Four-dimensional wall-crossing from three-dimensional field theory

Work done with Davide Gaiotto and Andy Neitzke

New England String Meeting
Brown University, Oct. 24, 2008

Based on
arXiv: 0807.4723
OUTLINE

1. INTRODUCTION
2. REVIEW OF BPS WALL-CROSSING
3. THE KS-FORMULA
4. COMPACTIFICATION OF $\mathcal{N}=2,\mathcal{D}=4$ THEORIES ON $\mathbb{T}^3 \times S^1$
5. TWISTOR SPACE
6. SINGLE PARTICLE Q.C.'S TO T.S.
7. MULTI-PARTICLE: RIEMANN-HILBERT
8. PHYSICAL PROOF OF THE KS FORMULA
9. TAKE-HOME SUMMARY
10. CONCLUSION
1. Introduction

This talk is about the BPS spectrum of $W=2$, $D=4$ field theories.

The BPS spectrum of the theory on $\mathbb{R}^4$ is a "piecewise constant" function of the boundary conditions at $\infty$ of vector/multiplet scalars.

Recently there has been some progress in understanding precisely how the spectrum depends on boundary conditions.

These are called wall-crossing formulae (WCF). This talk will give a physical interpretation and proof of a famous WCF of Kontsevich + Soibelman.
Consider a theory on $\mathbb{R}^4$ with $N=2$ super-Poincare' symmetry.

Let $\mathcal{H}$ be the one-particle Hilbert space.

As a representation of the $W=2$ super-Poincare' algebra $\mathfrak{D}$, $\mathcal{H}$ depends on the boundary values of fields at $\infty$.

These boundary conditions are valued in the moduli space of vacua: $\mathcal{M}_v$.

For $u \in \mathcal{M}_v$, write $\mathcal{H}_u$. 
For all $u \in \mathbb{M}$, there is an unbroken abelian gauge symmetry of rank $r$, so $\mathcal{H}$ is graded by the symplectic lattice $\Gamma$ of elec.+ mag. charges. (of rank $2r$).

$$\mathcal{H}_u = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_{\gamma,u}$$

On each subspace $\mathcal{H}_{\gamma,u}$ the central charge operator $Z \in \Delta$ is a scalar.

Denote the value $Z_{\gamma}(u)$ on $\mathcal{H}_{\gamma,u}$ $E \geq |Z_{\gamma}(u)|$. 
DEF: $H_{y_1, u}^{BPS} =$ SUBSPACE SATURATING THE BPS BOUND.

ON THIS SUBSPACE $E = |Z_y(u)|$

SOME BPS PARTICLES CAN BE VIEWED AS BOUNDSTATES OF OTHERS

[Cecotti et al.; Seiberg & Witten]

$Z_y(u)$ IS LINEAR IN $y = y_1 + y_2$ SO

$E(u) = |Z_y(u)| - (|Z_{y_1}(u)| + |Z_{y_2}(u)|) \leq 0$

$\Rightarrow$ DECAY ONLY HAPPPENS ALONG WALLS OF MARGINAL STABILITY:

$MS(y_1, y_2) := \{ u \mid Z_{y_1}(u)/Z_{y_2}(u) \in \mathbb{R}_+ \}$
WCF: Be more quantitative about "how many" states decay

Define the BPS index

\[ \Omega^\text{ph}(\gamma; u) = -\frac{1}{2} \text{Tr}_{\mathcal{F}_{\text{bps}}^{\gamma, u}} (2J_3)^2 (-1)^{2J_3} \]

Denef & Moore gave formulae for \( \Delta \Omega \) for decays \( \gamma \rightarrow \gamma_1 + \gamma_2 \)
where at least one of \( \gamma_1, \gamma_2 \) are primitive.

The derivation is based on Denef's multicentered solutions of \( \mathcal{N}=2 \) Sugra and quiver quantum mechanics.

The methods are difficult to use when both \( \gamma_1, \gamma_2 \) are non-primitive.
Kontsevich & Soibelman proposed a remarkable W.C.F. for an index $\Omega^\text{DT}(y;u)$.

"Generalized Donaldson-Thomas invariant of a Calabi-Yau 3-fold"

Their formula applies to all decays $y \to y_1 + y_2$.

We expect that

$$\Omega(y;u) = \Omega^\text{Ph}(y;u) = \Omega^\text{DT}(y;u)$$

So the KS WCF applies to physical BPS degeneracies.

