1. Introduction

Fluxes and branes have been playing a key role in string theory and M-theory for about 10 years now, and continue to be important for example in recent claims of moduli stabilization. As we heard from Lisa McAllister, that’s crucial to attempts to understand stringy cosmology.

Nevertheless, I feel that the mathematical formulation of these fluxes is not yet in a satisfactory state. So I continue to think about the general theory of fluxes. Let me list some of the issues which are still open and might have an impact on string cosmology and/or flux stabilization.

1. Dirac Quantization:

The formulation in $\mathbb{II}/M/hot$ is difficult. Until we understand how these views are actually compatible, we’re missing something deep and fundamental.

2. RR Action:

Has not been properly written in the literature. Work with D.F. and D.B. \(\rightarrow\) we now know it for $T(34)$.\[\]
3. Tadpole conditions: torsion corrections

Torsion flux or charge: \( N \times \text{flux} = 0 \)

Not exotic: Discrete Wilson lines, SSB,

any CY \( \pi_1 \neq 0 \rightarrow \text{Tors} K \neq 0 \).

M-theory: \( \left[ \frac{1}{2} G \times G - I_8 \right] = 0 \) \( \mathbb{Z} \) integral refined

4. Anomaly Cancellation: effects of fluxes

\( \otimes \) orientifolds:

In attempts to implement KKLT scenario people wrap D-branes on non-spin manifolds - raises many issues.

W/ D. Freed & B. Florea

5. Hamiltonian Formulation - important for string cosmology

HT Wavetracer for C-field, RR field.

\( V(x) \rightarrow C \quad \Psi(c) \in \mathcal{H} \).

long distance/weak coupling \( \Psi \sim \Theta \) - function

subtle phases from point 2.
Last year at Strings 2004 I reported on some aspects of the Hamiltonian formulation of M-theory. Today I want to address the case of type II string theory.

Summarize the result:

Consider type \( \text{II}/X \times \mathbb{R} \). Usually assumed that the superselection sectors - the flux sectors - are given

\[
\mathcal{H} = \bigoplus \mathcal{L}_x \\
\times \mathbb{C}K(X)
\]

\[
K(X) = \mathcal{K}^E(X) \quad \epsilon = 0/1, \quad \text{IIA/IIIB}
\]

D. Freed + G. Segal - Not true!

Basic observation even applies to 3+1 anti-Maxwell theory.

Real story: torsion fluxes don't commute.
Generate a Heisenberg group.
\( \mathcal{H} \)-reps of that Heisenberg group.
For me this is conceptually important because it means that our standard picture of a Dirac as a submanifold w/ v.b. and Connection needs to be modified.
Before we begin with technicalities— one general remark.
What I will say applies to a broad class of theories called "CAGTs."

Old subject of topology: Generalized Cohomology Theory,

\[ H^k(\text{pt}; G) = G \otimes \mathbb{R}_k \]

(I will write \( H^k(M) = H^k(M, \mathbb{Z}) \) unless explicitly said otherwise.)

Generalized K-theory: Drop the dimension axiom, K-theory,

To do physics we need local fields— constrained by topological considerations—

This is the new subject of "differential generalized cohomology."

Physicists should learn it

\[ \mathcal{H}(M) \quad \text{B-fields, M5 3-form, M-theory} \]

\[ \mathcal{K}^\mathcal{E}(M) \quad \text{type II RR} \]

\[ \mathcal{K}^0 \quad \text{type I, orientifolds} \]

Def: A CAGT is a field theory whose space of gauge-invariant fields

is a general cohomology theory.
2. Generalized Maxwell Theory

Maxwell theory = theory of a connection 1-form on a line bundle

For a fixed $L \rightarrow M$, the gauge invariant field space is

\[ \text{Conn}'s \rightarrow \mathcal{A}(L)/\mathcal{G} \leftarrow \text{gauge group} \]

Considering all line bundles together

\[ \{\text{gauge-invariant fields}\} = \bigcup_{c} \mathcal{A}(L, c)/\mathcal{G} \]

- We'd like to generalize to arbitrary form degree
- Above is actually an abelian group.

