcal{O}^{+}(\Gamma_{2}) \mathcal{O}^{-}(\Gamma_{1}) \mathcal{O}^{-}(\Gamma_{2})

= \mathcal{U}_{1,1} \mathcal{U}_{1,0} \mathcal{U}_{0,1} \mathcal{U}_{0,0}

= \mathcal{U}_{1,1,1}

\text{PRIMITIVE W.C. FORMULA}!
SU(2) SEIBERG-WITTEN THEORY

\[ \Gamma_1 = \text{MONPOLE} \]

\[ \Gamma_2 = \text{DYON} \]

\[ [e_{a,b}, e_{c,d}] = 2(bc-ad)e_{a+c, b+d} \]

**STRONG:** ±(1,0), ±(0,1) \( \Omega = +1 \) \( \mathbb{P} \mathbb{M} \)

**WEAK:** ±(1,1) \( \Omega = -2 \) \( \mathbb{V} \mathbb{M} \)
\[ \pm(n,n+1), \pm(n+1,n) \] \( \Omega = +1 \) \( \mathbb{P} \mathbb{M} \)

**STRONG:** \( U_{1,0} \cdot U_{0,1} \)

**WEAK:**
\[ (U_{0,1} U_{1,2} U_{2,3} \cdots) U_{1,1}^{-2} (\cdots U_{32} U_{21} U_{10}) \]

**EQUALITY APPEARS TO BE TRUE!**

\[ \text{NEW IDENTITIES FOR } N_f = 1, 2, 3 \]
4. D6D2DO SYSTEM

Identify $\Gamma$ with its image

$\epsilon \in K^0(X) \rightarrow \Gamma = \chi_{\epsilon} \sqrt{\Delta} \in H^{ev}(X)$

$H^0 \oplus H^2 \oplus H^4 \oplus H^6 \ni \Gamma = (p^0, p, q, q_0)$

consider: $\Gamma(\beta, n) := \Gamma = (1, 0, -\beta, n)$

$\beta = \text{PD}[\sigma]$ $\sigma \in X$ holomorphic curve

charge of (the dual of) an ideal sheaf:

$\chi_{\mathcal{I}} \sqrt{\Delta} = 1 - \beta + ndV$

consider binding these to D2DO particles with charge:

$\Gamma_h = (0, 0, -\beta h, n_h)$
PLOT MARGINAL STABILITY CURVE

\[ \mathcal{Z}(\Gamma(\beta, \eta); t) = \lambda \mathcal{Z}(\Gamma, t) \quad \lambda \in \mathbb{R}_+ \]

\[ \Pi A \quad \langle \Gamma_1, \Gamma_2 \rangle = \int \text{d} \varepsilon_1 \otimes \bar{\varepsilon}_2 \hat{A} \]

\[ \mathcal{Z}(\Gamma, t) = \frac{\langle \Gamma, \Omega \rangle}{\sqrt{\langle \Omega, \Omega^* \rangle}} \]

SUGRA REGIME: \[ \Omega = - e^t \]

\[ t = B + i J \]

\[ \frac{t^3}{6} - \beta \cdot t - n = \lambda \left( -\beta h \cdot t - n \eta \right) \quad \lambda \in \mathbb{R}_+ \]
THESE WALLS EXTEND TO $\infty$ IN THE KÄHLER CONE!

$$z = x + iy$$

Set $t = z^P$ for $P \in \mathbb{R}$.

$z = x + iy$

$x = \frac{n_h}{2P \cdot \beta_h}$

$n_h < 0$

$\begin{align*}
\text{STABLE} & \quad \text{UNSTABLE} \\
\text{STABLE} & \quad \text{UNSTABLE}
\end{align*}$

$x = \frac{n_h}{2P \cdot \beta_h}$

$n_h > 0$
Consider the halo bound states with central particle \( \pi(\beta, n) \) as we increase the B-field 

\[ B = xP \times \] increases halos of D2DO particles \((0, 0, -\beta_n, n_n)\). Appear & disappear.

