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Some Suggestions for the Term Paper for Physics 618: Applied Group Theory

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Abstract: Here are some ideas for topics for the term paper.
The paper is due WEDNESDAY MAY 11, 2022
If you do not get the paper to me by that date you will probably get an "Incomplete" in your grade, and it is a slow and difficult process to change it.
Because of the pandemic you must hand it in by emailing it to gmoore@physics.rutgers.edu. I strongly encourage you to send a pdf file. If I cannot open your file I will consider it as not handed in.
The topics range from easy to challenging.
For each topic I give some indication of a point of entry into the literature on the subject. These are not meant to be reference lists and part of the project is to do literature searches to see what is known about the subject.
This is an opportunity for you to explore some topic you find interesting. You can write a short ($\geq 10pp.$) expository account of what you have read, or go for a more in depth investigation in some direction. You do not need to do original research, although if you can do something original that would be great.
Feel free to choose your own topic (if it is not on this list please discuss it with me first). The only boundary condition is that the topic should have something to do with applications of group theory to physics. Version March 28, 2022.

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1. Wigner's Theorem

Wigner's theorem is the basic fact that leads to the application of group theory to quantum mechanics. Review the statement and various proofs of the theorem:

1. S. Weinberg, Quantum Field Theory, Vol. 1, Sec. 1
2. R. Simon, N. Mukunda, S. Chaturvedi, V. Srinivasan, "Two elementary proofs of the Wigner theorem on symmetry in quantum mechanics," arXiv:0808.0779 [quant-ph]
3. Freed, Daniel S.: *On Wigner's theorem.* arXiv:1112.2133

Find other proofs of the theorem.

2. The Spectral Theorem

Explain the statement and proof of the spectral theorem in functional analysis, and why it is necessary to define projection-valued measures in the general case.

This is standard textbook material. See, for example:
1. Reed and Simon, *Functional Analysis*, vol. 1
2. L. A. Takhtadjan, *Quantum Mechanics for Mathematicians*, Springer GTM 95
3. N. Weaver, *Mathematical Quantization*

Explain some uses of the spectral theorem in quantum mechanics.

3. C^*-algebras And Quantum Mechanics

Some approaches to quantum mechanics emphasize the theory of operator algebras. Explain what C^*-algebras are, and how one can formulate the Dirac-von Neumann axioms of quantum mechanics using C^*-algebras.

You could start with

N. P. Landsman, *Mathematical Topics Between Classical and Quantum Mechanics*

4. Jordan Algebras

P. Jordan wanted to reformulate quantum mechanics in terms of a certain kind of non-associative algebra now known as an *Jordan algebra*.

Explain the historical origin of Jordan algebras.

Explore the relation of Jordan algebras to exceptional groups.

5. Stabilizer Codes In Quantum Information Theory

One technique for constructing "quantum error correcting codes" makes heavy use of group theory. Explain how:

1. J. Preskill, Online notes on Quantum Information Theory, chapter 7.
2. Nielsen and Chuang, *Quantum Computation and Quantum Information*, chapter 10
3. A. Yu. Kitaev, A. H. Shen, M. N. Vyalyi, *Classical and Quantum Computation* (Graduate Studies in Mathematics v. 47), chapter 13.

6. Crystallographic groups and Bravais lattices

This is standard textbook material:

Explain the classification of symmetries of lattices in 3 dimensions.

Describe the work of Frankenheim and Bravais, and list the 14 Bravais lattices.

Illustrate the use of this classification in solid state physics.

Give an overview of the classification of the 230 3d crystals (and explain why there are 219 isomorphism classes).

Explain the "international notation" for point and space groups.

Discuss the 3d magnetic space groups (a.k.a. Shubnikov groups) and explain why there are 1561 types.

Refs.

1. Miller, *Symmetry Groups and Their Applications*
2. S. Sternberg, *Group Theory and Physics*

7. Lattice symmetries for $d > 3$

Describe symmetries of lattices in various dimensions.

Discuss Hilbert's 18th problem and Bieberbach's theorem.

8. Quasicrystals

Explain the possible lattice symmetries in three dimensions.

What can you say about higher dimensions?

One nice technique involves projecting lattices from higher dimensions to three dimensions.

1. Works of Penrose
2. Works of Steinhardt
3. *Introduction to the mathematics of Quasicrystals*, M.V. Jarić.

9. Occurances of icosahedral symmetry in nature

This is a broad topic.

Start by looking up articles on this topic in *Physics Today*, *American Journal of Physics*, *Reviews of Modern Physics*.

Explore the subject of local icosahedral symmetry. The icosahedron gives a good close packing and hence is preferred by some molecules. Yet it cannot be an exact symmetry of a lattice. See papers of S. Sachdev and D.R. Nelson.

One major application is in carbon chemistry – the chemistry of fullerenes.

Some references:

8. Dresselhaus, et. al. *Science of Fullerenes and Carbon Nanotubes*

9. Kroto, Heath, et. al. Nature **318**(1985) 354

10. Fowler and Manolopoulos, *An Atlas of Fullerenes*

There are some remarkable occurances of such structures in solitons in nonlinear sigma models in $3 + 1$ dimensions. See, for example,

11. Battye and Sutcliffe, "Solitonic fullerene structures in light atomic nuclei," hep-th/0012215.

There are also applications to biology in the structure of viruses.

For a nice description of the relation to groups over finite fields see Conway above and

12. B. Kostant, "Structure of the truncated icosahedron (such as fullerene or viral coatings) and a 60-element conjugacy class in $PSL(2, 11)$," Proc. Natl. Acad. Sci. USA, Vol. 91, pp. 11714-11717, 1994.

10. Use of Root Lattices in Condensed Matter Theory

Review the paper "Flux Hamiltonians, Lie Algebras and Root Lattices With Minuscule Decorations," on the Cond-Mat arxiv: 0802.3466.

11. Classification of defects in condensed matter physics using homotopy

This is a classic subject where point,line,surface etc. defects in condensed matter physics can be classified using ideas of topology, namely the fundamental groups. A good reference to start with is

Nakahara - textbook

D. Mermin, The topological theory of defects in ordered Media, Rev. Mod. Phys. **51**(1979)591.

12. The Braid group and anyons

Mathematics of knots and the braid group.

Braid group representations.

Representations of mapping class groups.

Physical realization of braid groups representations in the context of fractional statistics in 2+1 dimensions.

Explore relations to quantum groups.

Ref:

1. Reprint volume edited by Wilczek and Shapere.
2. Book on quantum groups and quantum invariants by Turaev.

13. Nonabelions

Write a thorough review the current knowledge about the "Pfaffian state" in the fractional quantum Hall effect. (Include the experimental status of the description of the state at filling fraction 5/2.)

Here are some resources to get you started:

1. Go to www.kitp.ucsb.edu and follow the links for the online talks for the workshop on topological quantum computation. There are many pedagogical lectures at this site.

2. Physics Today, Search and Discovery, October 2005

3. For background material on the quantum Hall effect see the reprint collection edited by M. Stone, or the Les Houches lectures by J. Frohlich in 1994 LH lectures, edited by David, Ginsparg, and Zinn-Justin.

4. Nick Read wrote a review in cond-mat/0011338

5. Focus on the relation to the Landau-Ginzburg effective field theory description using Chern-Simons theory. You should look up papers with coauthors Chetan Nayak + Mike Freedman in the last few years. There is also a good review by Tony Zee: cond-mat/9501022, but it only covers the abelian case.

6. You can also investigate the "Read-Rezayi" or parafermion states and/or new applications to rotating Bose-Einstein fluids. See papers on cond-mat by Wilkin and Gunn and cond-mat/0507064.

7. Das Sarma, Freedman, Nayak et. al. "Non-abelian Anyons and Topological Quantum Computation," Review article.

8. A. Kitaev, et. al. *Classical and Quantum Computation*

14. K-theory and topological insulators

Clifford algebras are at the heart of K-theory. Explain how classification of topological insulators is related to K-theory.