Right point of view: The gauge invariance in a gauge field is encoded by the holonomy function

\[ \mathbb{Z}(M) \rightarrow U(1) = \exp(2\pi i R/2) \]

\[ \Sigma \rightarrow \exp(2\pi i \int_{\Sigma} A) \]

The space of all homomorphisms $\mathbb{Z}(M) \rightarrow U(1)$ is an abelian group:

\[ H^2(M) = \bigcup_{c} \mathcal{A}(L, c)/\mathcal{G} \]
$\hat{H}^k(M) = \{ \text{Homomorphisms } \hat{X} : \mathbb{Z}(M) \to U(1) \}$

We can view this as a group of gauge inequivalent $k$-form gauge fields.

By "generalized Maxwell" I mean a field theory such that $f$ gauge inequiv. fields $\in \hat{H}^k(M)$

Sometimes I'll denote $[A] \in \hat{H}^k(M)$

So that

$\chi(\Sigma) = \exp(2\pi i \int A)$

But it's important to stress that $A$ is NOT a globally well-defined $(k-1)$-form.

Now I'll need to spend some time explaining some properties of $\hat{H}^k(M)$ - so we digress for a little math lesson.

Field strength

To a given $\chi \in \mathbb{Z}(M)$ s.t. if $\Sigma \cdot = \partial \Phi$ then

$\chi(\Sigma) = \exp(2\pi i \int F)$

as opposed to $A \cdot F$ is a globally well-defined form.
Now different $B$'s bound the same $\Sigma$

Small changes in $B$

$\Rightarrow dF = 0$  \textit{since F constitutes relevant ext.}

Large changes in $B$

$\Rightarrow F \in \mathcal{Z}_k(M)$

So $F$ has integral periods.
This is where physics often stop—but you can't. Some information is missing.

Suppose $\Sigma = \partial \mathcal{B}$ then

$$ (\exp 2\pi i \mathcal{A})^k = \exp (2\pi i F) $$

But knowing $F$ alone does not tell you how to take $k^{\text{th}}$ root. That extra info is encoded in the "torsion part" of the characteristic class

$$ \alpha(\mathcal{A}_T) \in H^2(M; \mathbb{Z}) $$

There is still more information: the topologically trivial flat fields.

There are two ways to summarize the structure of this group:

- These lie in

$$ H^2(M; \mathbb{R}/\mathbb{Z}) = \frac{\mathbb{R}^k}{\mathbb{Z}^{k-1}} $$

- Flat Wilson lines continuously tunable to zero.
This text is summariized in the two exact sequences:

\[ 0 \rightarrow H_e(M; \mathbb{R}/\mathbb{Z}) \rightarrow H^e(M) \rightarrow \Omega^e_{\mathbb{Z}}(M) \rightarrow 0 \]

\[ 0 \rightarrow \mathbb{S}^{d-1}/\mathbb{S}^{d-1} \rightarrow \mathbb{M}^e(M) \rightarrow H^e(M; \mathbb{Z}) \rightarrow 0 \]

\{ \text{topological trivial} \}

Notice

1. Flat fields need not be topologically trivial

\[ H^e_1(M; \mathbb{R}/\mathbb{Z}) = \text{compact abelian group} = \bigoplus_{\text{obj}} + C_{\text{obj}} \ldots + C_{\text{obj}} \]

Component of the identity = \[ H^e_1(M) \otimes \mathbb{R}/\mathbb{Z} \equiv \mathbb{S}^{d-1}/\mathbb{Z}^{d-1} \]

on the other hand

2. Topologically trivial fields do not just come u.s.

\[ \mathbb{S}^{d-1}/\mathbb{Z}^{d-1} \rightarrow \mathbb{S}^{d-1}/\mathbb{Z}^{d-1} \]

\[ \mathbb{S}^{d-1}/\mathbb{Z}^{d-1} \equiv \text{Im} d^+ = \text{vector space} \]
So the picture of $H^k$ is $[\cdot] 
\rightarrow H^k(M)$

Examples:

1) $H^1(M) = \{ f : M \rightarrow U(1) \}$

$a = \int f^*[d\theta] \quad F = \frac{1}{2\pi i} \ d(\log f)$

2) $\check{H}^2(M) = \{ \text{line bundles w/ connection} \}$

Finally we need a crucial fact

If $M$ is compact and oriented, then we have Poincaré duality: there is a perfect pairing

$$H^k(M) \times \check{H}^{n-k}(M) \rightarrow U(1)$$

$0 \rightarrow H^k(M, \mathbb{R}/\mathbb{Z}) \rightarrow H^k(M) \rightarrow \Omega^k_{\mathbb{Z}}(M) \rightarrow 0$