This talk proves the KS WCF for $\Omega^\text{Ph}(y;u)$ in $\mathcal{N}=2$ field theories.
DATA:

1. SYMPLECTIC LATTICE \( \Gamma \)

2. CENTRAL CHARGES \( Z_\gamma(u), u \in \mathbb{C} \)

3. PIECEWISE CONSTANT \( \Omega(\gamma; u) \in \mathbb{Z} \)

BPS RAYS: FOR \( u \in \mathbb{M}_v, \gamma \in \Gamma \)

\[
l_\gamma := Z_\gamma(u) \mathbb{R}_- = \{ s \mid s Z_\gamma(u) \in \mathbb{R}_- \}
\]
AS $u$ VARIES THE SLOPES OF THE BPS RAYS VARY

AS $u$ CROSSES A WALL $MS(\gamma_1, \gamma_2)$ BPS RAYS WILL COALESCE

\[ MS(\gamma_1, \gamma_2) \]

\[ u^+ \quad u^- \]

\[ l_{\gamma_1} \quad l_{\gamma_1 + \gamma_2} \quad l_{\gamma_2} = l_{\gamma_1 + \gamma_2} \quad l_{\gamma_1 + \gamma_2} \quad l_{\gamma_1} \]

\[ u^+ \quad U_{ms} \quad u^- \]
SECOND INGREDIENT: A SYMPLECTIC TORUS:

- **INTRODUCE THE COMPLEX TORUS**

\[ T = \Gamma^* \otimes \mathbb{C}^* \cong \bigotimes_{i=1}^{2r} \mathbb{C}^* \]

\[ \gamma \in \Gamma \implies \text{FUNCTION } X_\gamma : T \to \mathbb{C}^* \]

"HOLOMORPHIC FOURIER MODES"

CHOOSING A BASIS \( \gamma_i \) FOR \( \Gamma \implies \)

\[ (e^{\theta_1}, \ldots, e^{\theta_{2r}}) \in T, \ \theta_i \in \mathbb{C} \]

\[ X_\gamma = \exp \left[ \gamma \cdot \theta \right] \]
\[ \omega_T := \frac{1}{2} \epsilon^{ij}_\gamma \left( \frac{dX_{\gamma i}}{X_{\gamma i}} \wedge \frac{dX_{\gamma j}}{X_{\gamma j}} \right) \quad \epsilon^{ij}_\gamma = \langle \gamma_i, \gamma_j \rangle \]

For each \( \gamma \in \Gamma \) define a symplectomorphism:

\[ k_\gamma : X_{\gamma'} \rightarrow X_{\gamma'} (1 - X_\gamma) \langle \gamma'_i, \gamma'_j \rangle \]

\[ X_{\gamma'} \rightarrow X_{\gamma'} \exp \left[ \langle \gamma, \gamma' \rangle \log (1 - X_\gamma) \right] \]
 EXAMPLE

$r = 1 \implies \Gamma \cong \mathbb{Z} \oplus \mathbb{Z}$

$\langle (a, b), (a', b') \rangle = ab' - a'b$

$\Omega \cong \mathbb{C^*} \times \mathbb{C^*}$

$x = X_{1,0}, \quad y = X_{0,1}$

$\omega_T = \frac{dx}{x} \wedge \frac{dy}{y}$

$k_{a,b}: \left\{ \begin{array}{c}
  x \mapsto x(1 - (-1)^{ab} x^a y^b)^b \\
  y \mapsto y(1 - (-1)^{ab} x^a y^b)^{-a}
\end{array} \right.$
NOW CHOOSE A CONVEX CONE $\mathcal{V}$

\[ e_y = \mathbb{Z}^\vee \times \mathbb{R}_- \]

FOR EACH BPS RAY DEFINE

\[ S_y := \prod_{y' \parallel y} K_{y'} \Omega(y'; u) \]

AND THEN DEFINE:

\[ A_y := \prod_{e_y \in \mathcal{V}} S_y \]
\[ A_V := \prod_{\vec{e}_V \in V} S_{\vec{e}_V} = \prod_{\vec{e}_V \in V} K_{\vec{e}_V} \Omega(\vec{e}_V) \]

The product is taken over the rays in the clockwise order (decreasing slope).

\[ A_V \text{ depends on } u \text{ in two ways} \]

1. The ordering of factors depends on \( u \)

2. The \( \Omega(\vec{e}_V) \) depend on \( u \) ...
THE KS FORMULA STATES THAT

\[ A_\nu = \prod_{\kappa < \nu} k_\kappa \Omega(\gamma; \nu) \]

IS CONSTANT IN \( \nu \) AS LONG AS NO BPS RAY ENTERS OR LEAVES THE SECTOR \( \nu \).