For \( x > 0 \)

All \( \eta_n < 0 \) states have decayed. As \( x \to +\infty \) we move into the stable region for all \( \eta_n > 0 \), and ever larger "atoms" become stable.

**General picture: Bohr model**
When $\beta_1 = 0$ walls look different

\[ \Gamma = \frac{1 + \frac{q_0}{\Gamma_1}}{\Gamma_2} \]

\[ Z = \frac{t^3}{6} - q_0 \]

Set $t = (x + iy) \Rightarrow Zero \ if \ Z = \left(\frac{6q_0}{\rho^3}\right)^{1/3}$

$q_0 > 0$

$q_0 < 0$
**INTRODUCE GENERATING FUNCTION**

\[ Z_{D6D2D0}(u,v;t) := \sum_{\beta, n} \Omega(\Gamma(\beta, n); t) u^n v^\beta \]

**SEMI-PRIMITIVE WALL-CROSSING FORMULA:**

**CONTRIBUTION OF FOCK SPACE GENERATED BY**

\[ \Gamma_h = -\beta_h + n_h dV \]

**CROSSING INTO STABLE REGION:**

\[ Z_{D6D2D0} \rightarrow (1 - (-u)^{n_h} v^{\beta_h})^{n_h^1 n_{\beta_h}^0} Z_{D6D2D0} \]

\[ \Omega(-\beta_h + n_h dV) = \sum_{m_L, m_R} (-1)^{2m_L + 2m_R} N_{\beta_h}^{m_L m_R} \]

\[ = \ n_{\beta_h}^{\circ} \]

"**SPIN ZERO GV INVARIANT**" \( (\beta_h \neq 0) \)
EXAMPLE: $\text{D6DO}$

$$Z_{\text{D6DO}}(u) = \sum \Omega \left( l + q_0 dV : t \right) u q_0$$

$\Omega \left( q_0 dV \right) = -\chi(X)$

$$Z_{\text{D6DO}}(u) = \begin{cases} 
(M(-u)) \chi(X) & \text{arg } z < \frac{\pi}{3} \\
1 & \frac{\pi}{3} < \text{arg } z < \frac{2\pi}{3} \\
(M(-u^{-1})) \chi(X) & \frac{2\pi}{3} < \text{arg } z 
\end{cases}$$

$$M(u) := \prod_{k>1} (1 - u^k)^{-k}$$
Similarly, wall-crossings for the full \( \mathcal{Z}_{d6d2d0} \) as \( x \to \infty \) build up an infinite product similar to the infinite product form of \( \mathcal{Z}_{DT}(u,v) \).

On the other hand, an argument from M-theory [Dijkgraaf, Verlinde, Vafa; Denef, Moore] implies:

\[
\lim_{x \to +\infty} \mathcal{Z}_{d6d2d0}(u,v;z^p) = \mathcal{Z}_{DT}(u,v)
\]

\[
\lim_{x \to -\infty} \mathcal{Z}_{d6d2d0}(u,v;z^p) = \mathcal{Z}_{DT}(\bar{u}^1, v)
\]
- **States in Core Region Are Complicated Bound States**

- **Product of Wall-Crossings** ⇒

\[
Z_{DT}^{1, r > 0}(u, v) = \prod_{\beta > 0, k > 0} (1 - (-u)^k v^\beta)^{k \eta_\beta^0}
\]

- **Limit For \( x \to +\infty \):**

\[
Z_{DT}^{1}(u, v) = \frac{Z_{DT}^{1, r = 0}(u, v) Z_{DT}^{1, r > 0}(u, v)}{\text{Halos} \quad \text{Cores}}
\]

\[
Z_{DT}^{1, r > 0}(u, v) = \prod_{\beta > 0, k > 0} \prod_{l = 0}^{2r-2} \left(1 - (-u)^{r-2-1} v^\beta\right)^{(-1)^{r+l} \binom{2r-2}{l} \eta_\beta^r}
\]
LOCAL LIMIT: CHOOSE CURVE CLASS $\beta_h$

