Preprint typeset in JHEP style - HYPER VERSION

Physics 618: Applied Group Theory: Spring, 2022

Gregory W. Moore

Abstract: January 5, 2022

-TOC-

Contents

1. What the course is about

"A man who is tired of group theory is a man who is tired of life." – Sidney Coleman[1]

This is a course about groups and their representations, with an emphasis on topics arising in physical applications. This is a vast topic, with an unbelievably wide spectrum of applications to physics. I will cover basic definitions and examples, and also illustrate these with more advanced applications.

The course website is

http://www.physics.rutgers.edu/∼gmoore/618Spring2022/GroupTheory-Spring2022.html

2. Boundary Conditions

This course is primarily intended for advanced undergraduate and graduate students in physics intending to specialize in theory. There will be some bias towards particle theory, although there is much here that is useful to the nuclear and condensed matter theorist. It should also contain much of interest to the mathematics student with some interest in physics.

I will try to keep prerequisites to a minimum. Occasionally I will introduce examples or even sections based on more advanced material, but the main development will be kept elementary. Occasionally a knowledge of basic differential geometry and topology is useful, particularly for the material on Lie groups.

3. Tentative Plan

The main set of lecture notes I have been working with in recent versions of this course are

http://www.physics.rutgers.edu/ gmoore/618Spring2022/GTLect1-AbstractGroupTheory-2022.pdf

and

http://www.physics.rutgers.edu/ gmoore/618Spring2019/GTLect2-LinearAlgebra-2019.pdf

There is much more material here than can be covered in a single semester. Depending on the backgrounds and interests of the students I will select and alter this material.

In an ideal world a full course would cover the following material:

3.1 Abstract Group Theory

Basic definitions: Group, homomorphism, isomorphism. Permutation group and shuffles. Generators and relations. Cosets and conjugacy, Lagrange theorem, Sylow theorem. Exact sequences. Elementary number theory. Automorphisms. Group extensions. Group Cohomology. Heisenberg groups and Heisenberg extensions. Structure theorems: Kronecker structure theorem, finite simple groups. Categories and Groupoids. Lattice gauge theory.

We touch on some applications of symmetries to quantum mechanics, stressing the importance of extensions in passing from classical to quantum descriptions of a physical

[1]With apologies to Dr. Johnson.

system. Wigner's theorem. Extensions: Linear and anti-linear actions. Quantum mechanical implementation of symmetry.

3.2 Linear Algebra User's Manual

Rings and modules, vector spaces and linear transformations, real vs. complex vector spaces, kernel, image and cokernel, Jordan decomposition, nilpotent orbits, sesquilinear and hermitian forms, Hilbert space, unitary and hermitian operators, spectral theorem, putting matrices in canonical form, families of matrices and operators, WKB, determinants and pfaffians, super-linear algebra, integral quadratic forms and lattice, quadratic refinements, elementary homological algebra: Ext and Tor. Quivers.

Some material on Dirac von Neumann axioms for quantum mechanics.

3.3 Groups and Symmetry

Transformation groups and Cayley's theorem. Orbits. Spaces of orbits, bundles, orbifolds. Isometry groups of Euclidean space. Symmetries of regular objects in 2 and 3 dimensions. Platonic solids. Finite subgroups of SU(2) and O(3). Crystals. Rubik's cube. Simple singularities in 2 complex dimensions. Symmetric functions.

3.4 Introduction to representation theory

Basic definitions: Representations and co-representations. Unitary representations. Projective representations. Regular representation. Reducible and irreducible representations. Schur's Lemmas. Decomposition of the regular representation for finite groups: Peter-Weyl theorem. Fourier analysis. Bloch's theorem in solid state physics. Characters and character tables. Decomposing tensor products. Group ring and group algebra. 2D TFT and Frobenius algebras. Projection operators. Group theory classification of small oscillations, molecular frequencies. Representations of the symmetric group. Tensors and Schur-Weyl duality. Application to 2d bosonization. Induced representations and Frobenius reciprocity.