References:

1. Alexei Kitaev, Periodic table for topological insulators and superconductors, arXiv:0901.2686

2. arXiv:0803.2786 Title: Classification of topological insulators and superconductors in three spatial dimensions Authors: Andreas P. Schnyder, Shinsei Ryu, Akira Furusaki, Andreas W. W. Ludwig

3. Hasan and Kane, Rev Mod Phys 82, 3045 (2010).

4. Joel Moore, Nature 464, 194 (2010).

5. Qi and Zhang,Physics Today 63, 33 (2010).

6. Ryu, Schnyder, Furusaki and Ludwig, New J Phys 12, 065010 (2010).

7. Schnyder, Ryu, Furusaki and Ludwig, (Landau Memorial Conf.;arXiv:0905.2029), AIP Conf. Proc. 1134, 10 (2009).

8. Qi and Zhang, Rev. Mod. Phys. 83, 1057 (2011)

9. Hasan and Moore, Annual Reviews of Condensed Matter 2, 55 (2011).

10. D. Freed and G. Moore, Twisted Equivariant Matter, arXiv:1208.5055 [hep-th]

11. My own lecture notes at

    http://www.physics.rutgers.edu/∼gmoore/695Fall2013/CHAPTER1-QUANTUMSYMMETRY-OCT5.pdf

    http://www.physics.rutgers.edu/∼gmoore/PiTP-LecturesA.pdf

15. Crystalline Topological Insulators

1. D. Freed and G. Moore, Twisted Equivariant Matter, arXiv:1208.5055 [hep-th]

2. My own lecture notes at

   http://www.physics.rutgers.edu/∼gmoore/695Fall2013/CHAPTER1-QUANTUMSYMMETRY-OCT5.pdf

   http://www.physics.rutgers.edu/∼gmoore/PiTP-LecturesA.pdf

3. N. Okuma, M. Sato and K. Shiozaki, "Topological classification under nonmagnetic and magnetic point group symmetry: application of real-space Atiyah-Hirzebruch spectral sequence to higher-order topology," Phys. Rev. B **99**, no. 8, 085127 (2019) doi:10.1103/PhysRevB.99.085127 [arXiv:1810.12601 [cond-mat.mes-hall]].

4. Luuk Stehouwer, Jan de Boer, Jorrit Kruthoff, Hessel Posthuma, "Classification of crystalline topological insulators through K-theory," arXiv:1811.02592 [cond-mat.mes-hall]

5. Papers listing the thousands of possible crystalline topological insulators:

   NATURE NEWS 08 AUGUST 2018 Thousands of exotic topological materials discovered through sweeping search

   Vishwanath et. al. arXiv.org:1707.01903, arXiv.org:1807.09744

   J. Cano et. al. arXiv.org:1709.01935

   Bernivig, et. al. arXiv.org:1807.10271

16. Symmetry Protected Phases Of Matter

In general, phases of matter are related to phases associated with gapped Hamiltonians that cannot be continuously connected to each other. When one imposes a symmetry on the system, and only allows gapped Hamiltonians consistent with the symmetry then the classification changes in an interesting way.

A highly mathematical formulation of the classification, making use of "equivariant" topological field theory can be found in:

D. S. Freed and M. J. Hopkins, "Reflection positivity and invertible topological phases," arXiv:1604.06527 [hep-th].

See references therein for the physics literature.

17. Lie groups and random matrix theory

Explain the role of Lie groups and especially their Haar measures in the theory of statistical ensembles of matrices.

1. M.L. Mehta, *Random Matrices*, Elsevier, 2004

2. M.R. Zirnbauer, "Riemannian symmetric superspaces and their origin in random matrix theory," arXiv:math-ph/9808012

3. F. Mezzadri, "How to generate random matrices from the classical compact groups," Notices of the AMS, p.592, vol. 54, 2007

4. Series of 6 papers by F. Dyson, early 60's.

Random matrix ensembles have had a far-reaching impact in many areas of science. They were invented in the context of nuclear physics by Wigner and Dyson. They have been useful in describing electron transport in disordered media, in formulating models of quantum gravity, and in discussing the Riemann hypothesis and various generalizations thereof.

18. Altland-Zirnbauer classification of free fermion phases

The famous 3-fold way of Dyson was expanded to a 10-fold way by Altland-Zirnbauer and Heinzner-Huckleberry-Zirnbauer.

Review this beautiful bit of mathematical physics:

1. Dyson, Freeman J.: The threefold way. Algebraic structure of symmetry groups and ensembles in quantum mechanics. J. Mathematical Phys. **3** , 1199-1215 (1962).

2. Altland, Alexander, and Zirnbauer, Martin R.: Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures. Phys. Rev. B **55** , 1142-1161 (1997).

3. Heinzner, P., Huckleberry, A., and Zirnbauer, M.R.: Symmetry Classes of Disordered Fermions. Comm. in Math. Phys. **257** no. 3, 725-771 (2005),

4. M. Zirnbauer, "Symmetry Classes," in *The Oxford Handbook of Random Matrix Theory*

5. D. Freed and G. Moore, Twisted Equivariant Matter, arXiv:1208.5055 [hep-th]

6. G. Moore lecture notes:

   http://www.physics.rutgers.edu/∼gmoore/695Fall2013/CHAPTER1-QUANTUMSYMMETRY-OCT5.pdf

   http://www.physics.rutgers.edu/∼gmoore/QuantumSymmetryBook.pdf

19. Use of Lie group symmetries in nuclear structure theory

Start with

F. Iachello and A. Arima, *The Interacting Boson Model*

20. Quark models

Explain the Gell-Mann Okubo formula.

References to begin with: 1. H. Georgi, *Lie Algebras in Particle Physics*

2. S. Coleman, *Aspects of Symmetry*, ch. 1

3. Gibson and Pollard, *Symmetry principles in elementary particle physics*

4. S. Sternberg, *Group Theory and Physics*

5. J.J.J. Kokkedee, *The quark model*, W.A. Benjamin, 1969

6. M. Gourdin, *Unitary Symmetries and Their applications to high energy physics*, North Holland 1967.

7. Hagiwara, K., Phys. Rev. D66 010001 (2002), http://pdg.lbl.gov

8. H. Georgi, Weak Interactions, http://www.people.fas.harvard.edu/ hgeorgi/ weak.pdf, 2008.

9. S. Weinberg, The Quantum Theory of Fields II, Cambridge University Press, 2005.

Some further topics to consider reporting on:

A. Explain the relation of $SU(6)$ to the quark model.

B. Discuss proposals for exotic bound states of quarks, and how they are constrained by the quark model.

1. C.A. Meyer, "Light and exotic mesons", http://www.phys.cmu.edu/halld

2. R. Jaffe and F. Wilczek, Diquarks and Exotic Spectroscopy, http://arxiv.org/abs/hep-ph/0307341, 2003; R. Jaffe and F. Wilczek, "A Perspective on Pentaquarks," hep-ph/0401034; F. Wilczek, Diquarks as Inspiration and as Objects, http://arxiv.org/abs/hep-ph/0409168, 2004.

3. Find other references on the pentaquark and give a critical evaluation.

C. Heavy Quark Effective Field Theory:

1. A. Manohar and M. Wise, Heavy Quark Physics, Cambridge University Press, 2000

2. C. Davies, S.M. Playfer, Heavy Flavour Physics, The Scottish Universities Summer School in Physics, 2002

21. Solitons and Topological Stability

1. R. Rajaraman. Solitons and instantons : an introduction to solitons and instantons in quantum Field theory, Elsevier Science Pub. 1982.

2. Sidney Coleman. "Classical Lumps and their Quantum Descendents," in *Aspects of Symmetry* Cambridge University Press, February 1988.

3. S.C. Coleman, "The Magnetic Monopole 50 Years Later,"

4. A. M. Polyakov. Particle spectrum in quantum Field theory. Soviet Journal of Experimental and The- oretical Physics Letters, 20:194, September 1974.

5. G. 't Hooft. Magnetic monopoles in unified gauge theories. Nuclear Physics B, 79:276284, September 1974.

22. Skyrmions

In the 4-dimensional nonlinear sigma model with WZ term the "skyrmions" or solitons in the theory can be fermions.

Explain this phenomenon and discuss the applications to models of baryons in the nonlinear sigma model.

1. T.H.R. Skyrme, Proc. Roy. Soc. **A260** (1961) 127

2. E. Witten, Nucl. Phys. B223 (1983) 433

2. G. Adkins, C. Nappi, and E. Witten, Nucl. Phys. **B228** (1983) 552.

3. L.J.R. Aitchison, "Berry phases, magnetic monopoles and Wess-Zumino terms or how the skyrmion got its spin," Acta. Phys. Polonica, **B18**(1987) 207.

4. Series of papers by M. Mattis.

5. S. Coleman, "Classical Lumps and their Quantum Descendants."

23. Dynamical symmetries and scattering theory in nuclear physics

27) POTENTIAL SCATTERING, TRANSFER MATRIX, AND GROUP THEORY. By Y. Alhassid, F. Gursey, F. Iachello, Published in Phys.Rev.Lett.50:873-876,1983

24. Free fermions, bosonization, and representation theory

1. Pressley and Segal, *Loop Groups*, Chapter 10.

2. M. Stone, "Schur functions, chiral bosons, and the quantum-Hall-effect edge states," Phys. Reb. **B42** 1990)8399

3. S. Cordes, G. Moore, and S. Ramgoolam, "Lectures on 2D Yang-Mills Theory, Equivariant Cohomology and Topological Field Theories," Nucl. Phys. B (Proc. Suppl 41) (1995) 184, section 4. Also available at hep-th/9411210.