CHOOSE $P, \hat{\beta} \in \mathcal{K}$ WITH $P.\beta_h = 1, \hat{\beta}.\beta_h = 0$

$$t = e^{i\phi} \wedge \hat{\beta} + z \cdot P$$

$$\hat{\beta}^3 = 1, \quad 0 \leq \phi \leq \pi, \quad \wedge \to \infty$$

$$\lim_{\wedge \to \infty} \Omega(\Gamma(\beta,n); t) = \Omega(\infty)(\Gamma(\beta,n); \infty, \phi)$$

$\Omega_\infty$ WILL HAVE WC. FOR DECAYS:

$$\Gamma(\beta, n) \to \Gamma(\beta', n') + (-m \beta_h + n dV)$$

$$e^{3i\phi} \lambda^3 = \lambda (-mz - n) \quad \lambda \in \mathbb{R}_+$$

$$W_{m,n} = \left\{ \phi = \frac{1}{3} \arg(-mz - n) \mod \frac{2\pi}{3} \mathbb{Z} \right\}$$
STUDY THE PRODUCTS AS WE CHANGE $\phi$ AT CONSTANT $Z$

EXAMPLE OF CONIFOLD: $H_2 = \beta \cdot Z$

$$\Omega (-m\beta + ndV) = \begin{cases} -2 & m = 0 \\ 1 & m = \pm 1 \\ 0 & \text{else} \end{cases}$$

$Z_{\text{core}} = 1$
\[ \phi \rightarrow \]

\[ W_{n}^0 \quad \downarrow \quad W_{z}^m \quad \downarrow \quad W_{1}^m \quad \downarrow \quad W_{0}^m \quad \downarrow \quad W_{-1}^m \quad \downarrow \quad W_{-2}^m \quad \downarrow \quad W_{n}^0 \]

\[ \pi/3 \]

\[ Z = Z_{\text{core}} = 1 \]

\[ Z = \frac{a}{\prod_{k=1}^{m=1}} (1-(-u)^k v)^k \quad W_{a+1}^m < \phi < W_{a}^m \]

\[ \cdots W_{-2}^m - W_{-1}^m \quad W_{0}^m \quad W_{1}^m \quad W_{2}^m \quad \cdots \]

\[ 0 \]

\[ \pi/3 \]

\[ Z = M(-u)^2 \cdot \prod_{k=1}^{8} (1-(-u)^k v)^k \cdot \prod_{k=a}^{8} (1-(-u)^k v^{-1})^k \]
\[ Z = (M(-u))^2 \prod_{k=1}^{\infty} (1-(u)^{k})^k \prod_{k=1}^{\infty} (1-(u)^{k}v^{-1})^k \]

Interestingly – precisely such \( \infty \)-products have appeared recently in work of B. Szendroi.

\( \exists \) related results for \( C^2/\text{ADE} \times C \) connecting to B. Young \& J. Bryan.
5. THE D4-D2-DO SYSTEM: MODULARITY

Now consider \( p^0 = 0 \)

\[ \Gamma = P + Q + q^0 dV \]

**Regular Attractor Point:**

\[ p \text{ in Kähler cone } \wedge \quad \hat{q}^0_0 < 0 \]

\[ \hat{q}^0_0 = q^0_0 - \frac{1}{2} \left( \mathcal{D}_{ABC} P^C \right)^{-1} Q_A Q_B \]

**These are black holes:**

**Horizon Area** = 4 \( S(\Gamma) = 4\pi \left| Z_* (\Gamma) \right|^2 \)

\[ S(\Gamma) = \frac{2\pi}{\sqrt{6}} \cdot \sqrt{-\hat{q}^0_0 \chi(P)} \]

\( \chi(P) := P^3 + c_2 \cdot P > 0 \text{ for } P \in \text{Kähler Cone} \)