3.5 Survey of matrix groups: GL, SL, SO, SU, Sp etc.

Definition of a Lie group. Components, compactness, universal cover. GL and SL. Groups preserving sesquilinear forms. Grassmannians. Groups preserving symmetric bilinear forms. Orthogonal group O(p,q). Components, P and T. Spin. Groups preserving anti-symmetric bilinear forms: Symplectic groups. Lagrangian subspaces. Spaces of complex structures. Matrix groups vs. Lie groups - the example of the Heisenberg group. Statement of the classification of compact connected simple Lie groups.

3.6 The wonderful 2×2 matrix groups

$SU(2)$, $SL(2,\mathbb{R})$, and $SL(2,\mathbb{C})$. Relations to geometry of constant curvature metrics. Mobius groups. Relations to low-dimensional rotation and Lorentz groups. Finite dimensional reps. Massless wave equations. Twistors.

3.7 Quaternions and octonions

Division algebras. Composition algebras. Hurwitz theorem.

3.8 Lie algebras from Lie groups

Lie algebra and invariant vector fields. Expontential map and the Baker-Campbell-Hausdorff formula. Abstract Lie algebras. Adjoint representation. Lie's theorem. Survey of Lie algebras for the classical matrix groups. Central extensions and Lie algebra cohomology. Maurer-Cartan equation. Invariant metrics on Lie groups. Haar measure. Initial remarks on infinite-dimensional Lie algebras.

3.9 Harmonic oscillators: Symplectic and metaplectic groups.

Oscillator constructions of Lie algebras. Oscillator representations. Bogoliubov transformations.

3.10 Conformal groups and conformal algebras

Definitions and relations to orthogonal groups. Their relation to deSitter and anti-deSitter spaces (as used in the AdS/CFT conjecture). Wigner-Inonu contraction. Galilean group

3.11 Clifford algebras and spinors

Real and Complex. All signatures, all dimensions. Mod 8 periodicity. Relation of products of spinors to antisymmetric tensors. $\mathbb{Z}_2$-graded Clifford algebras.

3.12 Superconformal and Superpoincare algebras

Definitions and classification. Supersymmetric quantum mechanics.

3.13 Structure of semisimple Lie algebras.

Root systems and root lattices. Weyl groups. Cartan classification. Serre presentation.

3.14 Kac-Moody and affine Lie algebras, and beyond

3.15 Highest weight representations of semisimple Lie algebras

Verma modules. Weyl character formula.

3.16 Induced representations

3.17 Unitary Representations of the Lorentz and Poincaré groups

3.18 Representations of supersymmetry algebras

3.19 Nonlinear sigma models: Quantum field theories defined by group manifolds and homogeneous spaces.

3.20 Geometry and topology of Lie groups

4. Administrative- Spring 2022

1. Because of the pandemic all lectures this semester will be via Zoom. I will send a link to the members of the class. I strongly encourage you to keep your video on so I can judge if people are following. If the pandemic recedes significantly I will discuss with the class to assess the desirability of returning to in-person lectures.

2. Because of the pandemic I cannot hand out lecture notes. You can obtain them from the web page. I will also post a pdf of what I actually have written on the Ipad during the lecture as well as a video recording of the lecture on the course web page. There is a list of useful references and textbooks at the end of this handout.

3. The grade for those taking the course for credit will be based on a short paper and possibly a presentation given at the end of the semester. I will hand out topics towards the middle of the course.

4. I will probably not hand out problem sets to be graded. However, the lecture notes contain plenty of exercises. You are encouraged to do them in the strongest possible terms.

   You cannot hope to have any real knowledge without trying to do exercises and solve problems.

5. As a courtesy to others, PLEASE DO NOT EAT DURING CLASS. This is amplified by Zoom/WebEx. It is acceptable to have a drink available.