$0 \rightarrow \sum_{M, F} \bigoplus_{\mathbb{Z}} \rightarrow H^k(M) \rightarrow H^k(M) \rightarrow 0$

Pairing is $\quad \exp \left( 2\pi i \int_A \omega^\wedge F \right)$
Hamiltonian Formulation of Generalized Maxwell

Now \( M = X \times \mathbb{R} \)

\( \mathcal{A} \in \mathcal{H}^e(M) \quad \mathcal{S} = \int_{\frac{1}{2}} X^i F^i F \)

Completely straightforward \( \mathcal{H} = L^2(\mathcal{H}^e(X)) \supset \mathcal{U}(\mathcal{A}) \)

By grading by magnetic flux

\[ \mathcal{H} = \bigoplus_m \mathcal{H}_m \quad m \in \mathcal{H}^e(X) \]

because \( m \) labels the components of Config. space

There is an electro-magnetic dual formulation by

\[ [\mathcal{A}_e] \in \mathcal{H}^m(M) \quad \text{so} \]

\[ \mathcal{H} = \bigoplus_e \mathcal{H}_e \quad e \in \mathcal{H}^m(X) \]

But - can we simultaneously measure electric and magnetic flux?

\[ \mathcal{H} = \bigoplus_{e,m} \mathcal{H}_{e,m} \]

Your intuition should be "yes!"

Your intuation should be "yes!"
\[ \sum \left[ \int_{\Sigma_1} F, \int_{\Sigma_2} F \right] = \sum \left[ \int_{\Sigma_1} F, \int_{\Sigma_2} T \right] = \int_{\Sigma_1} \omega \cdot d\omega_2 = 0. \]

But these period integrals only measure flux/torsion.

To see what happens at the torsion level we need to understand the grading by electric flux in the $A$-formulation.

Diagonalizing $F = \text{Diagonalizing } T = \text{Translation eigenstate}$

We are only interested in the topological class of electric flux and therefore we define

**Defn**: A state of definite (topological) electric flux satisfies

\[ \forall \psi \in H^{n-1}(X, \mathbb{R}/2\mathbb{Z}) \quad \psi(A + \phi) = \exp(2i\pi \phi) \psi(A) \quad e \in H^{n-1}(X, \mathbb{Z}). \]
$\mathcal{H} = \bigoplus \mathcal{H}_e \quad e \in H^m(X; \mathbb{Z}) \quad \text{Diagonalizing } H^m(X; \mathbb{R}) \subset H^m(X)$

Dually:

$\mathcal{H} = \bigoplus \mathcal{H}_e \quad e \in H^m(X; \mathbb{Z}) \quad \text{Diagonalizing } H^{n-m}(X; \mathbb{R}) \subset H^{n-m}(X)$

Now, our main claim is that these translation operators don't commute.

Actually, this follows immediately since translation by a topologically nontrivial, but flat field changes the magnetic flux sector.
This actually follows immediately once translation by a topologically trivial flat field changes the magnetic flux sector.

But, to understand this more systematically, let's consider the following remark.

Let $S$ be any abelian group. Then $\mathrm{meas} = L^2(S) \subseteq L^1(S)$. If $\psi$ is a rep. of $S$, then for $s \in S$, $(L_s \psi)(s) = \psi(s + s)$. Thus for any $x \in S$, $(m_x \psi)(s) = \psi(x + s)$. Then $L_s m_x = m_{x(s)} L_s$. This is a rep. of $S \times S$.

In fact, Thm (S-vN): $\text{Heis}(S \times S)$ has a unique rep. $= L^2(S)$.

Apply this to $S = \tilde{H}^k(X)$.

Poincaré duality $\Rightarrow$ dual $\tilde{S} = \tilde{H}^{n-k}(X)$.