**Expect:** \( \log \Omega(\Gamma; t) \sim S(\Gamma) \)

For "large" \( \Gamma \) and "large" \( \text{Im} t \)
A. Rough Microscopic Description

For large $J$: Single D4 wraps $\Sigma \in |P|

\chi(P) = P^3 + c_2 \cdot P = Euler Character of $\Sigma$

Flux $F \in H^2(\Sigma, \mathbb{Z})$

AND N D6's

Compute Induced RR Charges:

D2: $Q = (2 \Sigma)_* (F)$

D6: $\hat{\phi}_0 = \frac{\chi(P)}{24} + \frac{1}{2} (F^-)^2 - N$

Susy $\Rightarrow N \geq 0$, $F^{2,0} = 0 \Rightarrow (F^-)^2 \leq 0 \Rightarrow$

$\hat{\phi}_0 \leq (\hat{\phi}_0)_{max} = \frac{\chi(P)}{24}$
\[ \mathcal{M}(p,f,n) := \text{moduli of such \$d4's} \]

\[ \text{Hilb}^n(\Sigma) \longrightarrow \mathcal{M}(p,f,n) \]

\[ \Sigma \overset{\text{smooth}}{\longrightarrow} \{ \Sigma \in \text{Pic} \mid f \in H^{m}(\Sigma) \} \]

"Moduli of stable objects \$E\$ in the derived category with specified Chern classes"

\[ \text{ch} E \sqrt{A} = p + q + q_0 \quad (\star) \]

\[ = \bigcup_{f,n \text{ s.t. } \star} \mathcal{M}(p,f,n) \]
B. Index of BPS States

\[ \Omega(\Gamma)_\infty \equiv \lim_{J \to \infty} \Omega(\Gamma; B + iJ) \]

\[ d(F, N) := (-1)^{\text{dim} M} \chi(M(P, F, N)) \]

\[ \Omega(\Gamma)_\infty = \text{Finite sum of } d(F, N) \]

Surprise: When \( h^{(1)}(x) > 1 \) there are splittings @ \( \infty \):

\[ \Gamma = P + Q + q_0 \omega \]

\[ = (P' + Q' + q'_0 \omega) + (P'' + Q'' + q''_0 \omega) \]

With:

\[ \sqrt{-\hat{\theta}_0''(P'')^3} > \sqrt{-\hat{\theta}_0} P^3 \]

\[ \Rightarrow \text{Even the leading order entropy is chamber dependent} \]

[E. Andriyash + G.M.]
For \( \Gamma = P + Q + q_* dv \), \( P \in \text{Kähler cone} \), \( \exists \text{ distinguished chamber} \):

\[
\Omega(\Gamma)_\lambda := \lim_{\lambda \to \infty} \Omega(\Gamma; B + i \lambda P)
\]

**Claim:** Limit exists \( \lambda \) is \( B \)-independent

(Finiteness of attractor flow trees)

Henceforth work in this chamber.
C. Modularity

\[ \tau \in \mathcal{H} \quad \xi \quad \mathcal{C} \in \mathbb{Z}^* \left( H^2(X, \mathbb{C}) \right) \]

\[ \mathbb{Z}(\tau, \bar{\tau}, \mathcal{C}) := \]

\[ \sum_{F, N} d(F, N) \exp \left\{ -2\pi i \tau \hat{q}_0 - 2\pi i \bar{\tau} \frac{1}{2}(F^+)^2 - 2m \cdot F \cdot (\mathcal{C} + \frac{p}{2}) \right\} \]

**SUSY PARTITION FUNCTION OF D3 INSTANTON**

**U-DUALITY \Rightarrow**

\[ \mathbb{Z}(\tau, \bar{\tau}, \mathcal{C}) \text{ IS A JACOBI FORM} \Rightarrow \]

\[ \mathbb{Z}(\tau, \bar{\tau}, \mathcal{C}) = \sum_{\mu \in L^*/L} H_\mu(\tau) \Theta_{\mu, L}(\tau, \bar{\tau}, \mathcal{C}) \]

\[ \text{SEIGEL-NARAIN} \]

\[ L := \bigoplus_{\Sigma} \left( H^2(X, \mathbb{Z}) \right) \subset \bigoplus_{\Sigma} \left( H^2(\Sigma; \mathbb{Z}) \right) \]