4. M. Douglas, "Conformal field theory techniques for large N group theory," hep-th/9303159.

5. M. Douglas, "Chern-Simons-Witten Theory as a topological Fermi liquid," hep-th/9403119

Look up the literature on limiting Young diagram shapes:

6. S.Kerov and A.Vershik

7. Logan and Shepp

25. Standard Model

Explain what nonabelian gauge theories are and describe the standard model of the strong, weak, and electromagnetic interactions based on $SU(3) \times SU(2) \times U(1)$ gauge theory.

This is standard textbook material. You can start with Weinberg's 3 volume work on Quantum Field Theory, or with Peskin and Schroeder.

26. Flavor symmetries

1. Explain what flavor symmetries are in the standard model.

2. Explain the role of flavor changing neutral currents in model building, and the GIM mechanism.

3. Discuss the Froggatt-Nielsen mechanism.

27. Noncommutative Geometry and the Standard Model

Explain Connes' formulation of the standard model using noncommutative geometry, and give a critique of the viability of this formulation.

Refs:

1. Alain Connes, *Noncommutative Geometry* Academic Press
2. Recent e-print arXiv papers of Connes, Chamseddine, Marcolli.

28. Grand Unified Model Building

Explain what is a "Grand Unified Theory" and discuss the examples of $SU(5), SO(10), E_6$ as unified gauge groups.

Explain clearly the group theory of the embeddings, and symmetry breaking in these examples.

Discuss the successes and difficulties of the theories.

1. Group Theory for Unified Model Building. R. Slansky, Phys.Rept.79:1-128,1981.
2. Langacker, Physics Reports
3. G. Ross, *Grand Unified Theories*
4. L. O' Raifeartaigh, *Group structure of gauge theories*, Cambridge Univ. Press, 1986.
5. S. Raby, *Grand Unified Theories*
6. E. Witten, "New perspectives on the quest for unification," hep-ph/9812208; "Quest for unifcation," hep-ph/0207124.
7. A. Zee, *Quantum field theory in a nutshell*, Princeton Univ. Press 2003
8. F. Gursey, P. Ramond, P. Sikivie, "A universal gauge theory model based on E_6," Phys. Lett. **60B** 1976.

29. The Hopf algebra of Feynman Diagrams and Renormalization

Describe the Connes-Kreimer-Marcolli reformulation of the BPHZ formulation of renormalization theory in terms of quantum loop group symmetries and Birkhoff factorization.

30. Discrete symmetries in particle physics

Discuss T,P reversal symmetry in physics, their physical implications, and evidence for the breaking of T, P symmetry in weak interactions.

Discuss how the structure of the Lorentz and Rotation groups is related to T,P symmetry and how it is implemented in quantum field theory.

Explain the CPT theorem.

1. Streater and Wightman, *PCT, Spin, Statistics, and all that.*

2. Steven Weinberg, "Causality, anti-particles, and the spin statistics connection in higher dimensions," Phys.Lett.B143:97,1984

3. Write about the experiments that prove that P is not a symmetry of nature, and about Pauli's reaction to the first news.

31. Lattice gauge theory

Explain the basic setup of lattice gauge theory and explain how group theory is used in deriving strong coupling expansions.

32. Chiral Fermions on the Lattice

One important outgrowth of the theory of anomalies is a technique for including chiral fermions on the lattice.

1. The fundamental topological nature of the difficulty was explained using homotopy theory in H. B. Nielsen and M. Ninomiya, "Absence Of Neutrinos On A Lattice. 1. Proof By Homotopy Theory," Nucl. Phys. B **185**, 20 (1981) [Erratum-ibid. B **195**, 541 (1982)].

2. An interesting reinterpretation of this was obtained in D. Friedan, "A Proof Of The Nielsen-Ninomiya Theorem," Commun. Math. Phys. **85** (1982) 481. Available at

3. A nice proposal of D. Kaplan uses anomaly inflow to get around the problem.

4. This idea has been successfully implemented in a series of papers by H. Neuberger in the past several years.

33. Global topology of gauge fixing and the Gribov "problem"

1. I.M. Singer, "Some remarks on the Gribov ambiguity," Commun. Math. Phys. **60**(1978)7.

2. M.F. Atiyah and J.D.S. Jones, "Topological aspects of Yang-Mills theory," Commun. Math. Phys. **61**(1978)97.

34. Two-dimensional Yang-Mills theory

A very beautiful illustration of the use of the heat kernel on compact Lie groups is the solution of Yang-Mills theory in two-dimensions.

1. Papers of Gross and Taylor.

2. Stefan Cordes, Gregory W. Moore, Sanjaye Ramgoolam, "Lectures on 2D Yang-Mills Theory, Equivariant Cohomology, and Topological Field Theories," Nucl.Phys.Proc.Suppl.41:184-244,1995 (Part 1 of the lectures is published in the Nucl. Phys. B, Proc. Suppl., and part 2 in the Les Houches proceedings. The entire set of lectures is posted as hep-ph/9411210) Also in Trieste Spring School 1994:0184-244 (QCD161:T732:1994) e-Print Archive: hep-th/9411210

3. J.A. Minahan and Polychronakos, Phys. Lett. B312 (1993) 155; hep-th/9303153

4. Polychronakos, Phys. Lett. B266 (1991) 29

35. Duality symmetries in gauge theory and string theory

Interesting discrete groups such as $SL(2,\mathbb{Z})$ have played an enormous role in gauge theory and string theory in recent years.

Write an introductory expository essay on electric-magnetic duality and "S-duality" in 4-dimensional gauge theory.

0. D. I. Olive, "Spin and electromagnetic duality: An outline," arXiv:hep-th/0104062.

1. D. I. Olive, "Introduction to electromagnetic duality," Nucl. Phys. Proc. Suppl. 58, 43 (1997).

2. D. I. Olive, "Exact electromagnetic duality," Nucl. Phys. Proc. Suppl. 45A, 88 (1996) [Nucl. Phys. Proc. Suppl. 46, 1 (1996)] [arXiv:hep-th/9508089].

3. E. Witten, "On S duality in Abelian gauge theory," Selecta Math. 1, 383 (1995) [arXiv:hep-th/9505186].

4. L. Alvarez-Gaume and S. F. Hassan, "Introduction to S-duality in N = 2 supersymmetric gauge theories: A pedagogical review of the work of Seiberg and Witten," Fortsch. Phys. 45, 159 (1997) [arXiv:hep-th/9701069].

36. The Langlands Dual Group and Magnetic Monopoles

There is a symmetry in the root systems of simple Lie algebras exchanging roots and coroots. At the Lie group level this leads to the notion of a Langlands dual group.

Discuss the mathematical construction of the Langlands dual group, and explain its relation to magnetic monopoles by reviewing the paper

P. Goddard, J. Nuyts and D. I. Olive, "Gauge Theories and Magnetic Charge," Nucl. Phys. B **125**, 1 (1977).

Also review the Montonen-Olive conjecture

C. Montonen and D. I. Olive, "Magnetic Monopoles as Gauge Particles?," Phys. Lett. B **72**, 117 (1977).

Find some modern reviews on the S-duality symmetry of N=4 SYM.

37. Geometric Langlands Program and its relation to $N = 4$ Supersymmetric Yang-Mills Theory

The S-duality of N=4 SYM has led to very deep relations between physics, number theory and geometric representation theory.

Review this development starting with

1. Anton Kapustin (Caltech) , Edward Witten, "Electric-Magnetic Duality And The Geometric Langlands Program," e-Print Archive: hep-th/0604151

2. Sergei Gukov and Edward Witten, " Gauge Theory, Ramification, And The Geometric Langlands Program," arXiv:hep-th/0612073

3. E. Witten, "Geometric Langlands And The Equations Of Nahm And Bogomolny," arXiv:0905.4795 [hep-th].

38. The role of E_{10} in supergravity theory

Describe the noncompact symmetries E_d of 11-dimensional supergravity compactified on a torus of dimension d.

Describe the hyperbolic Lie algebra E_{10}, and its role in compactifications of 11-dimensional supergravity on a 10-dimensional torus.

Describe attempts to realize E_{10} in terms of vertex operator algebras.

See papers of Axel Kleinschmidt and Hermann Nicolai

39. Exceptional groups and BPS black holes

Explain the role of the cubic and quartic invariants in formulating entropy of BPS black holes.