THIS IS A WALL-CROSSING FORMULA ...
As u crosses a wall

\[ l_\gamma = \mathcal{E}_\gamma(u), \quad \mathbb{R}_- \text{ rotates} \]

\[ \Rightarrow \text{ exchange order in} \]

\[ A_N = \prod_{l_\gamma \in \mathcal{V}} k_{l_\gamma}^{\Omega(\gamma;u)} \]

\[ \Omega(\gamma;u) \text{ makes a compensating change} \]
ONE CAN RECOVER THE
PRIMITIVE & SEMI-PRIMITIVE
WCF. FROM THIS FORMULATION...

[with Wu-yen Chuang]
4. Compactification of \( N=2, D=4 \) Field Theories

A. Seiberg-Witten Solution

\( G \) - Compact S.S. Gauge Group, rank = \( r \)

\[ \Rightarrow \ D = 4, N = 2 \ \text{Field Theory} \]

(can also include HM's)

\[ \mathcal{M}_V = (g_G)^G \cong \mathbb{C}^r \quad (u_2 = Tr \Phi^2, u_3 = Tr \Phi^3 \ldots) \]

S&W gave formulae for

- \( Z_y(u) \)

- Low energy Abelian Gauge Theory

in terms of

Special Kähler Geometry
VIEW $\mathcal{D}$ AS A LOCAL SYSTEM OVER $\mathcal{M}_v$

$$\mathcal{G}_u \rightarrow \mathcal{G}$$

$$\downarrow \quad \downarrow$$

$$\mathcal{U} \rightarrow \mathcal{M}_v$$

$$\mathcal{G}_u = \Gamma_u \otimes \frac{1}{2\pi i} (\mathbb{R}/2\pi \mathbb{Z}) \cong U(1)^{2r}$$

Fibers = Abelian Varieties

IN REGIONS OF $\mathcal{M}_v$ CHOOSE A DUALITY FRAME:

$$\Gamma = \Gamma_{el} \otimes \Gamma_{mag}, \quad \Gamma_{mag} = \Gamma_{el}^*$$

$$= \text{Span}\{\alpha^i\} \oplus \text{Span}\{\beta^i\}$$

$$\langle \alpha^i, \alpha^j \rangle = \langle \beta^i, \beta^j \rangle = 0 \quad \langle \alpha^i, \beta^j \rangle = \delta_{ij}^*$$
Choosing a duality frame, \( J_u \) has period matrix \( \tau_{IJ} \)

\[ L = -\frac{1}{4\pi} \text{Im} \tau_{IJ} (d\alpha^I \overline{d\alpha^J} + F^I \overline{F^J}) \]
\[ + \frac{1}{4\pi} \text{Re} \tau_{IJ} F^I \wedge F^J \]
\[ \alpha^I = Z_{\alpha^I}(u) \quad I = 1, \ldots, r \]

Local coord's on \( M_v \)

2. Central Charge Function

\[ Z_y(u) = a \cdot \gamma_{el} + a_D \cdot \gamma_{mg} \]

\[ \tau_{IJ} = \frac{\delta a^I}{\delta a^J} = \frac{\partial}{\partial x^I \partial x^J} \]
S. W. IDENTIFY $\Sigma_u$ AS JACOBIANS OF AN EXPLICIT FAMILY OF RIEMANN SURFACES

**Basic Example:** $G = SU(2)$

$\Sigma_u$: $y + \frac{u}{y} = x^2 - 2u$

\[
\begin{align*}
\mathbf{u} &\quad \mathbf{w} \\
\mathbf{u} &\quad \mathbf{w}
\end{align*}
\]

\[
\begin{align*}
\mathbf{MS} &\quad \mathbf{WEAK} \\
\mathbf{MS} &\quad \mathbf{WEAK}
\end{align*}
\]

\[
\begin{align*}
u &= -\lambda \quad \text{STRONG} \\
\lambda \quad \text{STRONG}
\end{align*}
\]

\[
\begin{align*}
a &= \frac{\sigma}{\lambda} \times \frac{dy}{y} \\
ad &= \frac{\sigma}{\rho} \times \frac{dy}{y}
\end{align*}
\]
\[ \mathcal{E}^{BPS}_{\text{weak}} = \bigoplus_{n \in \mathbb{Z}} \text{HM}(2n, 1) \bigoplus \text{VM}(2,0) \bigoplus \text{CONJUGATE} \]

\[ \mathcal{E}^{BPS}_{\text{strong}} = \text{HM}(2, -1) \bigoplus \text{HM}(0, 1) \bigoplus \text{CONJUGATE} \]

[Billal & Ferrari]

\[ k_{2,-1} k_{0,1} = k_{0,1} k_{2,1} k_{4,1} \ldots k_{2,0} \ldots k_{6,1} k_{4,1} k_{2,1} \]

**IT IS TRUE !!!**
B. COMPACTIFY ON A CIRCLE.

- Now consider the theory on $\mathbb{R}^2 \times S^1_R$.