**SELF-DUAL**

\[ l \in L \text{ IS ALWAYS IN } H^{1,1}(\Sigma) \Rightarrow \]

\[ d(F + l, N) = d(F, N) \quad \forall \ l \in L \]
• $H_\mu(\tau)$ is a vector-valued nearly holomorphic modular form of weight $w = -1 - \frac{h_\mu^2}{2}$ and multiplier system $M^*$ dual to that of $\Theta_\mu, L$

This has two interesting consequences:

• $w < 0 \implies H_\mu$ is determined by its polar terms.

Suppress $\mu$-index for simplicity:

$$H(\tau) = \sum_{\hat{g}_0} \Omega(\cdot \cdot \cdot) e^{-2\pi i \hat{g}_0 \tau}$$

$$= \sum_{0 < \hat{g}_0 \leq \frac{\chi(p)}{24}} (\cdot \cdot \cdot) + \sum_{-\infty < \hat{g}_0 \leq 0} (\cdot \cdot \cdot)$$

\[\text{Polar} \quad \text{Nonpolar}\]
1. **Fourier coeffs of cusp forms**
   **Weighted by polar degeneracies vanish.**

\[ \mathcal{P} = \text{vector space of polynomials} \]
\[ \text{in } e^{-2\pi i \hat{\varrho}_0 \tau} \text{ for} \]
\[ 0 < \hat{\varrho}_0 \leq (\hat{\varrho}_0)_{\max} = \frac{\chi(p)}{24} \]
THEN [ D. NIEBUR; J. MANSCHOT + G.M. ]

\[ 0 \rightarrow M_w(\Gamma, \mathcal{M}^*) \rightarrow \mathcal{P} \rightarrow \mathcal{S}_{2-w}(\Gamma, \mathcal{M}) \]
\[ e^{-(\hat{\varrho}_0 \tau)} \rightarrow G^{(\hat{\varrho}_0)}(\tau) \]
\[ G^{(\hat{\varrho}_0)}(\tau) = \sum_{\mathcal{D} \in \Gamma \backslash \mathcal{S}} (c\tau + d)^{-3-h/2} e^{2\pi i \hat{\varrho}_0 \chi(c)} \]

\[ \Rightarrow \text{Prediction of modularity:} \]

\[ \sum_{0 < \hat{\varrho}_0 \leq \frac{\chi(p)}{24}} \Omega(\Gamma), \ G^{(\hat{\varrho}_0)}(\tau) = 0 \]
2. Polar degeneracies determine all other degeneracies through an explicit formula—the Rademacher formula.

The nonpolar degeneracies are of physical interest for black hole entropy leads to

\[ \Omega(\Gamma)_\infty \sim \int d \omega e^{-2\pi i \hat{\omega}_0 \omega} H^{\text{polar}}(-1/\omega) + \ldots \]

So we want to compute the polar degeneracies...
C. MACROSCOPIC POLAR STATES

IF \( \Gamma = (0, P, Q, q_o) = P + Q + q_o dV \)

IS POLAR: \( 0 < \hat{q}_o \leq (\hat{q}_o)_{\text{max}} \)

THEN \( z(\Gamma; t) \) HAS A ZERO.

