Physics 618 2020

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Important remark I forgot to make:

Central Extensions & Projective Reps.

Suppose we have a proj. rep. of $G$

\[ \rho : G \longrightarrow \text{GL}(V) \]

\[ \in \text{U}(1) \]

\[ \rho(g_1) \rho(g_2) = \zeta(g_1, g_2) \rho(g_1 g_2) \]

proj. rep.

satisfies cocycle identity

\[ \tilde{G} \text{ is the true symmetry group.} \]

True rep. of $\tilde{G}$ is $(z, g)$

\[ (z_1, g_1) \cdot (z_2, g_2) = (z_1 z_2 \zeta(g_1, g_2), g_1 g_2) \]

\[ \tilde{\rho}(z, g) = z \rho(g) \]

$\tilde{\rho}$ TRUE REP of $\tilde{G}$. 
Weinberg allows a priori
\[ \rho(g_1) \rho(g_2) \cdot \psi = C(g_1, g_2, \psi) \rho(g_1 g_2) \cdot \psi \]
\[ \text{phase.} \]
Weinberg then shows that
\[ C(g_1, g_2, \psi) \text{ must be independent of } \psi \]
One could take
\[ V_1 \oplus V_2 \]
\[ \uparrow \quad \uparrow \]
\[ \text{inequivalent projective reps of } G. \]
\[ \tilde{G}_1 \quad \tilde{G}_2 \]
\[ \text{True rep of } \tilde{G}_1 \times \tilde{G}_2 \]
\[ S = \int \left( \frac{1}{2} I \dot{\phi}^2 + \frac{eB}{2\pi} \phi \right) dt \]

EOM \( \dot{\phi} = 0 \)

Classical symmetry group: \( G = O(2) \)

\( P : \phi \rightarrow -\phi \)

\( R(\alpha) : \phi \rightarrow \phi + \alpha \quad \alpha \sim \alpha + 2\pi \)

\( e^{i\phi} \rightarrow e^{-i\phi} \)

\( e^{i\phi} \rightarrow e^{i\alpha} e^{i\phi} \)
In general \( A_\mu = (\phi, \vec{A}) \)

Field strength tensor

\[
F_{\mu\nu} = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & -E_0 \\
0 & 0 & 0 & E_0 \\
0 & 0 & 0 & 0 \\
1 & 2 & 3 & 0
\end{pmatrix}
\]

Coulomb potential

\[
F_{0i} = \partial_0 A_i - \partial_i \phi = E_i
\]

\[
F_{ij} = \frac{1}{2} \epsilon_{ijk} B_k
\]

\[
F_{12} = B_3
\]

\[
F_{23} = B_1
\]

\[
F_{13} = -B_2
\]

\[
F_{\mu\nu} = \begin{pmatrix}
0 & 0 & 0 & -E_0 \\
0 & 0 & 0 & E_0 \\
0 & 0 & 0 & 0 \\
1 & 2 & 3 & 0
\end{pmatrix}
\]
\[ P^2 = 1 \]
\[ R(\alpha) R(\beta) = R(\alpha + \beta) \]
\[ P R(\alpha) P = R(-\alpha) = R(\alpha)^{-1} \]

**Quantum Theory:**

*Canonical momentum conj. to \( \phi \) is*

\[ L := \frac{\delta S}{\delta \dot{\phi}} = \mathcal{I} \dot{\phi} + \frac{eB}{2\hbar} \]

\[ H = \frac{\hbar^2}{2m} \left( -i \frac{\partial}{\partial \phi} - \mathcal{B} \right)^2 \]

\[ \mathcal{B} := \frac{eB}{2\pi \hbar} \in \mathbb{R} \]

\[ \mathcal{H} = L^2(S^1) \]
$H_B$ has a complete set of eigenvectors

$$\psi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \text{ for } m \in \mathbb{Z}$$

$$H_B \psi_m = E_m(B) \psi_m$$

$$E_m(B) = \frac{\hbar^2}{2I} (m-B)^2$$

For each $m$ only one e.v.

Long list of remarks:

1. Action makes sense $\phi \in \mathbb{R}$ rather than $\phi \sim \phi + 2\pi$.

Then $m$ is not quantized and the spectrum of $H_B$ is indpt of $B$. Topological term has no effect.
2. However, if \( \phi + 2\pi \) single valued position is \( e^{i\phi} \) then the topological term matters - even though it is a total derivative.

