Physics 618 2020

April 7, 2020
Adding a Potential to Particle on a Ring

\[ \langle \mathbf{z}, \mathbf{B} \rangle \]
\[ \mathbf{m}_e \rightarrow \mathbf{\phi}(t) \]

\[ \mathbf{S} = \int \frac{1}{2} I \mathbf{\phi}^2 dt + \int \frac{eB}{2\pi} \mathbf{\phi} dt \]
\[ \mathbf{H}_B = \frac{\hbar^2}{2I} \left( -i \frac{\partial}{\partial \mathbf{\phi}} - \mathbf{B} \right)^2 \quad \text{on } L^2(S^1) \]

Classical \( O(2) = \langle \mathbf{P}, R(\alpha) \rangle \)

\[ \mathbf{P}: \mathbf{\phi} \rightarrow -\mathbf{\phi} \quad \mathbf{R}(\alpha): \mathbf{\phi} \rightarrow \mathbf{\phi} + \alpha \]
\[ \mathbf{P} R(\alpha) \mathbf{P} = R(\alpha)^{-1} = R(-\alpha) \]

Quantum: \( 2B \in \mathbb{Z} \) (parity symmetry)

\[ \psi_m(\mathbf{\phi}) = \frac{1}{\sqrt{2\pi}} e^{im \mathbf{\phi}} \quad \text{e.v. of } H_B \]
\[ R(\alpha) : \psi_m \rightarrow e^{i m \alpha} \psi_m \]
\[ \mathcal{P} : \psi_m \rightarrow \psi_{2m^2 - m} \]
\[ \mathcal{P} R(\alpha) \mathcal{P} = e^{i 2m^2 \alpha} R(-\alpha) \]

\( 2B \in 2\mathbb{Z} \) modify \( R(\alpha) \) by coboundary.

\[ \hat{R}(\alpha) = e^{-i B \alpha} R(\alpha) \]

\( \mathcal{P} \hat{R}(\alpha) \mathcal{P} = \hat{R}(\alpha)^{-1} = \hat{R}(-\alpha) \)

realizes \( O(2) \) on Hilbert space.

But, if \( 2B \) odd integer

\[ \text{Spin}(2) \xrightarrow{\pi} \text{SO}(2) \]
\[ \hat{R}(\alpha) = \exp(\hat{\alpha} \sigma^1 r^2) \]
\[ \rightarrow R(2\alpha) \]
Extend to

\[ \text{Pin}^+(2) \xrightarrow{2:1} \text{O}(2) \]

\[ \langle \hat{P}, \hat{R}(\hat{\alpha}) \rangle \rightarrow \langle P, R(\alpha) \rangle \]

\[ \hat{P} \hat{R}(\hat{\alpha}) \hat{P} = \hat{R}(\hat{\alpha})^{-1} \]

We can represent \( \text{Pin}^+(2) \) on \( \mathcal{H} \)

\[
\begin{align*}
\rho(\hat{R}(\hat{\alpha})) &= e^{-2i(2\B)\hat{\alpha}} R(2\B) \\
\rho(\hat{P}) &= P
\end{align*}
\]

Satisfy defining rels of \( \text{Pin}^+(2) \).

E.g. \( \B = \frac{1}{2} \) \( \mathcal{H}_{\text{pqel}} = \text{Span} \{ \psi_0, \psi_1 \} \)

\[
\begin{align*}
\rho(\hat{R}(\hat{\alpha})) &\big|_{\mathcal{H}_{\text{pqel}}} = \begin{pmatrix} e^{-i\B} & 0 \\
0 & e^{i\B} \end{pmatrix} \\
\rho(\hat{P}) &\big|_{\mathcal{H}_{\text{pqel}}} = \begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix}
\end{align*}
\]
$H_B + U(\phi)$ where $2B = \text{odd}$

$U(\phi) = \sum u_n \cos(2n\phi)$

breaks $O(2) \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$

$r : \phi \mapsto \phi + \pi$

1 $\rightarrow \mathbb{Z}_2 \xrightarrow{i} D_4 \xrightarrow{\tau} \mathbb{Z}_2 \times \mathbb{Z}_2$

$\hat{r} = e^{iB\pi} R(\pi)$ where $R, P$ operate on $\mathbb{R}^2$.