1. BRANES, CENTRAL CHARGES AND U DUALITY INVARIANT BPS CONDITIONS. By Sergio Ferrara (CERN), Juan M. Maldacena (Rutgers U., Piscataway). RU-97-35, CERN-TH-97-112, Jun 1997. 14pp. Published in Class.Quant.Grav.15:749-758,1998 e-Print Archive: hep-th/9706097

2. ORBITS OF EXCEPTIONAL GROUPS, DUALITY AND BPS STATES IN STRING THEORY. By Sergio Ferrara, Murat Gunaydin (CERN),. CERN-TH-97-187, PSU-TH-188, Aug 1997. 11pp. Published in Int.J.Mod.Phys.A13:2075-2088,1998 e-Print Archive: hep-th/9708025

40. Duality symmetries in supergravity theory

1. C.M. Hull, P.K. Townsend, "Unity of superstring dualities," Nucl.Phys.B438:109-137,1995 e-Print Archive: hep-th/9410167

2. E. Witten, "String theory dynamics in various dimensions," Nucl. Phys. B **443**, 85 (1995) [arXiv:hep-th/9503124].

3. N. A. Obers and B. Pioline, "U-duality and M-theory," Phys. Rept. **318**, 113 (1999) [arXiv:hep-th/9809039].

For a more recent reference see

4. M. Gunaydin, "Realizations of exceptional U-duality groups as conformal and quasiconformal groups and their minimal unitary representations," Comment. Phys. Math. Soc. Sci. Fenn. **166**, 111 (2004) [arXiv:hep-th/0409263].

41. Triality symmetry

Explain what triality symmetry of the representations of $SO(8)$ is.

Investigate the role of triality symmetry in representation theory, differential geometry, and string theory.

For example,

1. Show how triality symmetry is used in proving equivalence of the Green-Schwarz and RNS formulations of string theory in light cone gauge. (Witten: Conference proceedings, c. 1982.)

2. Show how triality symmetry emerges in the flavor symmetries of $d = 4, N = 2$ supersymmetric gauge theory with 4 doublet flavors. (Seiberg-Witten II).

42. Cheeger-Simons Differential Characters and Gerbes

The theory of Cheeger-Simons characters is a mathematically precise way of formulating the gauge theories of abelian p-form gauge potentials. They have had some applications to supergravity theories in diverse dimensions. See also the following two projects.

1. Hitchin's review of gerbes.

2. J. Cheeger and J. Simons, "Differential Characters and Geometric Invariants," in *Geometry and Topology*, J. Alexander and J. Harer eds., LNM 1167.

3. K. Gawedzki, "Topological actions in two-dimensional quantum field theories," in *Nonperturbative quantum field theory* Procedings of the 1987 Cargese meeting, pp. 101-141.

4. B. Harris, "Differential Characters and the Abel-Jacobi Map", in *Algebraic K-theory: Connections with Geometry and Topology*, J.F. Jardine and V.P. Snaith eds., Nato ASI Series C: Mathematical and Physical Sciences – Vol. 279, Kluwer Academic Publishers, 1989.

5. R. Dijkgraaf and E. Witten, "Topological gauge theories and group cohomology," Commun.Math.Phys.129:393,1990

6. R. Zucchini, "Relative topological integrals and relative Cheeger-Simons differential characters," hep-th/0010110

43. Differential Cohomology Theories and Electro-Magnetic Duality

Explain what differential cohomology is and how it is used in formulating electro-magnetic duality:

1. D. Freed, " Dirac Charge Quantization and Generalized Differential Cohomology," arXiv:hep-th/0011220

2. Freed, Moore and Segal, hep-th/0605198

3. Freed, Moore and Segal, hep-th/0605200
4. G. Moore lecture online at the KITP:
http://online.itp.ucsb.edu/online/strings05/

If you are ambitious, explain the foundational paper

Quadratic functions in geometry, topology, and M theory. M.J. Hopkins, I.M. Singer (MIT) . Nov 2002. 99pp. Published in J.Diff.Geom.70:329-452,2005 e-Print Archive: math.at/0211216

Open problem: Extend this approach to electric-magnetic duality to noncompact manifolds. If you solve it you can publish a nice paper.

44. Differential Cohomology Theories and Chern-Simons Theory

Explain what differential cohomology is and how it is used in quantizing self-dual theories
1. Witten: 5-brane partition function hep-th/96...
2. Witten: Duality relations, hep-th/99...
3. Belov and Moore, holographic action, hep-th/0605...
4. Belov and Moore, Type II actions from Chern-Simons theories...

If you are ambitious, explain the foundational paper

Quadratic functions in geometry, topology, and M theory. M.J. Hopkins, I.M. Singer (MIT) . Nov 2002. 99pp. Published in J.Diff.Geom.70:329-452,2005 e-Print Archive: math.at/0211216

45. Representations of Conformal and Superconformal groups

This is an important subject in particle physics.

1. Describe the unitary representations of conformal groups $SO(2, n)$ for various n.
2. Describe the finite list of superconformal groups in dimensions 2 to 6.
3. Describe the unitary representations of the superconformal groups.
4. Explain how these are an integral part of the AdS/CFT correspondence.

References to begin with:

1. W. Nahm, Supersymmetries and their representations, Nucl. Phys. B135 (1978) 149.
2. S. Minwalla, hep-th/9712074
3. Large N Field Theories, String Theory, and Gravity, Ofer Aharony , Steven S. Gubser , Juan M. Maldacena , Hirosi Ooguri , Yaron Oz Published in Phys.Rept.323:183-386,2000 e-Print Archive: hep-th/9905111
4. Papers of Dolan and Osborne
5. Helgason, A good (but dense) math book for the geometry of symmetric spaces.
6. Gilmore, covers some of the relevant material on the differential geometry of symmetric spaces in a more expository style.

46. Conformal field theories in higher dimensions: AdS/CFT

The AdS/CFT correspondence is an important aspect of string theory and particle theory. Explain what this correspondence is about. See also Section **??** above.

1. Large N Field Theories, String Theory, and Gravity, Ofer Aharony , Steven S. Gubser , Juan M. Maldacena , Hirosi Ooguri , Yaron Oz Published in Phys.Rept.323:183-386,2000 e-Print Archive: hep-th/9905111

2. Lecture notes on holographic renormalization. Kostas Skenderis Class.Quant.Grav.19:5849-5876,2002 e-Print Archive: hep-th/0209067

47. Modular groups and modular forms in 2D CFT

Review modular forms for $SL(2,\mathbb{Z})$ and explain how modular invariance is used in 2D CFT.

Explain the derivation of the "Cardy formula" relating the asymptotic degeneracy of states with the central charge.

1. DiFrancesco et. al. book on conformal field theory
2. G. Moore, Trieste Lectures on modular forms and black hole physics

48. The two-dimensional WZW model for a compact group target

The WZW model for a compact group involves many very beautiful ideas in group theory and conformal field theory.

While this is textbook material, there is lots to explore.

At the most basic level one should define the Lagrangian and the model and explain the space of states of the theory, and discuss why it is a generalization of the Peter-Weyl theorem.

Explain the null vectors and character formulae.

References to start with:

1. J. Fuchs, *Affine Lie Algebras and Quantum Groups*, Camb. Univ. Press, 1992.
2. DiFrancesco, P. Mathieu, and D. Senechal , *Conformal Field Theory*, Springer Verlag, 1997.
3. Pressley and Segal, *Loop Groups*, Oxford Univ. Press, 1986.
4. P. Goddard and D. Olive, "Kac-Moody and Virasoro Algebras in Relation to Quantum Physics," Int. J. Mod. Phys. **A1** (1986)303.
5. V.G. Kac and D.H. Peterson, "Infinite-Dimensional Lie Algebras, Theta Functions, and Modular Forms," Adv. Math. **53**(1984)125.

Also read some of the foundational papers:

1. D. Gepner and E. Witten, “String theory on group manifolds,” Nucl. Phys. **B278**(1986)493

2. V.G. Knizhnik and A.B. Zamolodchikov, “Current algebra and Wess-Zumino model in two dimensions,” Nucl. Phys. **B247**(1984)83

3. A. Polyakov and P.B. Wiegmann, “Goldstone fields in two dimensions with multi-valued actions,” Phys. Lett. **141B**(1984)223; “Theory of nonabelian Goldstone bosons in two dimensions,” Phys. Lett. **131B** (1983) 121

4. Edward Witten , “Nonabelian Bosonization in Two Dimensions,” Commun.Math.Phys.92:455-472,1984 (Also in *Goddard, P. (ed.), Olive, D. (ed.): Kac-Moody and Virasoro algebras* 483-500)

5. Edward Witten, “Global Aspects of Current Algebra,” . Nucl.Phys.B223:422,1983 Also in Trieste Particle Phys.1983:III-2 (QCD161:W626:1983

49. Fusion rules of the WZW model

Conformal field theory introduces a novel tensor product representation on representations of affine Lie algebras of level k.

Explain how to take a tensor product of two modules of level k to produce a third of level k.

Explain how the fusion rules are related to the usual Clebsch-Gordon decomposition of finite dimensional representations of compact Lie groups.