$\Rightarrow \tilde{H} = \text{Unique rep of } \text{Heis}(\tilde{H}^k(X) \times \tilde{H}^{n-k}(X))$.
Now recall

\[ H^{e-1}(X, \mathbb{R}/\mathbb{Z}) = (H^e(X) \otimes \mathbb{R}/\mathbb{Z}) \times H^R_T(X) \]
\[ H^{\text{m-1}}(X, \mathbb{R}/\mathbb{Z}) = (H^{\text{m-1}}(X) \otimes \mathbb{R}/\mathbb{Z}) \times H^{n-1}_T(X) \]

Therefore

\[ \mathcal{H} = \bigoplus_{\mathcal{H}_{\mathcal{E}, \mathcal{M}}} \mathcal{H}_{\mathcal{E}, \mathcal{M}} \text{ Heis}(H^e_T \times H^{n-1}_T) \]

Example: Maxwell on \( S^3/\mathbb{Z}_k \times \mathbb{R} \)

\[ H^2(S^3/\mathbb{Z}_k) = \mathbb{Z}_k \]

Heis = \( \langle P, Q \mid PQ = \omega QP \rangle \)

In a basis we have magnetic fluxes

\[
\begin{pmatrix}
\omega & 0 \\
0 & \omega^{-1}
\end{pmatrix}
\]

and electric fluxes

\[
\begin{pmatrix}
0 & 0 & \cdots \\
0 & 0 & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]
Close with 2 Remarks

(1) A closely related observation in the special case of Maxwell on Lens spaces was made by Gukov-Rangamani-Witten: Noncommuting Wilson line operators on a Lens space.

D3/ S^3/Zk cannot simultaneously measure the F1 and D1 strings.

Present discussion places that remark in a wider context.

(2) Notice that our formulation of the Hilbert space is manifestly electric-magnetic dual.

This gives a very crisp formulation of a self-dual field:

\[ \dim X = 2p-1 \quad \left[ A \right] \in \check{H}^p (\mathcal{M}) \quad \text{self-dual} \]

\[ \mathcal{H} := \text{Heis} \left( \check{H}^p (X) \right) \]
41 RR Fields

Now let's apply these ideas to type II strings.

First we have \( [\mathcal{B}] \in \mathcal{K}^3_{8}(M_{10}) \)

RR fields \( [\mathcal{C}] \in \mathcal{K}^8_{8}(M_{10}) \)

We have a picture of differential K-theory similar to what we had before.

Characteristic class: \( x \in \mathcal{K}^8_{8}(M) \)

Field strength: \( \widetilde{F} = F_0 + \alpha F_2 + \cdots + (-1)^{p-2} F_{p-2} \)

\( d_+ \widetilde{F} = 0 \quad d_+ = d - H \)

Let \( R = TR[u, u^*] \) \( \deg u = 2 \).

\( \Omega^k_{d_+}(M; R) = \{ \text{closed forms valued in } R \} \)

Chern character \( ch_R : \mathcal{K}^8_{8}(M) \to \mathcal{H}^8_{d_+}(M) \)

Quantization of periods: \( [\tilde{F}]_{d_+} = ch_R(x) \sqrt{A} \)

defines \( \mathcal{K}^8_{d_+}(M; R) \)
The two exact sequences from before have precise analogs: (take IIA for definiteness)

\[
0 \to K^{-1}_B(M; \mathbb{R}/2) \to K^0_B(M) \to \mathbb{Z}_{\text{dilat}} \to 0
\]

\[
0 \to \mathbb{Z}(M)/\mathbb{Z}(M) \to K^0_B(M) \to K^0_B(M) \to 0
\]

But now there is a new ingredient: the RR field is self-dual, both in IIA and in IIB.

\[\Rightarrow\] Subtleties in formulating the action— but recent progress

\[\Rightarrow\] Hilbert space should be formulated along the above lines

Then let \( X \) be odd-dim., (cpt, \( K \)-oriented)

Then there is a perfect pairing

\[
K^0_B(X) \times K^0_B(X) \to \mathbb{R}/\mathbb{Z}
\]

Therefore \( \text{Heis}(K^0_B(X)) \) has a unique irrep, and we define this to be the Hilbert space \( \mathcal{H}_K \).
Now we can return to our main result mentioned at the beginning.

Is there a grading \( \mathcal{KE} = \bigoplus \mathcal{KE}_x \) ?