Indeed \( S(\Gamma) = \frac{2\pi}{\sqrt{6}} \cdot \sqrt{-\hat{q}_o} x(P) \)

SO NO SINGLE-CENTERED SOLUTION

But \( H(t) \) HAS \( W < 0 \) \( \Rightarrow \) SOME

POLAR DEGENERACIES ARE NONZERO

\( \Rightarrow \) THESE MUST BE REALIZED AS

SPLIT ATTRACTOR STATES.
SIMPLE EXAMPLE

PURE D4: \( \Gamma = P + q_0 \, dV \)

WITH \( q_0 = \hat{q}_0 = (\hat{q}_0)_{\text{max}} = \frac{\chi(P)}{24} \)

FIND ONLY ONE SPLITTING

\( \Gamma = P + q_0 \, dV = \Gamma_1 + \Gamma_2 \)

\[ = e^{S_1} \left( 1 + \frac{c_2(x)}{24} \right) - e^{S_2} \left( 1 + \frac{c_2(x)}{24} \right) \]

1D6 with flux = \( S_1 \)

1D6 w/ FLX \( S_2 \)

\( S_1 - S_2 = P \)

\[ t \to \infty \]

Diagram:

- \( S_1 \)
- \( S_2 \)
- MS(\( \Gamma_1 \), \( \Gamma_2 \))
Moreover - you can compute the polar degeneracy:

\[ \Omega(\Gamma, t_\infty) = (-1)^{I_{12}} / |I_{12}| \quad \Omega(\Gamma_1) \cdot \Omega(\Gamma_2) = (-1)^{I_{12}} / |I_{12}| \]

\[ I_{12} = \langle \Gamma_1, \Gamma_2 \rangle = \frac{p^3}{6} + \frac{c_2(X) \cdot p}{12} \]

Indeed = the correct answer for \( \chi \cdot \text{(moduli of pure D4)} = \chi(\overline{1P1}) \)
Describing the split attractor flows for $0 < \hat{\varphi}_0 < \frac{\pi(p)}{24}$ is much more complicated...

In general, polar states can be very complicated split attractors, realized in many different ways....

But in the limit $p \to \infty$ we can say something.
**EXTREME POLAR STATES**

\[ H^\text{polar}(\tau) = |\lambda| e^{-2\pi i \frac{\chi(p)}{24}} + \ldots + O\left(\frac{-2\pi i \tau}{e^{1\beta}p}\right) \]

"EXTREME POLAR" "BARELY POLAR"

E.P.S. CONJECTURE: \( \exists \epsilon < 1 \) so that

\[
\frac{\hat{\rho}_{b_0}^{\text{max}} - \hat{\rho}_{b_0}}{\hat{\rho}_{b_0}^{\text{max}}} < \epsilon \quad \Rightarrow
\]

Polar states split as \( \text{D6}$\text{D6}$ + \text{HALOS}:

\[ \Gamma_1 = e^{S_1(1 - \beta_1 n_1 dV)} \]

\[ \Gamma_2 = -e^{S_2(1 - \beta_2 n_2 dV)} \]
So, by the W.C.F. together with results on $Z_{D6D2D0}$ the extreme polar degeneracies are related to:

$$|Z_{DT}|^2 \overset{\text{MNOP}}{=} |Z_{\text{Top}}|^2$$

suggesting a relation like the GSV conjecture

$$\Omega(\Gamma)_\infty = \int d\phi \ |Z_{\text{top}}(g_{\text{top}}, t)|^2 e^{-2\pi g_0 \phi}.$$  

- $\exists$ strong arguments for $|\hat{q}_0| \gg p^3$
- $\exists$ potential counterexamples for $|\hat{q}_0| \lesssim p^3$: "entropy enigma"
6. Conclusion

With more time I would go on to describe the application to OSV.

Most important open problem is the behavior of barely polar degeneracies.

Turns out to be related to a sharp mathematical question:

$$\lim_{\lambda \to \infty} \frac{\log \log |N_{\delta T}(\lambda^2 \beta, \lambda^3 \lambda)|}{\log \lambda} = k(\beta, n)$$

Expect: \[ 2 \leq k(\beta, n) \leq 3 \]
6. Rough Sketch of OSV

Our Version of OSV:

If \( \text{ch} \sqrt{A} = P + Q + q_0 \) with \( P \) in the Kähler cone, then:

\[
\Omega (\Gamma)_\infty := \lim_{\lambda \to \infty} \Omega (\Gamma; t = B + i\lambda P)
\]

Limit is well-defined and \( B \)-independent.