References:

1. Gepner and Witten

2. G. Moore and N. Seiberg, Classical and Quantum Conformal Field Theory, Commun. Math. Phys.

3. V. Kac, ...

50. Theory of vertex operator algebras

Review the mathematical construction of a vertex operator algebra.

1. Borcherds, Richard (1986), ”Vertex algebras, Kac-Moody algebras, and the Monster”, Proc. Natl. Acad. Sci. USA. 83: 30683071

2. Frenkel, Igor; Lepowsky, James; Meurman, Arne (1988), Vertex operator algebras and the Monster,

3. Kac, Victor (1998), Vertex algebras for beginners, University Lecture Series, 10

51. The WZW model for $SL(2, R)$

This is an active topic of current research. Start with

1. J. Maldacena and H. Ooguri, "Strings in AdS3 and the SL(2,R) WZW model" hep-th/0001053
2. J. Maldacena, H. Ooguri, and J. Son, hep-th/0005183

52. Fuchsian groups and their applications in 2D CFT and string theory

Review the uniformization theory for Riemann surfaces.

Explore the uses of Fuchsian groups and Kleinian groups in perturbative string theory and in the AdS/CFT correspondence.

53. Two-Dimensional CFT And Sporadic Groups

1. Review the Frenkel-Lepowsky-Meurman construction of the Monster group.
2. Review the recent developments in Mathieu moonshine.

Some papers to start with:

1. Articles by Griess and Frenkel et. al. in *Vertex Operators in Mathematics and Physics*, J. Lepowsky, S. Mandelstam, and I.M. Singer, eds.
2. J. Harvey, "Twisting the Heterotic String," in *Unified String Theories*, Green and Gross eds.
3. L.J. Dixon, P.H. Ginsparg, and J.A. Harvey, "Beauty And The Beast: Superconformal Symmetry In A Monster Module," Commun.Math.Phys. 119 (1988) 221-241
4. T. Gannon, *Moonshine beyond the Monster: The Bridge Connecting Algebra, Modular Forms and Physics* Cambridge Monographs on Mathematical Physics, 2010.
5. M.C.N. Cheng, J.F.R. Duncan, and J.A. Harvey, "Umbral Moonshine," e-Print: arXiv:1204.2779 [math.RT]

Some reviews

1. https://arxiv.org/pdf/1411.6571.pdf
2. https://arxiv.org/pdf/math/0402345.pdf
3. https://arxiv.org/pdf/1807.00723.pdf
4. https://arxiv.org/pdf/1605.00697.pdf

54. K3 Surfaces, String Theory, And The Mathieu Group

Eguchi, Ooguri, and Tachikawa made a very surprising observation indicating the existence of some deep relation between K3 surfaces, string theory, and Mathieu sporadic groups. A full conceptual understanding still remains open. (April, 2018).

The initial observation has been broadly generalized in a series of observations known as "umbral Moonshine."

For reviews you can get started with

M. C. N. Cheng, (Mock) Modular Forms in String Theory and Moonshine, Lecture Notes for the Asian Winter School at OIST. (January, 2016).

There are also good online introductory lectures by J. Harvey and also by M. Cheng.

55. Pure spinors and string perturbation theory

In his investigations into Clifford algebras Chevalley introduced the notion of a "pure spinor."

Investigate the role pure spinors have played in geometry. Start with the book of Lawson and Michelson.

Review the recent proposals of N. Berkovits for a formulation of perturbative string theory based on "pure spinors."

See also

1. N. Nekrasov, "Lectures on curved $\beta\gamma$ systems," arXiv:hep-th/0511008

2.O. A. Bedoya and N. Berkovits, "GGI Lectures on the Pure Spinor Formalism of the Superstring," arXiv:0910.2254 [hep-th].

56. Modular Tensor Categories

Describe how braiding and fusing and modular transformation rules of conformal blocks in RCFT leads to the notion of a modular tensor category.

References:

1. G. Moore and N. Seiberg, "Lectures on RCFT," available from my homepage.

2. "On Witten's 3-manifold invariants" by Kevin Walker. This manuscript was written in 1991 and never finished or published. It is now available on-line at http://canyon23.net/math/.

3. Bakalov and Kirillov, *Lectures on Tensor Categories and Modular Functor*

4. James Lepowsky, "From the representation theory of vertex operator algebras to modular tensor categories in conformal field theory," Proc Natl Acad Sci U S A. 2005 April 12; 102(15).

5. V. Turaev, *Quantum Invariants of Knots and Three Manifolds*

57. Chern-Simons-Witten invariants of 3-manifolds

1. V. Jones, "The Jones Polynomial," 2005

2. E. Witten, Quantum field theory and the Jones polynomial, Commun. Math. Phys. 1988.

3. Perturbative expansions. Axelrod-Singer paper

4. L. Rozansky and E. Witten, “Hyperkahler manifolds and invariants of 3-folds,” hep-th/9612216

5. Vasiliev invariants and Kontsevich’s interpretation, and their relation to Yang-Baxter equations and Knizhnik-Zamolodchikov equations.

6. D. S. Freed, “Remarks on Chern-Simons Theory,” arXiv:0808.2507 [math.AT].

58. The relation of 3D Chern-Simons theory to 2D CFT

1. Edward Witten, “Quantum field theory and the Jones polynomial,” Commun.Math.Phys.121:351,1989

2. Scott Axelrod Steve Della Pietra Edward Witten, “Geometric quantization of Chern-Simons gauge theory,” J.Diff.Geom.33:787-902,1991

3. Shmuel Elitzur, Gregory W. Moore, Adam Schwimmer, Nathan Seiberg, “Remarks on the canonical quantization of the Chern-Simons-Witten theory,” Nucl.Phys.B326:108,1989

4. LECTURES ON RCFT. By Gregory W. Moore , Nathan Seiberg Presented at Trieste Spring School 1989. Published in Trieste Superstrings 1989:1-129 (QCD161:T732:1989) Also in Banff NATO ASI 1989:263-362 (QC20:N22:1989)

59. Chern-Simons theory for discrete gauge group and Dijkgraaf-Witten theory

The formal path integral of 3-dimensional Chern-Simons theory can be made quite rigorous in the case when the compact gauge group is a discrete group. This is done in

1. R. Dijkgraaf and E. Witten, Commun. Math. Phys. **129**, 393 (1990).

2. D. S. Freed and F. Quinn, “Chern-Simons theory with finite gauge group,” Commun. Math. Phys. **156**, 435 (1993)

60. Quantization of Spin Chern Simons Theories

An important generalization of topological field theory is “spin topological field theory” where amplitudes depend on the topology AND the spin structure of a manifold. These were first discussed in

1. R. Dijkgraaf and E. Witten, ‘Topological Gauge Theories And Group Cohomology,” Commun. Math. Phys. **129**, 393 (1990).

2. The case of SU(2) and SO(3) spin Chern-Simons is analyzed in

Jerome A. Jenquin, “Spin Chern-Simons and Spin TQFTs,” 31pp. e-Print Archive: math.dg/0605239

3. Already the case of an Abelian gauge group of the form $G = U(1)^r$ is nontrivial. See:

1. D. Belov and G. W. Moore, "Classification of Abelian spin Chern-Simons theories," hep-th/0505235.

2. A. Kapustin and N. Saulina, "Topological boundary conditions in abelian Chern-Simons theory," Nucl. Phys. B **845**, 393 (2011) [arXiv:1008.0654 [hep-th]].

3. D. S. Freed, M. J. Hopkins, J. Lurie and C. Teleman, "Topological Quantum Field Theories from Compact Lie Groups," arXiv:0905.0731 [math.AT].

4. S. D. Stirling, "Abelian Chern-Simons theory with toral gauge group, modular tensor categories, and group categories," arXiv:0807.2857 [hep-th].

61. Chern-Simons theory and Quantum Gravity in 2+1 Dimensions

- Achucarro and Townsend
- E. Witten, "2+1 dimensional gravity as an exactly soluble system," Nucl. Phys. 1988
- E. Witten, 3d gravity revisited, 2007

62. Freed-Hopkins-Teleman Theorem

This theorem identifies a certain twisted K-theory with the Verlinde algebra.

1. Start with review papers by Freed.

2. Give a physical interpretation: See

The Verlinde algebra and the cohomology of the Grassmannian. Edward Witten In *Cambridge 1993, Geometry, topology, and physics* 357-422. e-Print Archive: hep-th/9312104

K theory from a physical perspective. Gregory W. Moore Published in *Oxford 2002, Topology, geometry and quantum field theory* 194-234 e-Print Archive: hep-th/0304018

63. Quantum Groups

Quantum groups are a generalization of the notion of a group where one replaces the commutative algebra of functions on a group by a noncommutative Hopf algebra.

The extensive work of the Russian school (Zamolodchikov and Zamolodchikov, and the St. Petersburg group of Faddeev) has been re-interpreted in terms of quantum groups.