$\pm \hat{P} \leftrightarrow \hat{P}$

These operators commute with $H @ U_n = 0$
\[ <\rho, \hat{r}> \text{ generate a group of operators on } \mathcal{H} = D_4 \]

\[ D_4 : \langle x, y \mid x^2 = 1, y^4 = 1, xyx = y^{-1} \rangle \]

(Symmetries of the square)

\[ \rho^2 = 1, \quad \hat{r}^4 = 1, \quad \rho \hat{r} \rho = \hat{r}^{-1} \]

In addition, \( \hat{r}^2 = -1 \) on \( \mathcal{H} \) is not a relation of \( D_4 \), but it is compatible and it tells us what reps appear.

\[ y \rightarrow \hat{r}, \quad \hat{r}^2 = -1 \]

\[ \mathcal{H} = \bigoplus \text{2-dim eigenspaces all 2d irreps of } D_4 \]
Representations of $D_4$

1-dimensional representations

In a 1-dim rep.

\[ \rho(x) = x \in \mathbb{C} \]
\[ \rho(y) = \zeta \in \mathbb{C} \]

Satisfy relations defining a group

\[ x^2 = 1 \quad \zeta^4 = 1 \quad \zeta^2 x = \zeta^{-1} \]

\[ \zeta = \zeta^{-1} \Rightarrow \zeta^2 = 1 \]

\[ x^2 = 1 \quad , \quad \zeta^2 = 1 \]

Conclusion: There are four distinct 1-dim irreducible reps of $D_4$

\[ x = \pm 1 \quad \text{and} \quad \zeta = \pm 1 \]
Fact about reps of finite groups

Finite # of distinct irreps

\[ d_\mu \quad \mu = 1, \ldots, s \]

\[ |G| = \sum_{\mu} d_\mu^2 \]

\[ 8 = 4 \cdot 1^2 + 2^2 \]

2 dime irreps: Action on the Qubit ground state

\[ \rho(y) = \begin{pmatrix} 2 & -i \\ -i & 0 \end{pmatrix} \]

\[ \rho(x) = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix} \]

Irreducible. If it were reducible

\[ S \begin{pmatrix} 2 & -i \\ -i & 0 \end{pmatrix} S^{-1} = \begin{pmatrix} x_1 & x_2 \\ \bar{x}_1 & \bar{x}_2 \end{pmatrix} \]

\[ S \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix} S^{-1} = \begin{pmatrix} x_1 & \bar{x}_2 \\ x_2 & \bar{x}_1 \end{pmatrix} \]
Perturbatively, two-fold degeneracy

nonperturbatively these mix and we get a 1-dim ground state.
No such $S$ exists.

Ground state @ $u_n \neq 0$ is 2 diml irrep of Dy!!

(Again assuming decomposition into irreps is a continuous function of $u_n$.)

Remarkable:

In double-well potential
This group theory shows that 28 odd ground state degeneracy persists and tunneling effects do not spoil it!

Quantum Statistical Mechanics of this system:

\[ Z := \text{Tr}_\mathcal{H} \left( e^{-\beta H} \right) \]

\[ \beta = \frac{1}{kT} \quad T = \text{abs. temp} \]

\[ k = \text{Boltzmann const.} \]

\[ k = \hbar = 1 \], hence this.

In our example \( \mathcal{H} = L^2(\mathbb{S}^1) \)

\[ H = H_0 \]
\[
Z = \sum_{m \in \mathbb{Z}} e^{-\frac{\beta}{2\pi} (m-B)^2}
\]

function of \(\frac{\beta}{2\pi}\) and \(B\).

Note: Manifestly periodic in \(B\)

\(B \to B+1\) and that \(m \to m+1\)

\(U H_0 U^{-1} = H_{B+1}\)

\(e^{-\beta E/2}\)

\(\beta \to \infty\) interesting limits

\(\beta \to \infty, T \to 0\), dominant terms come from ground.

\(\beta \to 0, T \to \infty\), "all terms contribute about the same"

expect a divergence.
Underneath that divergence is a very interesting duality.

Evaluate $Z$ in a different way. Using a path integral.

$$Z = \int d\phi <\phi|e^{-\beta H_0}|\phi>$$

Special case if

$$<\phi_2|e^{-tE_H}\phi_1> @ t_E = \beta$$

In solving Schrödinger eq.

$$i\hbar \frac{\partial \psi}{\partial t} = H \psi$$

$$\psi(t) = U(t) \psi(0)$$
$U(t) = e^{-\frac{it}{\hbar}}$

Under good conditions, this admits an "analytic continuation" to a region in the complex $t$-plane

$U(z) = e^{-\frac{iz}{\hbar}}$ \hspace{1cm} z \in \mathbb{R} \subset \mathbb{C}$

so that $\mathbb{R} \subset \mathbb{C} \cap \overline{\mathbb{R}}$

Suppose $\mathbb{R}$ were the entire $\mathbb{C}$

$z = -it_E \quad t_E \in \mathbb{R}$

"Euclidean time"

$-(dt)^2 + (dx^i)^2 \quad t \rightarrow -it_E$

$(dt_E)^2 + (dx^i)^2$
$e^{-tEH}$

$H$ is bounded below; this is a "good" (trace-class) operator.