\( x \in K^G_8(X) \)

\[ \text{No! A } k^- \text{-theory class encodes both electric and magnetic flux, and these do not commute.} \]

More mathematically: such a grading would be diagonalizable translation by the flat fields \( k^G_8(X, \mathbb{R}/2) \)

But the Heisenberg pairing is nontrivial on these.

What we do have

\[ \mathcal{K}_{GR} = \bigoplus \mathcal{K}_{GR, x} \quad x \in K^G_8/\text{tors.} \]

Conclusion: Quantum states of the RR field cannot be in a definite \( k^- \)-theory class!
1. Introduction.

A. Dirac Quantization \[ \mathcal{II}/\mathcal{M} \] not different!


C. Torsion conditions: Torsion corrections?

\[ \text{any CY w/ } \pi_1 \neq 0 \rightarrow \text{TorsK} \neq 0. \]

\[ \left[ \frac{1}{2} G \wedge G - I_8 \right] = 0 \quad \exists \text{ integral refinement} \]

D. Anomaly cancellation: fluxes, orientifolds.

E. Hamiltonian Formulation - stringy cosmology

\[ Y \xrightarrow{X} C \quad \Psi(C) \in \mathcal{F}. \]

long distance \( \mathbb{H} \mathbb{K} \) coupling \( \Psi \sim \mathbb{H} \mathbb{K} \).

Summary: Type II/ \( X \times \mathbb{R} \)

\[ \mathcal{F} \neq \mathbb{H} \mathbb{K} \]

\( \mathbb{H} \mathbb{K} \times \mathbb{K}(X) \)

\[ H^2(M) = \bigcup_{\text{all} \ Y} \frac{1}{\text{gcd}(g, y)} \ (\text{cyclic}) \]

* this is an abelian group

Generalize to higher degree.

Differential G.C.T.

Applying to G.A.T.'s Generalized covariant theory: drops dimension

\[ H^0(G) \rightarrow \text{axioms} \rightarrow \text{dimension} \]

\[ H^0(G) = 0 \]

\[ H^0(G) = 0 \]

Closed 3-forms don't commute.

\[ \delta^2 = 0 \]

\[ 0 = \delta^2 \]

\[ k(x) = k(g)(x) \]

\[ e = 0 \]

\[ \frac{1}{2a} I \frac{1}{2b} \]
gauge invariance $\iff$ holonomy function

$\Sigma \rightarrow \exp\left(2\pi i \frac{\mathfrak{g}A}{\Sigma}\right)$

$\mathcal{Z}_1(M) \rightarrow U(1) = \exp(2\pi i \mathbb{R}/\mathbb{Z})$

**Def.**

$\mathcal{H}^l(M) = \left\{ \text{Hom's } \chi : \mathcal{Z}_l(M) \rightarrow U(1) \right\}$.

$= \left\{ \text{gauge inequiv fields for } \mathbb{I}-\text{form potential} \right\}$

"Generalized Maxwell"$

\Sigma$

**Properties.**

For fixed $\chi \equiv F \in \mathcal{E}^l(M)$ s.t.

$\chi(\Sigma) = \exp\left(2\pi i \int_\Sigma F_\chi\right)$

$\rightarrow dF = 0 \quad \rightarrow [F] \in \mathcal{H}^l_{\mathbb{R}}(M) \cong \mathcal{H}^l_{\mathbb{R};\mathbb{Z}}$

$F_{\mathbb{R}} \in \mathcal{E}_\mathbb{R}^l$

$k \Sigma = \partial B$

\[
\chi(\Sigma) = \left(\exp 2\pi i \frac{\mathfrak{g}A}{\Sigma}\right)^\chi = \exp 2\pi i \int_\Sigma F_

\text{Characteristic class } \chi^\tau a_\chi \in \mathcal{H}^l(M;\mathbb{Z})
\[
\begin{align*}
\Lambda &\xrightarrow[\Lambda]{} F \\
\hat{H}^l(M) &\xrightarrow{} \Omega^l_Z(M) \\
\alpha &\in H^l(M;\mathbb{Z}) \quad \Rightarrow \quad H^l(M;\mathbb{R}) \cong H^l_{DR}(M) [\mathcal{F}] \\
\bar{\alpha} &= [\mathcal{F}]
\end{align*}
\]

\[
\begin{align*}
0 \rightarrow &H^l(M;\mathbb{R}/\mathbb{Z}) \xrightarrow{} \hat{H}^l(M) \xrightarrow{} \Omega^l_Z(M) \rightarrow 0 \\
0 \rightarrow &\mathbb{Z} / \Omega^l_Z \rightarrow \hat{H}^l(M) \xrightarrow{} H^l(M,\mathbb{Z}) \rightarrow 0
\end{align*}
\]