3. Eigenspaces

If \( B \notin \mathbb{Z} \), all eigenspaces are 1-dim: know energy \( \Rightarrow \) unique \( m \).

\[
E_m(B) = \sim (m-B)^2
\]
If $2B \in \mathbb{Z}$ the energy eigenspaces can have degeneracies.

If $2B$ odd, e.g. $B = \frac{1}{2}$

$$E_m = E_{2B-m}, \quad 2B \in \mathbb{Z}$$

all eigenspaces are $\dim = 2$.

If $2B$ is even, all eigenspaces are $\dim = 2$ EXCEPT $m = B$

ground state is $1$-dime.
4. Spectrum is periodic in $\mathcal{B}$

\[ U H \mathcal{B} U^{-1} = H \mathcal{B} + 1 \]

"physics is periodic in $\mathcal{B}$"

5. Physically realized in mesoscopic systems "Coulomb blockade"

6. This system is a toy field theory

In field theory the "fields" are functions

\[ \Phi : M \rightarrow X \]

\[ \uparrow \quad \uparrow \quad \uparrow \]

\[ \text{spacetime} \quad \text{"target space"} \]

\[ F = \text{Space of fields} = \text{Map} (M \rightarrow X) \]
$S[\Phi]$ - action

**nonrelativistic**

e.g. particle moving on a Riemannian manifold:

\[ ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu \]

\[ S = \int \frac{m}{2} g_{\mu\nu}(x(t)) \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} dt \]

\[ F: \text{Map} \left( M \rightarrow X \right) \]

up

\[ \text{time} \quad M = \mathbb{R} \]

\[ M = [t_{\text{in}}, t_{\text{fin}}] \]

\[ M = S^1 \]
Generalizes further $M$ is $d+1$ dimensional with Lorentzian metric

$$ds^2_{M} = h_{ab}(\sigma) \, d\sigma^a \, d\sigma^b$$

$\sigma^a \quad a = 0, 1, \ldots, d$

$$ds^2_{x} = g_{\mu\nu}(x) \, dx^\mu \, dx^\nu$$

$$\int_M \frac{m}{2} \, h_{ab}(\sigma) \, g_{\mu\nu}(x(\sigma)) \, \bar{\partial}_a x^\mu \, \partial_b x^\nu \, \sqrt{\text{det} h_{ab}} \, d^{d+1} \sigma$$

Nonlinear $\sigma$-model.

$$F = \text{Map} \left( M \rightarrow X \right)$$

Q.M. of a particle is a 0+1 dil$$
\text{d field theory.} \quad (M = 1+1 \text{ dink is basis for string theory.})$

If in addition $X$ has a

gauge field on it, and particle
has charge $e$ then action

$$S = \int \frac{1}{2} m g_{\mu\nu}(x(t)) \dot{x}^\mu \dot{x}^\nu \, dt$$

$$+ e A_\mu(x(t)) \dot{x}^\mu \, dt$$

Our case: field is a map

$$e^{i\phi} : \mathbb{M} \rightarrow S^1 = \text{unit complex number}$$

$$\mathbb{R}$$

$$\phi \rightarrow -\phi \quad \text{"parity"} \quad \text{particle on ring viewpoint}$$

Charge conjugation:

$$e^{i\phi} \rightarrow (e^{i\phi})^*$$

field theory viewpoint.

Also there are world volume symmetries:

$$t \rightarrow t + t_0 \quad \text{and} \quad t \rightarrow -t$$

We might talk about these later
7. $\theta$-terms in other field theories

Maxwell's theory in 1+1 dim's on a cylinder

$$M$$

$$A_\mu \quad \mu = 0, 1$$

$$F_{\mu\nu} = \text{only } F_{01} = -F_{10} \text{ can be nonzero}$$

$$S_{\text{Max}} = \int \frac{1}{2e^2} dx^0 dx^1 F_{01}^2 = \frac{E^2}{E^2}$$

$$E^2 - \Box$$

but we can add a topological term.
\[
S = \int \frac{1}{2e^2} F_{01}^2 \, dx^0 dx^1 + \int \frac{\theta}{2\pi} F_{01} dx^0 dx^1
\]

\[\partial_0 A_1 - \partial_1 A_0\]

Choose \( A_0 = 0 \) gauge

In KK reduction

\[A_1 = \sum_{n \in \mathbb{Z}} e^{2\pi i n x^1 / \rho} A^{(n)}\]

Working with the zeroth faster coeff.

\[e^{i \phi(t)} = e^{i \oint_{s'} A} = e^{i \oint_{s'} A_{0} (dx^1)}\]

\[\Rightarrow \text{get the above Quantum System} \quad \Theta \sim e^{2\pi B}\]

\[\Theta = 2\pi B\]
Differential forms: Proper mathematical formalism for discussing gauge theories

\[ F = \frac{1}{2!} F_{\mu\nu} \, dx^\mu \wedge dx^\nu \]

\[ dF = 0 \]
\[ d*F = 0 \] (in vacuum)

Maxwell's eqs.
There is also an important topological term in 3+1 gauge theory.