for $t_E > 0$ for $t_E < 0$ $H$ unbonded above

"Analytic continuation to Euclidean time"

"Wick rotation"
In QM matrix element for propagation

\[ \langle \phi_2 | e^{-i\tau H} | \phi_1 \rangle \]

has a path integral interpretation.

\[ = \int [d\phi(t')] \phi(t) = \phi_2 \quad e^{i S[\phi(t')]} \]

\[ \phi(0) = \phi_1 \]

This path integral realization has a Wick rotation.
\[ \langle \phi_2 \mid e^{-\beta H} \mid \phi_1 \rangle := Z(\phi_2, \phi_1 | \beta) \]

\[
= \int \left[ d\phi(t) \right] \phi(\beta) = \phi_2 \\
\phi(0) = \phi_1
\]

\[
\exp \left[ -\frac{1}{\xi} \int_0^\beta \frac{1}{2} \mathcal{L} \phi^2 \, dt - 2i \int_0^\beta \mathcal{B} \phi \, dt \right]
\]

\[
Z = \int d\phi \langle \phi \mid e^{-\beta H} \mid \phi \rangle
\]

\[
\phi_2 = \phi(\beta) = \phi_1 = \phi(0)
\]

\[
\phi(t) : [0, \beta] \rightarrow \mathbb{R} / 2\pi \mathbb{Z}
\]

\[
\phi(0) = \phi(\beta), \quad \phi(t) \text{ is defined on } S^1 = \text{Circle of Euclidean time radius } \beta.
\]
\[ e^{i\phi(t)} : M \rightarrow X \] target space

Euclidean time manifold in this case

\[
\text{Path integral is an integral over the space of maps } S^1 \rightarrow S^2.
\]

**General fact:** For all Gaussian integrals the saddle point approximation (physics: semiclassical approximation) is **EXACT**

**First step:** Find the saddle points - i.e. find the solutions to eqs. of motion.
\[ e^{-\frac{1}{\hbar} \int_0^\beta H \phi^2 dt} - i \int_0^\beta \partial_t \phi dt \]

\[ \phi(0) = \phi_1, \quad \phi(\beta) = \phi_2 \]

**Eqs:** \[ \dot{\phi} = 0 \]

**Solution:**

\[ \phi(t) = \phi_1 + \frac{t}{\beta} (\phi_2 - \phi_1), \quad 0 \leq t \leq \beta \]

Not the only solution because \( \phi(t) \in \mathbb{R} \) rather than \( \phi(t) \in \mathbb{R} / 2\pi \mathbb{Z} \)

\[ e^{i\phi(t)} : S^1 \rightarrow S^1 \]

Can have **WINDING MODES**
The classical solutions are:

\[ \phi_{cl}(t) = \phi + \frac{\phi_2 - \phi_1 + 2\pi W}{\beta} t \]

\( W \in \mathbb{Z} \)

\[ e^{i \phi_{cl}(t)} : S^1 \xrightarrow{\sim} S^1 \]

Composition failure

Winding \( \neq W \).

\[ \dot{\phi} = 0 \quad \phi(0) = \phi_1 \mod 2\pi \mathbb{Z} \]

\[ \phi/\beta = \phi_2 \mod 2\pi \mathbb{Z} \]

In fact there are an \( \infty \) of solutions to classical equations of motion.
For historical reasons these are referred to as "instantons."

Going back to the path integral

\[ \phi(t) = \phi_1 \quad \text{mod} \quad 2\pi \\]
\[ \phi(t) = \phi_2 \quad \text{mod} \quad 2\pi \]

In S^7/S^5 analysis we expand around each stationary point and do the Gaussian integral around the stationary point and then sum over S.P. / Solutions of eqs. of motion. If \( \phi_{\text{cl}}(t) \) is a solution:

\[ \phi(t) = \phi_{\text{cl}}(t) + \phi_q(t) \]
\[ S(\phi(t)) = S[\phi_{el}(t)] + S[\phi_{\delta}] \]

\[ Z(\phi_2, \phi_1 | \beta) = Z_0 \cdot \frac{1}{Z_0} \sum_{w \in \mathbb{Z}} e^{-S[\phi_{el}(t)]} \]

\[ \sum_{w \in \mathbb{Z}} e^{-\frac{2\pi^2 T}{\beta} \left( w + \frac{\phi_2 - \phi_1}{2\pi} \right)^2} \cdot e^{2\pi i \beta \left( w + \frac{\phi_2 - \phi_1}{2\pi} \right)} \]

\[ Z_0 = \text{path integral over } \phi_{\delta}. \]
\[
\phi(t) = \phi(t)_{\text{cl}} + \phi(t)_{\text{c}}
\]

\[
\phi_{\text{c}}(0) = \phi_{\text{c}}(\beta) = 0
\]