1.) Flat need not be top. trivial

\[
H^{l-1}(M;\mathbb{R}/\mathbb{Z}) = \Omega^{l-1} + \bigoplus (\cdots)
\]

\[
H^l(M;\mathbb{R}/\mathbb{Z}) = \text{free} / \Omega^l Z
\]

2.) Top. trivial fields not just a vs.
\[ A \in \mathbb{S}^{l-1} \quad \chi_A(\xi) = \exp \left( 2\pi i f_A \right) \]

\[ A \rightarrow A + \omega. \]

\[ \mathcal{H}^l / \mathcal{H}_2^{l-1} = \mathbb{S}^{l-1} \rightarrow \mathbb{S}^{l-1} / \mathbb{S}_2^{l-1} \]

\[ \mathbb{S}^{l-1} / \mathbb{S}_2^{l-1} = \text{Ind}^+ = \text{vector space} \]

\[ \{0\} \quad \{0\} \quad \{0\} \]

\[ \text{Explos:} \quad 1) \quad \mathcal{H}^l(M) = \{ f : M \rightarrow U(1) \} \]

\[ a = f^* \{ \theta \} \quad F = \frac{i}{2\pi} \, d(\log f) \]

\[ 2) \quad \mathcal{H}^2(M) \]

Poincaré duality If \( M \) cpt, oriented \( \dim M = n \)

\[ \mathcal{H}^l(M) \times \mathcal{H}^{n+1-l}(M) \rightarrow U(1) \]

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Hamiltonian Formalism

\[ M = X \times \mathbb{R} \]

\[ [\tilde{A}] \in H^0(M) \]

\[ S = \int \frac{1}{2} \psi^* F \star F \]

\[ H = L^2(\mathcal{H}(X)) \psi(X) \]

\[ \mathcal{H} = \bigoplus_m H^e_m \quad m \in H^0(X, \mathbb{Z}) \quad H^e(X, \mathbb{Z}) \]

\[ = \bigoplus_e H^e_e \quad e \in H^{n-1}(X, \mathbb{Z}) \]

\[ H^e = \bigoplus_{e, m} H^e_{e, m} \]

\[ \left[ \begin{array}{c} F \\ \Sigma_1 \end{array} \right] , \left[ \begin{array}{c} \Sigma_2 \\ \Sigma_2 \end{array} \right] \right] = \left[ \begin{array}{c} F \\ \Sigma_1 \end{array} \right] , \left[ \begin{array}{c} \Sigma_2 \\ \Sigma_2 \end{array} \right] \right] \]

\[ = \left[ \begin{array}{c} \omega_1 F \\ \omega_2 \Sigma_1 \end{array} \right] , \left[ \begin{array}{c} \omega_2 \Sigma_2 \\ \omega_2 \Sigma_2 \end{array} \right] \right] \]

\[ = \int_X \omega_1 d\omega_2 = 0. \]

**Def.:** A state of definite (topological) electric flux

\[ \forall \phi \in H^{n-1}(X, \mathbb{R}/\mathbb{Z}) \quad \psi(\tilde{A} + \phi) = \exp(2\pi i \int_X \phi \mu) \psi(\tilde{A}) \]

\[ e \in H^{n-1}(X, \mathbb{Z}) \]
$S$ - any abelian group w/ measure $\mathcal{H} = L^2(S)$

rep. of $S$ $s_o \in S$ $(L_{s_o} \chi) (s) = \chi (s + s_o)$

rep. of $\tilde{S}$ $\chi \in \tilde{S}$ $(m_{\tilde{S}} \chi) (s) = \chi (s) \tilde{4}(s)$

Not rep. of $S \times \tilde{S}$ $L_{s_o} m_{\tilde{S}} = m_{\tilde{S}} L_{s_o}$

$1 \rightarrow U(1) \rightarrow \text{Heis}(S \times \tilde{S}) \rightarrow S \times \tilde{S} \rightarrow 1$