Then

\[
\Omega (\Gamma)_\infty = \int d\phi \, \mu(\phi) \left| Z_{\text{top}}^e (g_{\text{top}}, t) \right|^2 e^{-2\pi q_0 \phi} \cdot (1 + o(e^{-\Delta}))
\]

Where:
1. \( g_{\text{top}} = \frac{2\pi}{\phi_0} \quad t^A = \frac{1}{\phi_0} (\phi^A + i \frac{P^A}{2}) \)

2. \( Z_{\text{top}} (g,t) = \text{top. sing p.f.} \)
   \[ = \sum_{\beta, n} N_{\beta T} (\beta, n) (-e^g)^n e^{2m_J t} \]

3. \( Z_{\text{top}}^\epsilon (g, t) = \sum_{\beta, P < \epsilon P^J, |n| < \epsilon |P^J|} (\ldots \ldots \ldots ) \)

4. \( \mu (P, \phi) = \frac{1}{g_{\text{top}}^2} \text{Re} \left( X^A \frac{\partial F_{\text{top}}^\epsilon}{\partial X^A} \right) \)
   \[ = \frac{1}{g_{\text{top}}^2} e^{-K} \quad (b_1 (X) = 0) \]

5. \( \Delta = \text{FUNCTION of} : \epsilon, P, \phi_0 ; \)

If \( (g_{\text{top}})^{\text{s.p.}} \approx \sqrt{-\frac{g_0}{\phi_0}} \gg 1 \)

Then \( e^{-\Delta} = \exp \left( -\frac{\pi}{12\mu} \frac{\epsilon}{\phi_0} P^3 \right) \)
• Above assumes the truth of the "EPS conjecture".

• Moreover, the phenomenon of "Swing states" ⇒ we must take
  \[ \epsilon = \delta |P|^{-3/2}. \]

  \( c_d = \text{"Core dump exponent"} \)
  known to be \( \leq 3. \)

• But for W.S. instantons to be relevant we need \( c_{d} \leq 2. \)
  ⇒ Open Problem.
1. Fareytail:

\[ Z_{D4D2D0} = \sum \text{mod. tmns. of } Z_{D4D2D0}^{\text{polar}} \]

2. \[ Z_{D4D2D0}^{\text{polar}} = Z_{D6D6}^E (t_{ms}) + \text{ET}(e) \]
   \[ \uparrow \]
   \[ \text{EPS CONJECTURE} \]

3. \[ Z_{D6D6}^E (t_{ms}) = Z_{D6D2D0}^E (t_{ms}) Z_{D6D2D0}^E (t_{ms}) \]
   \[ \uparrow \]
   \[ \text{W.C. Formula} \]

4. \[ Z_{D6D2D0}^E (t_{ms}) = Z_{DT}^E \]

\[ \epsilon \sim |P|^{-\xi_{cd}} \]
\[ P \rightarrow \infty \]

\[ \text{SWING STATE CONJECTURE: } \xi_{cd} < 2 \]

5. \[ Z_{DT} = Z_{GW} = Z_{top} \]
   \[ \text{MNOP CONJ.} \]

\[ \text{STEPS 1 \& 2 MAKE IMPORTANT APPROXIMATIONS} \]
7. PROBLEMS AT WEAK COUPLING

IN THE CHARGE REGIME

\[ g_{\text{top}} \sim \sqrt{-\frac{g_s}{\rho}} \leq O(1) \]

THE ABOVE DERIVATION BREAKS DOWN

- MODIFICATIONS OF THE "MODERN F.T." BECOME IMPORTANT
- BARELY POLAR DEGENERACIES BECOME LARGE

TOY MODEL: \( \chi \sim \frac{\rho^2}{24} \)

\[ \frac{1}{\eta \chi} = \sum_{n=0}^{\infty} P_X(n) e^{2\pi i(n - \chi/24)} \]

\[ P_X(\frac{\chi}{24} + l) \sim \exp \left[ k \cdot \chi + k' \cdot l \right] \]