Regard \( \phi_{\text{c}} \in \mathcal{T} \).

\[
Z_{\phi} = \int [d\phi(t)] \left. \phi_{\text{c}}(\beta) = 0 \right. e^{-\int_0^\beta \frac{1}{2} \phi^2 dt}
\]

**Gaussian Integrals**

\[
\int_{-\infty}^{\infty} dx \ e^{-\frac{1}{2} x^2} = \sqrt{2\pi}
\]

\[
\int_{-\infty}^{\infty} dx \ e^{-a x^2} = \frac{1}{\sqrt{a}}
\]

\( \text{Re}(a) > 0 \) for other values

Use analytic continuation.
A_{ij} \text{ symmetric matrix } k_{ij} \leq n \\
Re (A_{ij}) > 0.

\[ \int d\phi \prod_{i=1}^{n} \frac{dx_i}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \sum_{i,j} x^i A_{ij} x^j + b_i x^i \right) \]

= \frac{1}{\sqrt{\det A}} \ e^{\frac{1}{2} b^i (A^{-1})^i j b_j}

(\infty \text{ if } A \text{ has a zero mode })

Generalize to \( \infty \)-dimensions

\[ \int [d\phi_\theta(t)] \exp -\int_0^1 \phi_\theta A \phi_\theta \]

\[ A = -\frac{1}{2\beta} \frac{d^2}{dt^2} = 0 \text{ on } L^2[0,1] \]
$A = 0$ has zero mode $\Phi^a(t) = \text{const.}$

$\Phi^a(0) = 0 = \Phi^a(\beta).$

$$Z_{\Phi} = \frac{1}{\sqrt{\text{Det}(0)}}$$

$$\mathcal{O} = -\frac{T}{2\beta} \frac{d^2}{dt^2}$$

Finite dimensions if $A$ is diagonal.$^2$

Then $\text{Det}(A) = \prod \lambda_i$

Won't work in $\infty$ dimensions.

$8\sin(n\pi)$ are eigenfunctions

$$\prod_{n=1}^{\infty} \left( \frac{T}{2\beta} n^2 \right)$$
Define the determinant using $S$-function regularization:

$$\frac{d}{ds} \left| \chi^{-s} \right|_{s=0} = - \log \chi \geq 0$$

For $\Theta$ define a $S$-function

$$S_\Theta(s) := \sum_{\lambda \neq 0} \chi^{-s}$$

Formally,

$$\text{Det} \Theta = \exp \left( - S_\Theta'(s) \bigg|_{s=0} \right)$$

If $S_\Theta(s)$ converges for $\text{Re}(s)$ sufficiently large and positive and admits an a.e. to $s = 0$ then we **define**
$\text{Det } O := \exp(-S_0(0))$

For $O = -\frac{1}{2\pi\beta} \frac{d^2}{dt^2}$

acting on functions on $[0, 1]$ with $\phi(0) = \phi(1) = 0$

$S_0(s) = 2 \left( \frac{\Gamma \pi^2}{2\pi\beta} \right)^s S(2s)$

Riemann.

$S(s) = -\frac{1}{2} + s \log \frac{1}{\sqrt{2\pi}} + O(s^2)$

near $s = 0$

$\text{Det}(O) = \frac{\beta}{\Gamma} \uparrow$ up to $2\pi\beta$. 
\[ \beta \to 0 \]

\[ Z \to Z_0 e^\left( -\frac{1}{2} (\phi_2 - \phi_1)^2 + i \delta (\phi_2 - \phi_1) \right) \]

\[ (1 + O(e^{-k/\beta})) \]

\[ \exp \text{ small as } \beta \to 0. \]

Leading term is standard quantum propagator from \( \phi_1 \) to \( \phi_2 \)

Huge action suppression
\[ Z_0 = \sqrt{\frac{I}{2\pi + \beta}} \]

Net result:

\[ Z(\phi_2, \phi_1 | \beta) = \sqrt{\frac{I}{2\pi + \beta}} \]

\[ \sum_{\omega \in \mathbb{Z}} e^{-\frac{2\pi^2 I}{\beta} \left( \omega + \frac{\phi_2 - \phi_1}{2\pi} \right)^2 + 2\pi i \omega \left( \frac{\phi_2 - \phi_1}{2\pi} \right)} \]

\[ \sum_{\omega \in \mathbb{Z}} \]

\[ Z(\phi_2, \phi_1 | \beta) = \langle \phi_2 | e^{-\beta H} | \phi_1 \rangle \]

\[ = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} e^{-\beta \left( m - \beta \right)^2 + im(\phi_2 - \phi_1)} \]

Hamilton viewpoint:

\[ = \sum_{\psi_n} \langle \phi_2 | \psi_n \rangle \langle \psi_n | e^{-\beta H} | \psi_n \rangle \]