1. Chari and Pressley, *A guide to quantum groups*, Cambridge Univ. Press 1994

2. M. Jimbo, Yang-Baxter Equation in Integrable Systems," Adv. Ser. in Math.. Phys. **10** WorldScientific 1989

3. A.B. Zamolodchikov and Al. B. Zamolodchikov, Ann. Phys. 120 (1979) 253

4. V.G. Drinfel'd, "Quantum Groups," Proceedings of the International Congress of Mathematicians, Berkeley, 1986

5. P. Etingof, I. Frenkel, and A. Kirillov, "Lectures on Representation Theory and Knizhnik-Zamolodchikov equations," American Math. Soc. , Providence, 1998.

6. J. Baez, "Braids and Quantization" http://math.ucn.edu/home/baez/braids.html

7. C. Kassel, Quantum Groups, Springer-Verlag, New York, 1995.

8. A. Klimyk, K. Schmudgen, Quantum Groups and Their Representations, Springer, New York, 1997.

9. P. Aschieri, Lectures on Hopf Algebras: Quantum Groups and Twists. Lectures given at 2nd Modave Summer School in Theoretical Physics, Modave, Belgium, 6-12 Aug 2006. arXiv:hep-th/0703013

64. Variations on the theme of supersymmetric quantum mechanics

Character-valued index theorems

Fixed point theorems, proofs using supersymmetric quantum mechanics.

Representative papers:

1. FERMION QUANTUM NUMBERS IN KALUZA-KLEIN THEORY. By Edward Witten (Princeton U.). PRINT-83-1056 (PRINCETON), Oct 1983. 78pp. Published in Shelter Island II 1983:227 (QC174.45:S45:1983

2. E. Witten, Holomorphic Morse inequalities

3. M. Goodman, Proof of Character Valued Index Theorems, CMP

65. Exact Stationary Phase: Localization In Equivariant Cohomology

A very important technique for exact evaluation of path integrals is the concept of localization.

Some extensive reviews include:

1. https://arxiv.org/pdf/hep-th/9411210.pdf

2. https://arxiv.org/pdf/1608.02952.pdf

Topics could include:

1. Duistermaat-Heckman theorem

2. Nonabelian localization theorem and 2d Yang-Mills theory.

3. M.F. Atiyah and R. Bott, "The moment map and equivariant cohomology," Topology **23** (1984) pp. 1-28

4. R. Szabo, "Equivariant localization of path integrals," arXiv:

5. Nikita Nekrasov, Andrei Okounkov, "Seiberg-Witten theory and random partitions," arXiv: hep-th/0306238

5. Papers of N. Nekrasov, deriving the Seiberg-Witten prepotential from integration over moduli space of instantons.

66. Donaldson theory of 4-manifolds and Floer theory of 3-manifolds

Formulation of Donaldson and Floer theory using supersymmetric Yang-Mills.

19) TOPOLOGICAL QUANTUM FIELD THEORY. By Edward Witten (Princeton, Inst. Advanced Study). IASSNS-HEP-87/72, Feb 1988. 61pp. Published in Commun.Math.Phys.117:353,1988

Representative reviews:

1. Cordes, Moore, Ramgoolam, and refs therein

2. Blau et. al. "Topological field theory," and refs therein

3. hep-th/9703136 Title: Les Houches Lectures on Fields, Strings and Duality Author: R. Dijkgraaf Comments: 152 pages, 31 epsf figures, latex

4. J. M. F. Labastida, Carlos Lozano, Lectures in Topological Quantum Field Theory, hep-th/9709192

5. Van Baal review (good for Floer theory) .

6. Mathematical formulation of Donaldson theory: Donaldson and Kronheimer, *Geometry of Four-Manifolds*, Oxford

67. Applications of the Seiberg-Witten solution to 4-manifolds

Explain how the SW solution of $N = 2$ SYM allows one to compute Donaldson invariants of 4-manifolds.

Explain relation to the "geography problem" for 4-manifolds.

1. E. Witten, "Monopoles and four-manifolds," hep-th/94mmnnn

2. G. Moore and E. Witten, "Integration over the u-plane in Donaldson Theory," hep-th/97mmnnn

3. Review by M. Marino.

4. Book by Labastida and Marino.

68. The computation of $\mathrm{Tr}(-1)^F$ in supersymmetric Yang-Mills theory

By compactifying on a torus one reduces this to a SQM problem, such as that discussed in this course. There are some very interesting and subtle issues connected to classifying flat connections on tori of dimension larger than 2.

Two representative papers:

1. CONSTRAINTS ON SUPERSYMMETRY BREAKING. By Edward Witten (Princeton U.). PRINT-82-0163 (PRINCETON), (Received Mar 1982). 116pp. Published in Nucl.Phys.B202:253,1982

2. E. Witten, Toroidal Compactification Without Vector Structure, hep-th/9712028 (appendix)

3. E. Witten, Supersymmetric Index in Four-Dimensional Gauge Theories, hep-th/0006010, and references therein.

4. Classification of flat connections in 3 and 4 dimensions. Borel-Friedman-Morgan.

69. Seiberg-Witten theory and integrable systems

Donagi-Witten

Martinec and Warner

Papers of Nekrasov, Marshakov et. al.

70. Hitchin Systems

The reduction of the self-dual Yang-Mills equations to a Riemann surface gives a set of differential equations known as the *Hitchin equations* which have very beautiful relations to integrable systems. This began with the classic paper:

1. Hitchin, Nigel (1987), "Stable bundles and integrable systems", Duke Mathematical Journal 54 (1): 91114.

Given the relations to instantons there have been many applications of Hitchin systems to particle physics in recent years. See some of the subsequent topics below.

71. Hitchin Systems and Seiberg-Witten theory

1. R. Donagi and E. Witten, "Supersymmetric Yang-Mills Theory And Integrable Systems," Nucl. Phys. B **460**, 299 (1996) [arXiv:hep-th/9510101].

2. E. Witten, "Solutions of four-dimensional field theories via M-theory," Nucl. Phys. B **500**, 3 (1997) [arXiv:hep-th/9703166].

3. Wall-crossing, Hitchin Systems, and the WKB Approximation. Davide Gaiotto, Gregory W. Moore, Andrew Neitzke, . e-Print: arXiv:0907.3987 [hep-th]

72. Quantization of Teichmuller Space

The Teichmuller space of Riemann surfaces is a phase space whose quantization is related to the Liouville quantum field theory.

Review the ideas of Kashaev, Fock, and Teschner. A good place to start is

1. J. Teschner, "An analog of a modular functor from quantized Teichmuller theory, I," arXiv:math/0405332.

See also:

2. J. Teschner, "Quantization of the Hitchin moduli spaces, Liouville theory, and the geometric Langlands correspondence I," arXiv:1005.2846 [hep-th].

3. Fock and Goncharov, arXiv:math/0311149v4 [math.AG]

73. Geometrical Structures and supersymmetric sigma models

Review N=1,2,4 supersymmetry of nonlinear sigma models in different dimensions and how this constrains the target space geometry. Give proofs of the classic results:

1. A d=4, N=1 model requires a Kahler target space.

2. A d=4, N=2 model of (hypermultiplet) scalars requires a hyperkahler target space.

and explain the relation to the results:

3. A d=2 (2,2) model requires a Kahler target space.

4. A d=2 (4,4) model requires a hyperkahler target space.

Discuss the geometry associated with d=2 (0,2) and (0,4) models.

Interpret the N=2 superconformal algebra in terms of Kahler geometry of loop space.

A recent review is:

U. Lindström, "Uses of Sigma Models," arXiv:1803.08873 [hep-th].

See also the textbook

A. van Proeyen and D. Z. Freedman, *Supergravity*

Be sure to look the classic paper of L. Alvarez-Gaumé and D. Freedman and see references to that. See also papers of M. Roček et. al.

74. The elliptic genus and the index theorem on loop space

Generalization of the SQM approach to index theory to the Dirac operator on loop space. A nice application of string theory to new topological invariants and new cohomology theories.

Some representative papers:

1. Hirzebruch book: Manifolds and modular forms
2. ANOMALY CANCELLING TERMS FROM THE ELLIPTIC GENUS. By W. Lerche, B.E.W. Nilsson, A.N. Schellekens, N.P. Warner (CERN). CERN-TH-4765/87, Jun 1987. 39pp. Published in Nucl.Phys.B299:91,1988
3. LATTICES AND STRINGS. By W. Lerche, A.N. Schellekens, N.P. Warner (CERN). CERN-TH-5155-88, Aug 1988. 197pp. Published in Phys.Rept.177:1,1989
4. THE INDEX OF THE DIRAC OPERATOR IN LOOP SPACE. By Edward Witten (Princeton U.). PUPT-1050, Mar 1987. 30pp. To appear in Proc. of Conf. on Elliptic Curves and Modular Forms in Algebraic Topology, Princeton, N.J., Sep 1986.
5. ELLIPTIC GENERA AND QUANTUM FIELD THEORY. By Edward Witten (Princeton U.). PUPT-1024, Nov 1986. 20pp. Published in Commun.Math.Phys.109:525,1987
6. G. Segal's Bourbaki talk on elliptic genera.