AND THERE IS GOOD REASON THE DERIVATION BREAKS DOWN ...
Now choose $q_0 < 0$, $P$ ample so
\[ \Gamma = (0, P, 0, q_0) \]
has a regular attractor point

Nevertheless! We can choose $q_0, QA$ so that $\exists$ a two-centered solution with
\[ \Gamma = \Gamma_1 + \Gamma_2 \]

\[ \Gamma_1 = \left( r, \frac{1}{2} P,QA, \frac{1}{2} q_0 \right) \quad \Gamma_2 = \left( -r, \frac{1}{2} P, -QA, \frac{1}{2} q_0 \right) \]

Both solutions exist.
So... compare entropies

\[ S(\Gamma) \text{ vs. } S(\Gamma_1) + S(\Gamma_2) \]

In fact,

\[ \exists \text{ family of charges} \]

\[ \lambda \Gamma = \lambda(0, p, 0, q_0) = \Gamma_1^\lambda + \Gamma_2^\lambda \]

\[ \Gamma_1^\lambda = (\gamma, \frac{1}{2} p, \lambda^2 q, \frac{\lambda}{2} q_0) \quad \Gamma_2^\lambda = (-\gamma, \frac{1}{2} p, -\lambda^2 q, \frac{\lambda}{2} q_0) \]

Scaling of entropies:

\[ S(\lambda \Gamma) = \lambda^2 S(\Gamma) \]

But!

\[ S(\Gamma_1^\lambda) = S(\Gamma_2^\lambda) \sim \frac{(\lambda p)^3}{\Gamma} \sim \lambda^3 \]

\[ \Rightarrow \text{ many implications for physics & mathematics} \]
8. Some Open Problems

a.) Physical derivation of the KS formula

b.) How to compute polar degeneracies effectively?

c.) Resolve the question of the Entropy Enigma: Are there cancellations bringing $e^{\lambda^3} \rightarrow e^{\lambda^2}$?

d.) Is there an OSV-like relation for $\Omega(\Gamma, t_\star(\Gamma))$? Do these enjoy automorphy properties?

e.) Our proof of (strong coupling) OSV suggests a nonptive definition of $Z_{top}$. Is it physically natural?

f.) Are BPS algebras useful to these ideas?
B. MATH APPLICATIONS
(WITH E. DIACONESCU)

Consider the case where $D^4$ wraps a rigid surface $S$ in CY.

$P$ is not in Kähler cone.

Example: $r = 2$ bundle on $S$ with $b''(S) > 1$

As a function of $T$ can have mathematical (“slope”) instability:

$$0 \to \mathcal{E}_2(n_2) \to \mathcal{E} \to \mathcal{E}_1(n_1) \to 0$$

Such bundles become unstable across walls in Kähler cone.
Walls of Physical Stability Asymptote to Walls of Mathematical (slope) Stability.

\[ \Omega(y; \Gamma; t) = \text{Tr}_{\mathcal{H}(\Gamma; t)} (-y)^{2J_3} \]

\[ \mathcal{H}(\Gamma; t) = H^* \{ \text{moduli of branes} \} \]

\[ \Omega(y; \Gamma; t) = \text{Poincaré polynomial} \]

\[ \Delta \Omega = (-y)^{\langle \pi, \pi \rangle + 1} \frac{1-y^{2\langle \pi, \pi \rangle}}{1-y^2} \Omega(y; \Gamma_1) \Omega(y; \Gamma_2) \]

\[ \Rightarrow \text{reproduce results of Göttsche and Yoshioka on moduli of bundles,} \]
Moduli of D4

Wrapping a rigid holo. surface is not moduli space of (slope stable) sheaves!! even when ws. inst. corrections are neglected.

Contradicts statements found in the literature. (including my papers)

Presumably the right concept is the moduli space of stable objects in the derived category (yet to be constructed)