- Low energy theory is a 3D $\sigma$-model: $\mathbb{R}^3 \rightarrow \mathcal{M}$

  \[
  \alpha^I(x, x^4) \rightarrow \alpha^I(x)
  \]

  \[
  \phi_\epsilon^I = \sum_{s^1} A^I_{4} \, dx^4
  \]

  \[
  \phi_m, I = \sum_{s^1} (A_{D, 4})_I \, dx^4
  \]  \[\text{PERIODIC!}\]

- SUPERSYMMETRY $\Rightarrow$

  $\mathcal{M}$ must carry a hyperkähler metric

  LET US TRY TO DESCRIBE IT
Topologically $\mathcal{M}$ is a torus fibration over $\mathcal{M}_v$:

It is exactly $\mathcal{M} = \mathcal{G} = \mathbb{T}^* \otimes \mathbb{R}/2\pi \mathbb{Z}$ that appeared above:
THE SEMI-FLAT METRIC

LEADING $R \to \infty$ APPROXIMATION:

USE DIMENSIONAL REDUCTION

+ DUALIZATION OF 3D GAUGE FIELD:

\[
    \mathcal{L}^\text{(3)} = -\frac{R}{2} \text{Im} \tau_{IJ} \, d\alpha^I \ast d\bar{\alpha}^J \\
    \quad - \frac{1}{8\pi^2 R} (\text{Im} \tau)^{-1} \tau_{IJ} \, d\bar{z}^I \ast d\bar{z}_J
\]

\[
    d\bar{z}^I = d\varphi_m^I - \tau_{IJ} \, d\varphi_e^J
\]

THIS DEFINES THE SEMI-FLAT METRIC

\[
    g^\text{sf}_J = R \left( \text{Im} \tau \right) \left| d\alpha \right|^2 + \frac{1}{4\pi^2 R} \left( \text{Im} \tau \right)^{-1} \left| d\bar{z} \right|^2
\]
C. The Key Idea

- The metric $g_{st}$ receives quantum corrections from BPS particle world-lines wrapping $S^1$.

- Therefore the quantum corrections depend on the BPS spectrum.

- The true metric $g$ should be a smooth metric on $M$ away from the locus in $M_{v}$ where BPS particles become $M = 0$.

- Smoothness of $g$ across walls of $M_{s}$ implies a WCF. CLAIM: IT IS THE KS WCF.
5. TWISTOR SPACE APPROACH

WE WILL USE HITCHIN’S THEOREM: KNOWING \((M,g)\)
IS EQUIVALENT TO KNOWING TWISTOR SPACE \(Z := M \times \mathbb{CP}^1\)
AS A HOLOMORPHIC MANIFOLD.

**Theorem**: If \((M,g)\) is HK of dimension 4, then:
1. \( \exists \) Holo. FIBRATION

\[ p : \mathbb{Z} \rightarrow \mathbb{C}P^1 \]

\[ \mathcal{M}_5 = p^{-1}(5) = \mathcal{M} \text{ in complex structure } 5 \]

2. \( \exists \) Holomorphic Section

\[ \omega \text{ of } \bigotimes^2 \Omega^2_{\mathbb{Z}/\mathbb{C}P^1} \otimes \mathcal{O}(2) \]

\[ \widetilde{\omega}_5 := \omega \big|_{\mathcal{M}_5} = \text{holomorphic symplectic form on } \mathcal{M}_5 \]

3. \( \forall x \in \mathcal{M} \), \( \exists \) Holomorphic Section

\[ s_x : \mathbb{C}P^1 \rightarrow \mathbb{Z} \text{ with normal bundle } \mathcal{O}(1)^{\otimes 2r} \]

4. \( \exists \) Anti-Holomorphic \( \sigma : \mathbb{Z} \rightarrow \mathbb{Z} \)

Covering \( \mathbb{I} \rightarrow -1/\mathbb{I} \)
CONVERSELY,

GIVEN $1, 2, 3, 4$ ONE CAN RECONSTRUCT THE METRIC:

FOR $\mathbb{C}^*$:

$$\omega_5 = -\frac{i}{25} \omega_+ + \omega_3 - \frac{i}{2} \bar{z} \omega_-$$

$\uparrow$

KAHLER FORM

$$\omega_+ = \omega_1 + i \omega_2$$

OUR STRATEGY IS TO CONSTRUCT $\omega_5$ EXPLICITLY FOR $\mathcal{F}_5$ USING A "NICE" SET OF HOLomorphic FUNCTIONS ON twistor SPACE:

$$\chi_y, y \in \mathcal{F}$$
USE THE TORUS FIBRATION OF $\mathcal{G}$:

$\mathcal{G} \xrightarrow{\mu} \mu$ with $\mathcal{G}_\mu \cong (U(1))^2$

FOR $s \neq 0, \infty$ $\mathcal{G}_\mu$ IS NOT HOLOMORPHIC

BUT CONSIDER $\tilde{\mathcal{G}} := \Gamma^* \otimes \mathbb{C}^*$

- $\tilde{\mathcal{G}}$ HAS A FIXED COMPLEX STRUCTURE WITH HOLOMORPHIC FIBERS $\cong (\mathbb{C}^*)^2$

- $\tilde{\mathcal{G}}$ HAS HOLOMORPHIC FUNCTIONS $X_y$

- $\tilde{\mathcal{G}}$ HAS A FIBERWISE HOLOMORPHIC SYMPLECTIC FORM:

$$\omega_{\tilde{\mathcal{G}}} := \frac{1}{2} \epsilon^{i,j} \frac{dX_{y_i}}{X_{y_i}} \wedge \frac{dX_{y_j}}{X_{y_j}}$$
WE SEARCH FOR A HOLONOMIC MAP

\[ \chi: \mathbb{Z} \rightarrow \mathcal{T} = \mathbb{R}^* \otimes \mathbb{C}^* \]

\[ \mathcal{J}_u \cong (U(1))^2 \]

\[ \chi(\cdot, \delta) \]

\[ \mathcal{T}_u \cong (\mathbb{C}^*)^2 \]

So that: \[ \omega_\delta = \chi^*(\omega^\mathcal{T}) \]

If we define \[ \chi_y := \chi^*(X_y) \]

\[ \Rightarrow \quad \omega_\delta = \frac{1}{2} \varepsilon^{ij} \frac{dX_{y_i}}{X_{y_i}} \wedge \frac{dX_{y_j}}{X_{y_j}} \]
WE CAN VIEW

\[ \Omega_j^i = \frac{1}{2} \varepsilon^{ij} \frac{dX_{\gamma i}}{X_{\gamma i}} \wedge \frac{dX_{\gamma j}}{X_{\gamma j}} \]

IN TWO WAYS:

• KNOW THE METRIC \( \Rightarrow \) CONSTRUCT \( X_\gamma \)

— DO THIS FOR THE SEMI-FLAT METRIC AND FIRST QUANTUM CORRECTION

• ULTIMATELY, WE DEFINE THE \( X_\gamma \) AND USE THEM TO DEFINE \( \Omega_j^i \) (AND HENCE THE HK METRIC)
EXAMPLE: SEMI-FLAT LIMIT

WE KNOW $\varphi^{sf} \Rightarrow$ COMPUTE

$$\omega_5 = -\frac{i}{2j} \omega_+ + \omega_3 - \frac{i}{2} s \omega_-$$

$$= \frac{i}{8\pi^2} \, da^I \wedge dz_I + \ldots + \frac{j}{8\pi} \, da^I \wedge dz_I$$

FIND $X_i$ so THAT:

$$\bar{\omega}_5 = \frac{1}{8\pi^2 R} \in \frac{i}{j} \frac{dX_i}{X_i} \wedge \frac{dX_j}{X_j}$$

SOLUTION: DEFINE $\Theta_y : \Gamma^* \rightarrow \mathbb{R}/2\pi\mathbb{Z}$

$$X^{sf}_y = \exp \left[ \pi R S^{-1} Z_y + i \Theta_y + \pi R S \bar{Z}_y \right]$$

[A. NEITZKE & B. PLOLINE]

- LEADING APPX. TO $X_y$ FOR $R \rightarrow \infty$
- NO Q.C.'S FROM BPS STATES.
6. SINGLE-PARTICLE CORRECTIONS

Now we include the first Q.C.

* For simplicity consider $r = 1$.

* Consider a point $u_\ast \in M_\nu$

Where a single HM has $n \to 0$.

⇒ Dominant contribution near $u_\ast$.