75. Formulating Elliptic Cohomology in terms of Quantum Field Theory

Explain Stolz and Teichner's approach to Elliptic Cohomology Theory.

1. Contribution to *Topology, Geometry, and Quantum Field Theory*, U. Tillmann ed.
2. See on-line talks at the KITP workshop on mathematics of string theory: Fall 2005.

76. String structures

Explain the topological conditions for the loop space of a manifold LM to be orientable, spin. These are conditions on the manifold M and are sometimes referred to as "string structures"

Refs:

1. Segal, Bourbaki talk on elliptic genera
2. Pilch and Warner, CMP
3. T. Killingback

77. Borel-Weil-Bott theorem.

A beautiful way of constructing representations of compact Lie groups is via the space of holomorphic sections of G^c/B, known as the Borel-Weil-Bott construction.

Describe this construction (Prerequisite: Some knowledge of the theory of holomorphic line bundles.)

Go on to describe the generalization of this procedure to representations of loop groups:

1. Pressley and Segal, *Loop Groups*, Oxford
2. O. Alvarez, I. Singer, et. al., path integral derivation of same.
3. P. Woit, hep-th/02....
4. A.A. Kirillov, *Elements of the Theory of Representations.* Springer 1976
5. R. Bott, "On Induced Representations"
6. R. Szabo, "Equivariant localization of path integrals,"
7. Alekseev, Faddeev, Shatashvili,

78. The Atiyah-Bott-Shaprio Construction - and the physicist's interpretation

Discuss Bott periodicity using Clifford algebras. Explain how this is related to questions of solitons in string theory, and the classification of D-branes by K-theory.

1. Atiyah, M. F., Bott, R., Shapiro, A. "Clifford modules," Topology 3 1964 suppl. 1, 3–38. A classic paper on Clifford algebras. Very readable.

2. Lawson and Michelson, *Spin Geometry*, Textbook summary of the ABS paper.

3. E. Witten, "D-Branes And K-Theory," JHEP **9812**:019, 1998; hep-th/9810188. Interprets the construction as a field configuration for a tachyon in a $D-\bar{D}$ system.

4. Harvey and Moore, "Noncommutative tachyons and K theory," hep-th/0009030. Generalizes the construction to noncommutative field theory.

79. Global anomalies in supergravity

Even when local anomalies cancel, gauge theories coupled to fermions can be inconsistent.

1. E. Witten, "An SU(2) anomaly" Phys. Lett. B, 1982

2) GLOBAL ANOMALIES IN STRING THEORY. By Edward Witten (Princeton U.). Print-85-0620 (PRINCETON), Jun 1985. 39pp. To appear in Proc. of Argonne Symp. on Geometry, Anomalies and Topology, Argonne, IL, Mar 28-30, 1985. Published in ANL Symp.Anomalies 1985:0061 (QC20:S96:1985)

3) GLOBAL GRAVITATIONAL ANOMALIES. By Edward Witten (Princeton U.). PRINT-85-0246 (PRINCETON), Jan 1985. 66pp. Published in Commun.Math.Phys.100:197,1985

4. Kervaire and Milnor's papers on exotic differential structures and exotic spheres.

In recent years S. Monnier has written a series of interesting papers re-visiting the global anomalies of several theories relevant to supergravity and string theory.

80. Baum-Connes conjecture

Explain what the conjecture is. This is a challenging project, requiring mastering noncommutative differential geometry, especially the Chern character for crossed-product algebras.

1. Ref: Connes' book.
2. Chern character for discrete groups. A fte of topology, 163–232, Academic Press, Boston, MA, 1988. (Chern character for discrete groups. A fte of topology, 163–232, Academic Press, Boston, MA, 1988.
3. T. Schick, "Operator algebras and their topology," arXiv:math.GT/0209164

81. Spin And Pin Structures

Explain what spin and pin structures are on manifolds and the obstructions to defining them, and the choices involved in defining them. Start with:

Lawson and Michelson, *Spin Geometry*

T. Friedrich, *Dirac Operators in Riemannian Geometry*

Explain how to define the Dirac operator on a manifold admitting a pin structure but not a spin structure.

This subject has been important in investigations into phases of matter. See:

1. E. Witten, "Fermion Path Integrals And Topological Phases," Rev. Mod. Phys. **88**, no. 3, 035001 (2016) [arXiv:1508.04715 [cond-mat.mes-hall]].
2. E. Witten, "Three Lectures On Topological Phases Of Matter," Riv. Nuovo Cim. **39**, no. 7, 313 (2016) [arXiv:1510.07698 [cond-mat.mes-hall]].
3. E. Witten, "The "Parity" Anomaly On An Unorientable Manifold," Phys. Rev. B **94**, no. 19, 195150 (2016) [arXiv:1605.02391 [hep-th]].

and references therein.

This topic is also important in the theory of "orientifolds" in string theory. See

J. Distler, D. S. Freed and G. W. Moore, "Spin structures and superstrings," arXiv:1007.4581 [hep-th].

82. Toric Geometry

Review basic ideas from Fulton's book.

Then explain Batyrev's construction of mirror duals in terms of reflexive polyhedra.

GKZ systems etc.

1. Klemm's review.
2. Aspinwall-Greene-Morrison papers.
3. Book by Cox and Katz
4. Danilov's review.
5. Articles in Essays on Mirror Symmetry, vol II

83. Chern-Simons Theory, Matrix Models and Topological Strings

A big subject. Start with reviews by Marcos Marino:

1. M. Marino, "Chern-Simons theory and topological strings," Rev. Mod. Phys. **77**, 675 (2005) [arXiv:hep-th/0406005].

2. M. Marino, "Les Houches lectures on matrix models and topological strings," arXiv:hep-th/0410165.

84. String theory, conformal field theory, and manifolds of special holonomy

Manifolds of special holonomy play an important role in discussions of string compactifications.

You could discuss the Mathematics of Special holonomy manifolds:

Construction of noncompact manifolds of special holonomy:

1. Bryant. R.L. "Metrics with exceptional holonomy," Ann. Math. 126 (1987) 525

2. Bryant, R.L. and Salamon, S.M., Duke Math. J. 58 (1989) 829

Construction of compact manifolds of special holonomy:

1. Joyce, Dominic D. Compact Riemannian 7-manifolds with holonomy G_2. I, II. J. Differential Geom. 43 (1996), no. 2, 291–328, 329–375. (

2. Joyce, Dominic D Compact Riemannian 7-manifolds with holonomy G_2. I, II. J. Differential Geom. 43 (1996), no. 2, 291–328, 329–375

3. Joyce, D. D. Compact 8-manifolds with holonomy Spin(7). Invent. Math. 123 (1996), no. 3

4. Joyce, D.D., *Compact Manifolds with Special Holonomy*, Oxford Univ. Press.

Explain how such structures are encoded in the CFT with such a target space:

5. Shatashvili and Vafa, "superstrings and manifolds of exceptional holonomy, " hep-th/9407025

Use citations to learn about follow-up papers.

Discuss the low energy effective Supergravity KK reduced on smooth G_2 manifolds:

1. Papadopoulos and Townsend, Phys. Lett. B365 (1995)300

2. Harvey and Moore, hep-th/9907026

3. Papadopoulos, hep-th/00mmnnn

Recent work on applications of G_2 manifolds to phenomonology:

4. Atiyah and Witten,

5. Acharya and Witten,

6. Recent papers of Gibbons, Cvetic et. al., Gukov et. al. ....

Nice review: B. Acharya and S. Gukov, hep-th/....

The subject of special holonomy has recently (c. 2016-2018) enjoyed some important advances see: ...

85. Calibrated submanifolds in special holonomy manifolds

Conditions for D-branes to wrap cycles etc:

Becker, Becker, and Strominger

Nonlinear instantons from supersymmetric p-branes. Marcos Marino (Yale U.) , Ruben Minasian , Gregory W. Moore (Yale U.) , Andrew Strominger (Jefferson Lab) . YCTP-P28-99, Nov 1999. 35pp. Published in JHEP 0001:005,2000 e-Print Archive: hep-th/9911206

Discuss "co-isotropic branes" – see papers of A. Kapustin, e.g.