Maxwell case: $U(1)$ gauge theory

$$S' = \int d^4x \left( \frac{1}{4e^2} \text{tr}(F_{\mu\nu}F^{\mu\nu}) + \int \frac{\Theta}{8\pi} \epsilon_{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda} d^4x \right)$$

$\sim \left( E^2 - B^2 \right)$

Show: this is a total derivative!

Effective theory of electromagnetic fields in materials such $\Theta$-terms appear.

If we have $P$-inv. or Time Reverse $\theta \rightarrow -\theta$ and $\theta \sim \theta + 2\pi$

(Remember "physics is periodic in $B"\)
If Material is an insulator

\[ \Theta = 0 \] “normal insulator”
\[ \Theta = \pi \] “topological insulator”

If we consider nonabelian gauge fields

\[ SU(3) \times SU(2) \times U(1) \]

Strong \hspace{1cm} Electroweak

Topological terms matter and \[ \Theta_{SU(3)} \]
induces an intrinsic electric dipole moment for the neutron: linear in \[ \Theta \]

Experimental: \[ |\Theta_{SU(3)}| < 10^{-9} \]

"Because of principle of naturalness" if no symmetry reason excludes a term in action - it should be there.
Field space of connections on a principal bundle $\mathcal{A}/\mathcal{G}$ has nontrivial topology: So fieldspace has nontrivial topology and topological terms matter.

Back to particle on the ring

Wigner tells us that some extension of $O(2)$ by $U(1)$ should act on Hilbert space commuting with Hamiltonian.

$R(\alpha) : \phi \rightarrow \phi + \alpha$

$\Psi_m \sim e^{im\phi} \rightarrow e^{im\alpha} e^{im\phi}$

$R(\alpha) \cdot \Psi_m = e^{im\alpha} \Psi_m$

$\rho(R(\alpha)) = \mathbf{R}(\alpha)$
\[ p(p) = p \quad \text{P: } \phi \rightarrow -\phi \]

\[ e^{im\phi} \rightarrow e^{-im\phi} \]

does not commute with Hamiltonian

\[ H_8 = \frac{1}{2\pi} \left( -i \frac{\partial}{\partial \phi} - \phi \right)^2 \]

\[ \phi \neq 0 \quad \text{H} \rightarrow -\phi \quad \phi \rightarrow -\phi \]

\[ H_\phi \text{ is not unitarily equiv. to } H_{-\phi} \]

P. \[ \psi_m = \sum \psi \quad 2\phi - m \]

If \[ a\phi \in \mathbb{Z} \]

If \[ 2\phi \notin \mathbb{Z} \quad \text{O(2) is broken by } g \quad \text{to } \text{SO}(2) \]
If $Z \in \mathbb{Z}$ then

\[ \mathcal{P} \cdot \psi_m = \psi_{2Z - m} \]

implies parity/charge conjugation and commutes with $H_B$.

Recall the classical operator relations:

\[ P^2 = 1 \]

\[ R(\alpha) R(\beta) = R(\alpha + \beta) \]

\[ PR(\alpha) P^{-1} = R(\alpha) = R(\alpha)^{-1} \]

What about the quantum ops?

\[ P^2 = 1 \quad \checkmark \]

\[ R(\alpha) R(\beta) = R(\alpha + \beta) \quad \checkmark \]
But - exercise! - you check

\[ PR(\alpha) P = e^{i(2\pi)\alpha} R(-\alpha) \]

Note: equation only makes sense when \( \alpha \in \mathbb{Z} \).

Now consider the group of operators generated by

\[ \mathbb{Z}, R(\alpha), z I_L \quad z\in U(1) \]

How is this related to the naive answer \( U(1) \times O(2) \)?

\[ O(2) = SO(2) \times \mathbb{Z}_2 \]

\[ \text{Aut}(SO(2)) \subseteq \tau \]

\[ \sigma: R(\alpha) \rightarrow R(-\alpha) \]