Choose duality frame so it has charge $(q, 0)$, $q > 0$

KK reduction ⇒ Target

Space metric is a Gibbons-Hawking Ansatz: [Seiberg Witten; Ogura Vafa; Seiberg Shenker]

$$g = V^{-1}(\vec{x}) \left( \frac{d\phi_\mu}{2\pi} + A \right)^2 + V(\vec{x}) \; d\vec{x}^2$$

$$F = * dV \quad \vec{x} \in \mathbb{R}^3$$
INTEGRATE OUT KK TOWER:

\[ V(\vec{x}) = \frac{g^2 R}{4\pi} \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{g^2 R |a|^2 + (\frac{q \phi_e}{2\pi} + n)^2}} \]

\[ \alpha = x^1 + i x^2 \]

\[ \phi_e = 2\pi R x^3 \quad \text{PERIODIC} \]

\[ V(\vec{x}) = V^{sf} + V^{\text{inst}} \]

\[ V^{sf} = -\frac{g^2 R}{4\pi} \left( \log \frac{\alpha}{\Lambda} + \log \frac{\bar{\alpha}}{\Lambda} \right) \]

\[ V^{\text{inst}} = \frac{g^2 R}{2\pi} \sum_{n \neq 0} e^{i n \phi_e} K_0 \left( 2\pi R |\ln q| a^1 \right) \]

\[ \sim e^{-2\pi R |\ln q| a^1} : \text{INSTANTON CONTRIBUTION} \]
Now, what are the holo. functions on twistor space?

Algebra of holo functions \( \{ X_y \} \) on twistor space is generated by:

\[
X_e := X_{(1, 0)} = \exp \left\{ i \varphi_e + \cdots \right\}
\]

\[
X_m := X_{(0, 1)} = \exp \left\{ i \varphi_m + \cdots \right\}
\]

\[
X_{(k_1, k_2)} = (X_e)^{k_1} (X_m)^{k_2}
\]

\[
(k_1, k_2) \in \mathbb{Z}^2 = \Gamma \]
Determine $\chi_e$ and $\chi_m$

from a differential equation

**HK Structure:** $\alpha = 1, 2, 3$:

$$\omega^\alpha = dx^\alpha \wedge \left( \frac{d\varphi_m}{2\pi} + A \right) + \frac{1}{2} \nabla \epsilon^{\alpha\beta\gamma} dx^\beta dx^\gamma$$

$\Rightarrow$ Compute

$$\omega_5 = -\frac{i}{2\pi} \omega_+ + \omega_3 - \frac{i}{2} \omega_-$$

$$\overrightarrow{\omega_5} = -\frac{1}{4\pi^2 R} \frac{d\chi_e}{\chi_e} \wedge \frac{d\chi_m}{\chi_m}$$
WE FIND:

\[ \chi_e = \chi_e^{s.f.} = \exp \left[ \frac{\pi R}{\delta} a + i \varphi_e + \pi R S \overline{a} \right] \]

**BUT**

\[ \chi_m = \chi_m^{s.f.} \cdot \chi_m^{\text{inst.}} \]

\[ \chi_m^{s.f.} = \exp \left[ \frac{\pi R}{\delta} \cdot a_\sigma + i \varphi_m + \pi R S \overline{a_\sigma} \right] \]

\[ a_\sigma = \frac{q^2}{2\pi i} \left( a \log \frac{a}{e\lambda} \right) \]

\[ \chi_m^{\text{inst.}} = \text{INSTANTON CONTRIBUTION} \]
\[ \chi_{m}^{\text{inst}}(s) = \exp \left\{ \frac{i q}{4 \pi} \int \frac{d s'}{s'} \frac{s' + s}{s' - s} \log \left( 1 - \chi_e(s')^2 \right) \right\} \]

\[ \int \chi_{(q,0)}^l = a \cdot R_+ \]

\[ \chi_e(s) \text{ EXP. SMALL} \]

\[ \chi_e(s) \text{ EXP. LARGE} \]
EMERGENCE OF THE KS TRANSFORMATION

As a function of $\xi$, $\chi_m$ is discontinuous across the BPS rays of the hypermultiplet of charge $(\pm 9, 0)$

$$\ell_\gamma := \{ \xi \mid \frac{Z_\gamma}{\xi} \in \mathbb{R}^- \}$$

Across these rays:

$$(\chi_e, \chi_m)^{cw} = K_{(0, \pm 9)}^{(\pm 9 \mp 9)} (\chi_e, \chi_m)^{ccw}$$

$$= (\chi_e, \chi_m (1 - \chi_e^{\mp 9})^{\mp 9})^{cw}$$
KEY FEATURES OF $X_\gamma$:

1. $X_\gamma$ ARE HOLomorphic ON $\mathbb{C}$

2. $X_\gamma \cdot X_{\gamma'} = X_{\gamma + \gamma'}$

3. $X_\gamma(\bar{s}) = \overline{X_{\bar{\gamma}}(-\frac{1}{2}\bar{s})}$

4. $X_\gamma \sim X_{\gamma}^{s.f.}$ FOR $R \to \infty$

5. $\lim_{S \to 0} X_\gamma \exp\left(-\frac{\pi R^2}{S} Z_\gamma(u)\right)$ FINITE

6. $X_{\gamma'}(S)$ TRANSFORMS

BY $K^\Omega_{\gamma'}(\gamma;u)$ ACROSS THE BPS RAY $E_\gamma$. 
7. MULTI-PARTICLE CONTRIBUTIONS

To take into account all BPS particles we cannot use a low energy effective Lag., because the particles will be mutually nonlocal.

Proposal: Properties 1–6 hold for the exact functions $\chi_{\mathbf{y}}$, using all the BPS rays $\ell_{\mathbf{y}}$ with discontinuity $K_{\mathbf{y}}^{\Omega}(\mathbf{x}; u)$.

This will determine them uniquely.
Observation: $X_{\delta}$ are the solution of a Riemann-Hilbert problem.

(R-H Problem: Find a piecewise holomorphic function with prescribed singularities and asymptotics.)

Summarizing the $X_{\delta}$ by a single map $X$ (recall $X_{\delta} = X^*(X_{\delta})$)

$\Rightarrow$ A Riemann-Hilbert problem in the $\mathbb{S}$-plane for the map

$X(\mathbb{S}): \mathbb{S} \rightarrow \mathbb{T} = \Gamma^* \otimes \mathbb{C}^*$

Piecewise holomorphic in $\mathbb{S}$
RIEMANN–HILBERT PROBLEM:

1.) $X(5)$ IS DISCONTINUOUS ACROSS BPS RAYS $l_y$:

$$X^{cw} = S_y(X^{ccw})$$

[RECALL: $S_y = \prod_{l_{y'} = l_y} K_{y'}$]

2.) $X(5)$ HAS ASYMPTOTICS FOR $5 \to 0, \infty$ GIVEN BY $X^{sf}(5)$, UP TO $O(1)$ CORRECTIONS

$$Y := (X^{sf})^{-1} X : \mathcal{G} \to \mathcal{G}$$

i.e.

$$Y_0 = \lim_{5 \to 0} Y(5) \quad \exists \quad Y_\infty = \lim_{5 \to \infty} Y(5)$$

EXIST
Solution:

\[ \chi_\gamma(5) = \chi^{sf}_\gamma(5) \cdot \exp \left\{ -\frac{i}{4\pi} \sum_{\gamma' \in \Gamma} \Omega(\gamma'; u) \langle \gamma, \gamma' \rangle \right\} \]

\[ \cdot \int \frac{d5'}{l_{\gamma'}} \frac{5'+5}{5'-5} \log \left[ 1 - \chi_\gamma(5') \right] \]

Iterating this equation (as a sum over trees...) gives the full instanton expansion!

⇒ Explicit construction of twistor coord's
• **WE RECONSTRUCT THE METRIC FROM**

\[ \omega = \frac{1}{4\pi^2 R} \chi^*(\omega^*) \]

• **AS u CROSSES A WALL OF MS BPS RAYS PILE UP**

\[ \Sigma \]

**BUT THE JUMP OF \( \chi \) IN THE RH PROBLEM IS CONTINUOUS**

**AS A FUNCTION OF u**

**THAT IS THE KS FORMULA**
Thus: The KS formula guarantees the continuity of the HK. metric across walls of MS.!

The resulting metric passes a number of consistency tests.

But... why is our proposal the right one?

Why is the metric the right one for the physical problem?
RH IS EQUIVALENT TO A DIFF. EQ.: 

\[ A_5 = x^{-1} J_{d_5} x \]

IS CONTINUOUS IN S-PLANE:

ACROSS \( \gamma \)

\[ x^{-1} J_{d_5} x \rightarrow (sx)^{-1} J_{d_5} (sx) \]

\[ = x^{-1} J_{d_5} x \]

\[ \Rightarrow A_5 \text{ IS HOLOMORPHIC FOR } s \in \mathbb{C}^* \]
\[ \Rightarrow S S_2 S = A_5 \Rightarrow \]

**STRUCTURE GROUP: SYMPL(T)**

**ASYMPTOTICS** \[ \Rightarrow \]

\[ A_5 = S^{-1} A_5^{(-1)} + A_5^{(0)} + S A_5^{(+1)} \]