6. Please note: Auditors are welcome. However, if you are taking the course for credit then attendance at the lectures IS NOT OPTIONAL. I will take attendance via chat. If you are a true genius and do not need to hear the lectures to write a brilliant breakthrough term paper, then you are excused. Please be aware that I will then judge your term paper by those standards.

5. Some sources

There is no formal textbook. The following is a list of sources I have used. There are many other texts available. Go to the library and find your favorite. If you find something great that is not on this list please let me know!

Basic math texts:

1. D. Joyner, *Adventures in Group Theory*, a gentle introduction with lots of fun applications, some of which I borrowed.

2. I.N. Herstein, Topics in Algebra: Chapter 2 has an excellent summary of basic group theory for the mathematically inclined. Chapters on rings and fields are also excellent. A little bit like having your mathematician uncle sit down and explain things to you.

3. N. Jacobsen, Basic Algebra I, also for the mathematically inclined. Much more demanding than Herstein and Joyner. No nonsense.

4. J.-E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer GTM: A very fine introduction to the structure of semisimple Lie algebras.

5. R. Carter, G. Segal, and I. MacDonald, *Lectures on Lie Groups and Lie Algebras*, London Mathematical Society Students Texts, 32. Beautifully concise.

6. F.R. Gantmacher, *Applications of the Theory of Matrices*, Interscience, 1959

7. P. Lancaster, *Theory of matrices*, Academic Press, 1969.

8. Bourbaki: The gold standard of ultra-rigorous mathematics.

Books written about group theory by physicists for physicists:

1. Daniel Arovas has an excellent set of lecture notes on applications of group theory to physics. They are somewhat complementary to the lecture notes for this course but I highly recommend them. Go to

   https://courses.physics.ucsd.edu/2018/Spring/physics220/lectures.html

2. R.N. Cahn, *Semi-Simple Lie Algebras and Their Representations*, Frontiers in Physics. Basic material on Lie algebras designed for a particle physicist audience. Also see the book by Howard Georgi.

3. DiFrancesco, P. Mathieu, D. Senechal, *Conformal Field Theory.* This book is about conformal field theory in two dimensions with an emphasis on the WZNW model and related CFT's. It has a nice summary of some aspects of Lie algebra and affine Lie algebra theory since that is an essential background for studying the 2d WZNW conformal field theory.

4. J. Fuchs and C. Schweigert, *Symmetries, Lie Algebras and Representations: A graduate course for physicists*, Cambridge. This is a very good source for material on semisimple Lie algebras. The authors are very careful.

5. J. Fuchs, *Affine Lie Algebras and Quantum Groups*, Cambridge. This continues where Fuchs and Schweigert left off and discusses in depth affine Lie algebras and applications to conformal field theory.

6. H. Georgi, Lie Algebras in Particle Physics: Group representation theory for particle physicists. Very readable with good exercises, but sometimes it is a bit sloppy.

7. R. Gilmore, Lie Groups, Lie Algebras, and Some of Their Applications. Dover. Excellent down-to-earth discussion of matrix groups and Lie groups and homogeneous spaces of Lie groups, and other related geometry.

8. M. Hamermesh, Group Theory: a good reference but a bit turgid.

9. C.J. Isham, Modern Differential Geometry for Physicists: a very gentle introduction to the differential geometry of Lie groups and their cosets.

10. D. Mermin, Rev. Mod. Phys. **51** (1979)591. An unusual summary of group theory which has much interesting information.

11. L. Michel, Symmetry, invariants, topology, Physics Reports, Volume 341, Issues 16, Pages 3-396 (February 2001) Among many other things, this nice monograph covers crystallography from a precise mathematical viewpoint.

12. Miller, *Symmetry groups and applications*: Much of the lecture material on crystallography and discrete subgroups of the group of Euclidean isometries was drawn from this book. The book is now out of print but can be accessed electronically at:

    http://www.ima.umn.edu/∼ miller/symmetrygroups.html

13. L. O' Raifeartaigh, *Group Structure of Gauge Theories*, Cambridge. Gives applications to unified field theory model building.