Then $(S \times \tilde{N})$ $\text{Heis}(S \times \tilde{S})$ has unique irrep $\tilde{4}(S)$

$S = \check{H}^\ell(X)$ Poincaré dual $\Rightarrow$ Poincaré in dual group $\Rightarrow \check{S} = \check{H}^{m-\ell}(X)$

$\mathcal{H} =$ Unique irrep of $\text{Heis}(\check{H}^\ell(X) \times \check{H}^{m-\ell}(X))$

$H^{\ell-1}(X, R/2) = (\check{H}^{\ell}(X) \otimes R/2) \times H^{\ell}_T \leftarrow$

$H^{m-\ell-1}$ $\Rightarrow$ (trivial pairing)

$\mathcal{H} = \bigoplus_{\ell, m} \check{H}_{\ell, m} \hookrightarrow \text{Heis}(H^\ell_T \times H^{m-\ell}_T)$

perfect pairing
Explicit: $S^3/Z_k \times \mathbb{R}$

$$H^2(S^3/Z_k) = \mathbb{Z}_k$$

Heis = $\langle P, Q | PQ = \omegaQP \rangle$

\[
\begin{pmatrix}
\omega \\
\vdots \\
\omega_k
\end{pmatrix}
\]

\[
\begin{pmatrix}
\mathfrak{p} \\
\vdots \\
\mathfrak{p}
\end{pmatrix}
\]

(1.) Gukov-Rajamani-Witten '98
$AdS_5 \times S^5/Z_N$
D3/ $S^3/Z_k$
F1 + D1 string

(2.) $dim X = 2p - 1$ $[\mathfrak{A}] \in H^p(M)$
$$\mathcal{H}_{SD} = \text{Heis}(H^p(X))$$

4/1 RR fields type II
$[\tilde{B}] \in H^2(M_{10})$

$[\tilde{\chi}] \in k_8^E(M_{10})$

Chowenke class $x \in k_8^E(M)$

Fieldstrength $\widetilde{F} = \widetilde{F}_0 + d\mathfrak{p}\tilde{F} + \cdots + C_{\mathfrak{p}_7} \tilde{F}
$

$d_H \tilde{F} = 0$
$d_H^2 = d - H.$
\[ R = \{ [u, u^*] \} \quad \text{deg} u = 2. \]

\[ \mathcal{S}_{d_H}^k(M; R) = \{ d_H \text{-closed forms on } R \} \quad \text{deg} = k \]

Direct quant. \[ \tilde{c}_B : \mathcal{K}_B^e(M) \to \mathcal{H}_{d_H}^e(M) \]

Field strength \[ F \in \mathcal{S}_{d_H2}^0(M; R) \]

Direct quant. \[ c_B : \mathcal{K}_B^e(M) \to \mathcal{H}_{d_H}^e(M) \]
\[ \tilde{F} = \int_{\partial_0} = c_B(x) \sqrt{\text{det} \theta} \]

Let

\[ 0 \to \mathcal{K}_B^{-1}(M; R/\mathbb{Z}) \to \mathcal{K}_B^0(M) \to \mathcal{S}_{d_H2}^0(M; R) \to 0 \]

\[ 0 \to \mathcal{S}_{d_H2}^0(M) \to \mathcal{K}_B^0(M) \to \mathcal{S}_{d_H2}^0(M) \to 0 \]
\[ C = c_1 + c_2 + c_3 + \cdots \]

Selfdual : \[ \to \text{action} \ldots \]
\[ \to \text{Hilbert space.} \]
Since $X$ is odd-dimensional (cpt, $K$-oriented), there exists a perfect pairing

$$
\mathcal{K}_B^e(X) \times \mathcal{K}_B^e(X) \to \mathbb{R}/2
$$

implies that $\text{Heis}(\mathcal{K}_B^e(X))$ has a non-trivial representation.

$$
\mathcal{H}_{RR} = \bigoplus_{x} \mathcal{H}_{RR,x} \cong \text{Heis}(\text{Tors} \mathcal{K}_B^e(X))
$$

$\mathcal{K}_B^e(X; \mathbb{R}/2)$ has non-trivial pairing.

Conclusion: Quantum states of the RR field cannot be in a definite K-Theory class!

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