Last time: We gauged the $S\mathcal{O}(2)$ symmetry:

\[ \Phi(t) = e^{i\phi(t)} \rightarrow e^{i\alpha} \Phi(t) \]

\[ S' = \int \left[ \frac{1}{2} \bar{\phi}(\dot{\phi} + A^{(e)})^2 + \mathcal{B}(\phi + A^{(e)}(t)) \right] dt \]

- $A^{(e)}(t)$: "external" "nondynamical"
- $\Phi(t)$: "dynamical": Integrate over in the path integral.

* Geometrically $d + iA = \text{connection}$

On principal $G$-bundle over spacetime $M^{\text{time}} = \mathbb{R}^{0+1}$

- $G = \mathbb{R}, \mathcal{O}(2), O(2)$
- $\phi(t) \rightarrow \phi(t) + \alpha(t)$
- $A^{(e)}(t) \rightarrow A^{(e)}(t) - dt \alpha(t)$

$A = A^{(e)}(t) dt$ $d + iA^{(e)}$
In general \( P \rightarrow G = \text{Lie group} \).

3 defs of connection —

* Rule for lifting paths in \( M \) to paths in \( P \) satisfies gluing —

Locally

\[
\begin{align*}
U_\alpha \subset M \\
\downarrow \pi \\
U_\alpha \times G \leftrightarrow (\delta(t), g(t)) \\
\downarrow \\
U_\alpha \leftrightarrow \gamma(t)
\end{align*}
\]

A connection locally looks like \( \text{Lie}(G) \).

\[
\nabla = d + A_\alpha, \quad A_\alpha \in \mathfrak{sl}(U_\alpha, \mathbb{g})
\]

\[
\begin{align*}
\dot{g}(t) & \left( \frac{d}{dt} + \gamma^\mu(t)(A_\alpha)_{,\mu}(\gamma(t)) \right) g(t) = 0
\end{align*}
\]
Change conjugation?

\[ \phi(t) \rightarrow -\phi(t) \]
\[ B \rightarrow -B \]
\[ A^a(t) \rightarrow -A^a(t) \]

Quantum mechanically \( B \rightarrow -B \) can only be a symmetry when \( 2B \in \mathbb{Z} \).

* Periodicity in "0-angle" \( B \)

\[ B \rightarrow B + 1 \]

Quantum Theory: Value of the action matters

We restore a version of this symmetry by adding a "Chern-Simons term".
\[ e^{-S'} = e^{-\int \frac{1}{2} (\partial^2 + iA)^2 - iB(\phi^2 + A) + ik\int A(t) dt} \]

Combined with c.c. \[ A^{(e)} \rightarrow -A^{(e)} \]
\[ k \rightarrow -k \]

\[ (B, k) \rightarrow (B+r, k+r) \text{ re } \mathbb{Z} \]

Q.M. \( r \in \mathbb{Z} \) for \( \eta \) periodicity

Combine with c.c. \[ (B, k) \rightarrow (-B, -k) \]
\[ = (B+N, k+N) \]

Only hope for c.c. when
\[ B = k = N/2 \in \frac{1}{2} \mathbb{Z} \]

**Issue**: Gauge invariance of Chern-Simons term.
Space of Gauge Fields + Gauge Transformations

In general

gauge group $G$ — compact Lie group.

$\mathcal{B} := \text{Map} (M \rightarrow G)$

$\mathcal{B}$ — group of gauge transformations

$(\text{Aut} (P \rightarrow M), \text{if } P \text{ is trivializable then } \cong \mathcal{B})$

$A = \text{space of gauge fields}$

$A/\mathcal{B} = \text{gauge inequivalent fields}$

Case 1: $G = \mathbb{R}$ $\alpha(t) \in \mathbb{R}$

$\mathcal{N} : \text{Map} (M \rightarrow \mathbb{R})$

$t \mapsto \alpha(t)$

(Not! $\mathbb{R}/2\pi\mathbb{Z}$)
Acts on $A$:

$$A^{(e)}(t) \rightarrow A^{(e)}(t) - d_t \alpha(t)$$

So $M = [t_1, t_2]$ with free b.c.'s

$$d_t \alpha(t) = A^{(e)}(t)$$ has a soln

We can always gauge $A^{(e)}(t) = 0$.

Case 2: $G = \mathbb{R}$, $M = \mathbb{R}$

impose $\alpha(t) \rightarrow 0$ as $t \rightarrow \pm \infty$

$$\mathcal{Y}' = \left\{ \text{Map } t \rightarrow \alpha(t) \mid \alpha(t) \rightarrow 0 \text{ as } t \rightarrow \pm \infty \right\}$$

$$\int_{-\infty}^{\infty} A^{(e)}(t) dt$$ is gauge invariant

$$\frac{\mathbb{A}}{\mathcal{Y}'} = \mathbb{R}$$
Case 3 \[ G = \text{SO}(2) \cong U(1) \]
\[ M = S^1 \]

\[ \mathcal{Y} = \text{Map} \left( M \rightarrow G \right) \]
\[ = \text{Map} \left( S^1 \rightarrow U(1) \right) \]

as a set of continuous maps, we can assign a winding number

\[ g(t) : S^1 \rightarrow S^1 \]

\[ 1 \rightarrow \mathcal{Y}_0 \rightarrow \mathcal{Y} \xrightarrow{\pi} \mathbb{Z} \rightarrow 1 \]

winding number \( \# = 0 \) g.t. i.e.

\[ g(t) = e^{i\alpha(t)} \]
\[ \alpha(t) : S^1 \rightarrow \mathbb{R} \]

single-valued.