**SINCE S_2 IS INDPT. OF R, u, \wedge \ldots**

**SAME ARGUMENT** \[ \Rightarrow X \text{ SATISFIES A} \]

**SET OF DIFFERENTIAL EQUATIONS:**
\[ \frac{\partial}{\partial u} x = A_{u} \cdot x \]
\[ \frac{\partial}{\partial \xi} x = A_{\xi} \cdot x \]
\[ \frac{\partial}{\partial \lambda} x = A_{\lambda} \cdot x \]
\[ \frac{\partial}{\partial \kappa} x = A_{\kappa} \cdot x \]
\[ \frac{\partial}{\partial R} x = A_{R} \cdot x \]
\[ \frac{\partial}{\partial S} x = A_{S} \cdot x \]

\[ A_{i} = S^{-1} A_{i}^{(-1)} + A_{i}^{(0)} + J A_{i}^{(+1)} \]

**Key Point:** These equations all follow from the physics of the 4D gauge theory!!
\[ \frac{\partial}{\partial u} \chi = A_u \cdot \chi \]\n
\[ \frac{\partial}{\partial \bar{u}} \chi = A_{\bar{u}} \cdot \chi \]\n
\[ \wedge \frac{\partial}{\partial \lambda} \chi = A_{\lambda} \cdot \chi \]\n
\[ \bar{\wedge} \frac{\partial}{\partial \bar{\lambda}} \chi = A_{\bar{\lambda}} \cdot \chi \]\n
\[ R \frac{\partial}{\partial R} \chi = A_R \cdot \chi \]\n
\[ S \frac{\partial}{\partial S} \chi = A_S \cdot \chi \]

\textit{Holomorphism on } \mathcal{M}_5 \textit{... also holomorphism... view } \wedge \textit{ as background vev of a VM.}

\textit{Anomalous scale and } R\textit{-symmetry}
THE S - DIFF. EQ. HAS AN IRREGULAR SINGULAR POINT AT $S = 0, \infty$; SOLUTIONS EXHIBIT STOKES PHENOM.

$A_s^{(-1)}$ IS CONJUGATE TO $Z \Rightarrow$

- STOKES RAYS = BPS RAYS $\ell_\gamma$

DENOTE STOKES FACTORS BY $\delta_\gamma$

REMAINING EQUATIONS:

ISOMONODROMIC DEFORMATION

$\Rightarrow$ STOKES FACTORS $\delta_\gamma$ ARE INDP'T OF $R, u, \Lambda, ...$

$\Rightarrow$ CHECK AT LARGE $R$ IN 1 - INSTANTON APPROXIMATION:

$\delta_\gamma = S_\gamma^{k_\gamma}$. 
9. TAKE-HOME SUMMARY

1. WE CONSTRUCT THE HK METRIC FOR CIRCLE-COMPACTIFICATION OF $\mathcal{N} = 2, D = 4$ FIELD THEORIES.

2. QUANTUM CORRECTIONS TO THE DIMENSIONAL REDUCTION METRIC COME FROM BPS STATES.

3. CONTINUITY OF THE QUANTUM-CORRECTED METRIC IS EQUIVALENT TO THE KS WCF.

4. USE TWISTOR TRANSFORM AND WRITE HOLOMORPHIC FUNCTIONS AS AN EXPLICIT SUM OVER BPS INSTANTONS.
10. CONCLUSION

— OTHER THINGS WE HAVE DONE —

• THERE ARE STRONG CONNECTIONS WITH THE $tt^*$ EQUATIONS OF CECOTTI & VAPA.

• THE FUNCTIONS $X_\gamma$ ARE 'T HOOFT-WILSON-MALDAENA LOOP OPERATOR VEV'S; MOREOVER THERE IS A NICE INTERPRETATION IN TERMS OF A 3D TFT [TO APPEAR]
• The moduli space \((M,g)\) is a moduli space of a Hitchin system [S. Cherkis & A. Kapustin].

For Hitchin systems with gauge group \(SU(2)\) with singularities on \(TP^1\) we have constructed the functions \(X_\phi\).

The KS transformations and \(\phi \to 0, \infty\) asymptotics emerge very naturally...
- TO DO -

• **Singularities at Superconformal Points Remain to be Understood**

• **Relations to Integrable Systems and the "Motivic W.C.F."**

• **Relation to the Work of Joyce & Bridgeland / Toledano Laredo**

• **Some of the Discussion Generalizes Nicely to SUGRA, but Puzzles Remain**

• **Might Give Explicit Formulation of Q.C.'s to Hypermultiplet Moduli Spaces.**