14. P. Ramond, *Group Theory: A Physicist's Survey.* A very beautiful book by a renowned string theorist. It has a stress on applications to particle physics. It includes some nice material on exceptional structures and Chevalley groups not readily available elsewhere.

15. I.V. Schensted, *A Course on the Application of Group Theory to Quantum Mechanics*: very readable.

16. J. Shapiro taught this course at Rutgers many times. His choice of topics is rather different. He has a nice set of lecture notes at

    https://www.physics.rutgers.edu/shapiro/618/lects.shtml

17. R. Slansky, "Group Theory for Grand Unified Model Building," Physics Reports, Vol. 79, pp. 1-128. This classic review was a major source of information for a generation of particle theorists. It has many useful tables for working with representations of simple Lie algebras.

18. J.D. Talman and E.P. Wigner, Special Functions: This book is a set of lecture notes on a course given by the legendary physicist Eugene Wigner. It explains the group theoretic approach to the theory of special functions.

19. Wu-Ki Tung, Group Theory in Physics. Good basic text.

Books written about group theory by mathematicians for physicists:

1. S. Sternberg, *Group theory and physics.* Sternberg is a matmematician and the book is written from a mathematicians perspective of applications to physics. It has some very nice material.

2. P. Woit, "Quantum Theory, Groups and Representations: An Introduction,"

   https://www.math.columbia.edu/ woit/QM/qmbook.pdf.

   A course covering quantum mechanics through the standard model of particle physics, with the stress on group theory all the way through.

For some history and culture associated with group theory see

1. H. Weyl, *Symmetry.* Beautiful nontechnical exposition.
2. H. Weyl, *The Theory of Groups and Quantum Mechanics.* Dover. There is a legal (I think) pdf version on the web. Hermann Weyl was one of the great mathematicians of the 20th century. He made important contributions to physics and had important interactions with physicists. This book, based on lectures in Princeton in 1929 offers an interesting historical snapshot of the interactions between math and physics from just after the great quantum mechanics revolution of 1925-1926.
3. E.T. Bell, *Men of Mathematics*
4. Mario Livio, *The Equation That Couldn't Be Solved: How Mathematical Genius Discovered the Language of Symmetry*
5. Hans Wussing *The Genesis of the Abstract Group Concept byThe Genesis of the Abstract Group Concept*

Online resources for computations with group theory and lattices:
http://www.gap-system.org/
https://en.wikipedia.org/wiki/Magma(computer algebra system)

Some of the later topics will touch on quantum field theory and supersymmetry. A very few references for these topics include:

Quantum mechanics:

1. P.A.M. Dirac, *Quantum Mechanics*, Fourth edition. Oxford. This is one of the first texts, and still a classic. Many more were to follow.
2. L. Takhtadjan, Quantum Mechanics for Mathematicians
3. P. Woit's book above.
4. S. Weinberg, *Lectures On Quantum Mechanics*,
5. T. Banks, *Quantum Mechanics: An Introduction*

Quantum Field Theory:

1. Ramond
2. Peskin and Shroeder
3. Banks, Modern Quantum Field Theory (A Concise Introduction)
4. S. Weinberg, *Quantum Theory of Fields, vols. 1-3*

Supersymmetry:

1. J. Bagger and J. Wess, *Supersymmetry and Supergravity.* Princeton
2. P. Freund, *Supersymmetry.* Cambridge; Nice summary of super-Lie algebras
3. J. Lykken - TASI lectures hepth/9612114
4. P. West, *Introduction to supersymmetry and supergravity.*
5. M. Sohnius, "Introducing supersymmetry," Phys. Rep. **128**(1985) 39.
6. J. Strathdee, "Extended superpoincare supersymmetry" Int. J. Mod. Phys. **A2**(1987) 273
7. A. Van Proeyen, "Tools for supersymmetry," hep-th/9910030
8. A. Bilal, "Introduction to supersymmetry," hep-th/0101055
9. Freedman and Van Proeyen, Supergravity