Anton Kapustin (Princeton, Inst. Advanced Study) , Dmitri Orlov, " Vertex algebras, mirror symmetry, and D-branes: The Case of complex tori," Commun.Math.Phys.233:79-136,2003 e-Print Archive: hep-th/0010293

86. Twistor transform

Explore various aspects of twistors and twistor transforms:

1. Ward and Wells, *Twistor Geometry and Field Theory*
2. Explain the use of Twistors in the ADHM construction

Harmonic Superspace

κ-supersymmetry.

Superspace formulation of $N = 4$ Super Yang-Mills equations.

Edward Witten, "Twistor - Like Transform In Ten-Dimensions," Nucl.Phys.B266:245,1986

87. String theory in twistor space and perturbative Yang-Mills theory

An important recent development in mathematical physics. See

Perturbative gauge theory as a string theory in twistor space. Edward Witten, Commun.Math.Phys.252:1
258,2004 e-Print Archive: hep-th/0312171

and investigate the subsequent literature.

88. Geometry of BRST cohomology and BV formalism

Discuss the geometrical meaning of BRST cohomology and other ways of understanding the phase space structure of gauge theories, and of the quantization of gauge theories.

This involves a discussion of odd-symplectic geometry, i.e. symplectic geometry of supermanifolds.

1. Henneaux and Teitelboim, book
2. Various papers by Stasheff.
3. Kevin Costello, *Renormalization and Effective Field Theory*, AMS 2011

89. Geometry of supergravity constraint equations

Explain the "torsion constraints" of supergravity from a geometrical point of view in terms of integrability conditions.

1. Wess and Bagger

2. Manin, Gauge theories and complex geometry
3. Witten, Notes on super Riemann surfaces

90. Special Geometry

An important class of geometries intimately associated with N=2 supersymmetry:

1. Papers of Ferrara and van Proeyen on special geometry
2. Freed, Special Geometry
3. See the IAS volumes on mathematical physics.

91. Derivation of the Weyl and the Weyl-Kac character formulae from geometry

Derive using

a.) Fixed point theorems.

b.) Stationary phase approximation of a suitable path integral.

92. Instantons

Discuss the instanton solution of $SU(2)$ Yang-Mills theory on $\mathbb{R}^4$, or S^4 discovered by Belavin, Polyakov, Schwarz, and Tyupkin.

1. M.F. Atiyah, "The Geometry of Yang-Mills Fields" in Collected Works, vol. 5
2. S. Coleman, "The Uses of Instantons" in *Aspects of Symmetry.*

93. Hyperkahler quotients , the ADHM construction and the Nahm transform

These are very powerful techniques for studying instantons and monopoles and their moduli spaces in certain spacetimes.

These constructions have played an important role in the development of D-brane theory.

See papers by Atiyah in Collected works, vol. 5

See papers of N. Hitchin

See the paper of Kronheimer and Nakajima

94. Representations of Quivers

The theory of quiver representations is an interesting branch of linear algebra.

1. A. King, Moduli of representations of finite-dimensional algebras, Quart. J. Math. 1994

2. M. Reinecke, Hardar-Narasimhan system...

3. M. Auslander, I. Reiten, and S. O. *Representation theory of Artin algebras*, volume 36 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. Corrected reprint of the 1995 original.

Quivers have found extensive use in mathematical physics and string theory in recent years. For a tiny sampling see:

1. D-branes, quivers, and ALE instantons. Michael R. Douglas, Gregory W. Moore e-Print Archive: hep-th/9603167

2. Papers of Cecotti and Vafa on wall-crossing and BPS degeneracies

3. Geometric engineering of (framed) BPS states Wu-yen Chuang, Duiliu-Emanuel Diaconescu, Jan Manschot, Gregory W. Moore, Yan Soibelman, e-Print: arXiv:1301.3065

95. Construction of infinite dimensional algebras from quivers

One of the more remarkable applications of the ADHM construction has been the construction of representations of affine Lie algebras using the moduli space of instantons on ALE spaces by Nakajima.

This topic has been nicely reviewed in expository works by Prof. H. Nakajima. See his home page, especially his lectures on Hilbert schemes of points.

Look up Ringel-Hall algebras.

Papers of D. Joyce, M. Kapranov,...

This topic is part of a broader subject known as "geometric representation theory." See the book of V. Ginzburg.

96. McKay correspondence and its D-brane interpretation

The McKay correspondence is a beautiful story about the relationship between the geometry and topology of the resolution $\mathbb{C}^2/\Gamma$, where Γ is a discrete subgroup of $SU(2)$ (labelled by A,D,E) and the corresponding A,D,E Lie groups and Lie algebras.

1. Explain the classical McKay correspondence, especially, explain how the resolution of the singularity $\mathbb{C}^2/\Gamma$ is a manifold whose intersection form describes the A,D,E root lattice.

2. Consider the theory of D-branes on $\mathbb{C}^2/\Gamma$ and explain the McKay correspondence in this setting. (A paper of Aspinwall and Plesser on the arxiv might be useful in getting started on this.

3. Discuss generalizations of the the McKay correspondence

a. higher dimensions: Bridgeland, King, and Reid.

b. quantum Mckay correspondence: Martinec and Moore, Moore and Parnachev

97. Singularities Of Elliptic Fibrations, Kodaira Classification, And F-Theory

An amazing connection between algebraic geometry, Lie group theory, gauge theory and string theory appears in the vast subject of F-theory. (This project is quite ambitious.)

1. Explain what an elliptic fibration is and the relation of the Kodaira classification to simple Lie algebras.

2. Explain the relation to string theory and string theory compactification

Some references:

1. Rick Miranda, "The Basic Theory Of Elliptic Surfaces"
2. Paul Aspinwall, P. S. Aspinwall, "K3 surfaces and string duality," [arXiv:hep-th/9611137 [hep-th]].
3. T. Weigand, "F-theory," [arXiv:1806.01854 [hep-th]].
4. W. Taylor, "TASI Lectures on Supergravity and String Vacua in Various Dimensions," [arXiv:1104.2051 [hep-th]].

98. Analogies between geometry and number theory

Explain the geometry of schemes and what an arithmetic variety is. Local/global principle. Analogies between localization at a prime ideal and power series near a point of a Riemann surface. The geometric Langlands program. The arithmetic analog of π_1.

Review the the Dessin des Enfants of Grothendieck and its relation to $Gal(\bar{Q}/Q)$.

99. Hyperbolic geometry in 3 dimensions

Explain Thurston's program for classification of 3-manifolds. (This is a hard project.)

Explore implications for

– Possible fundamental theories of elementary membranes.

– 3d quantum gravity.

1. Thurston's book.
2. Review by Scott.
3. Book on quantum gravity in 2+1 dimensions by S. Carlip.
4. Book by John Morgan on Perelman's proof of the Poincare conjecture.
5. See online lectures by John Morgan at the MSRI website.

Explore applications to string theory compactifications and black holes in 3 dimensions.

100. Discrete subgroups of $SL(n, C)$ for $n > 2$

Describe what is known about the classification of discrete subgroups for higher rank Lie groups.

101. Freudenthal-Tits Magic Square and the Construction of Exceptional Groups

1. N. Jacobsen, *Exceptional Lie Algebras*

2. P. Ramond, Notes on Exceptional groups (unpublished, I can give you a copy).

102. BMS group

Describe the Bondi-Metzner-Sachs group of asymptotics symmetries in general relativity.

103. Asymptotic symmetries of AdS3: Brown-Henneaux Virasoro action on the Hilbert space of three-dimensional gravity

104. Gravity near a cosmological singularity, chaos, and the BKL phenomenon

Near a spacelike singularity the modes of the gravitational field decouple along spatial slices, and one obtains Kasner like solutions. Remarkably, the evolution of these solutions is closely related to the mathematics of the Weyl group and Weyl chambers of certain infinite dimensional Lie algebras. See

1. V.A. Belinskii, I.M. Khalatnikov, and E.M. Lifshitz, "Oscillatory approach to a singular point in the relativistic cosmology," Adv. Phys. **19**(1970)525; "A general cosmological solution of the Einstein equations with a time singularity, " Adv. Phys. **31** (1982)639.

2. T. Damour and M. Henneaux, "E_{10}, BE_{10} and Arithmetical Chaos in Superstring Cosmology," hep-th/0012172

3. T. Damour, M. Henneaux, B. Julia, and H. Nicolai, "Hyperbolic Kac-Moody Algebras and Chaos in Kaluza Klein Models," hep-th/0103094

4. T. Damour, M. Henneaux, and H. Nicolai, "Cosmological Billiards," hep-th/0212256

5. T. Damour, "Cosmological singularities, billiards, and Lorentzian Kac-Moody algebras," arXiv:gr-qc/0412105. This is a nice review.

105. The Geroch group

The Geroch group is an infinite dimensional group generating solutions of Einstein's equations with a two-dimensional group of isometries.

The most modern treatments can be found in papers of H. Nicolai.

1. R. Geroch, "A Method for generating new solutions of Einstein's equations II,"