\( \mathcal{Y}_0 = \text{group of small gauge transformations} \)
anything in \( G \) that has nonzero winding number is called a “large gauge transformation”

\[ S(w) = g_w \]

\[ g_w(t) = \exp\left(\frac{2\pi i w t}{\beta}\right) \]

\[ S_{s.t.} = \left[0, \beta\right] / \beta \sim 0 \]

\[ g_w \cdot g_{w'} = g_{w+w'} \]

\[ \int A^{(e)}(t') dt' \] is inut under \( L_0 \)

is not inut under \( L \)!!

\[ \exp\left(i \int A^{(e)}(t') dt'\right) \] is inut under \( L \).

Complete gauge inut:
$A^{(e)}(t)$ periodic in $t \sim t + \beta$

$$A^{(e)}(t) = \sum_{n} e^{2\pi i n t / \beta} A^{(e)}_n$$

$$= A^{(e)}_0 + \tilde{A}^{(e)}$$

We can solve

$$\frac{\partial}{\partial t} \chi(t) = \tilde{A}^{(e)}(t)$$

for a single-valued $\chi(t)$.

i.e. Using $\mathcal{G}$, we can always gauge $A^{(e)}(t) = \mu / \beta$ constant.

---

Previous notation $\int_{t} A^{(e)}_t(t) \, dt$.

Simplify $A^{(e)}_t(t) \rightarrow A^{(e)}(t)$

Note:
Under the large gauge tranfors
\[ g_w(t) \rightarrow A^{(e)} \rightarrow g_w^{-1} \frac{d}{dt} g_w \]
shifts
\[ \mu \rightarrow \mu - 2\pi w \quad w \in \mathbb{Z} \]
\[ A/Y_0 \cong \mathcal{TR} = A^{\text{red}} \]
\[ A/Y \cong \mathcal{TR}/\mathbb{Z} \cong U(1) \]
\[ \mu \rightarrow \mu + 2\pi w \quad w \in \mathbb{Z} \]

Case 4: \( G = O(2) \) gauge c.c.
\[ O(2) = SO(2) \times \mathbb{Z}_2 \]
\[ H_0(2) = H_{SO(2)} \times \mathbb{Z}_2 \]
\[ 1 \rightarrow H_0 \rightarrow H_{O(2)} \rightarrow \mathbb{Z} \times \mathbb{Z}_2 \cong D_{\infty} \]
action on \( A^{\text{red}} \)
\[ \sigma : \mu \rightarrow -\mu \quad \text{and} \quad S : \mu \rightarrow \mu + 2\pi \]
\( \mathbb{Z} = \{(A/y) \}_{\text{coarse}} = [0, \pi] \)

Arbifold singularities

\( \mu \to -\mu \) around \( \mu = 0 \)

\( \mu \to 2\pi - \mu \) around \( \mu = \pi \)

Better \( A/y = " \text{stack"} \)

"stack"

"groupoid"

About the Chern-Simons term

\( M = S^1 \mathbb{S}_{\text{s.t.}} \quad G = \text{U}(1) \}

\[ \exp(i k \oint_M A^{(e)}(t') dt') \]  

Inv. under \( G \)

but for \( gw \)

\[ \exp(i k \oint_M A^{(e)}(t') dt') \]  

\[ \exp(2\pi i k w) \]  

Not under \( G \);
\[ e^{2\pi i k w} = 1 \quad \text{for all } w \in \mathbb{Z} \quad \text{iff } k \in \mathbb{Z}. \]

In fact, for \( G = U(1) \) on some spacetimes the action \( S \) is NOT gauge invariant!

But this is ok for the quantum theory because in the path integral only \( \exp(-S) \) enters

And for quantized level \( k \in \mathbb{Z} \)

\[ e^{-S} \] is gauge invariant.
Anomalies: In general in field theory - space $F$ of "fields" $\mathcal{F} \rightarrow \mathcal{F}$

\[ F_{\text{background fields}} = \mathcal{A}/g \times \{ \mathcal{B} \} \]

Often $F = F_{\text{back}} \times F_{\text{dyn.}} \times \{ k \}$

"control" all parameters can be considered "fields"

Integrate (path integral) over $F_{\text{dyn.}}$

$Z_{[\phi_{\text{back}}]} = \int_{F_{\text{dyn}}} e^{-\int_{F_{\text{dyn}}} S_{[\phi_{\text{dyn}}, \phi_{\text{back}}]} \frac{\text{Vol}(\phi_{\text{dyn}})}{\text{Vol}(\phi_{\text{dyn}})}}$

$Z(\mathcal{B}) \overset{\text{previous section}}{=} Z[\phi_{\text{back}}]$ in the gauged QM
Suppose $\mathcal{G}$ acts on $F$ preserves $S[\phi_{\text{dyn}}; \phi_{\text{bck}}]$. Suppose it formally preserves the measure $\text{val}(\phi_{\text{dyn}})$. Then we expect $Z[\phi_{\text{bck}}]$ to be $\mathcal{G}$-invariant.

$$Z[\phi_{\text{bck}}] = Z[\phi_{\text{bck}}].$$

Path integrals are formal things and need to be defined e.g. $Z[\phi_{\text{bck}}]$ as a function.

It can happen that after defining $Z$ carefully
(regularization, renormalization)

It turns out that well-defined
\[ \hat{\mathcal{Z}}[\phi^{\text{bck}}] \] is NOT \( G \)-inv.

"Potential anomaly"

* Sometimes the lack of invariance can be removed by physically unimportant redefinitions.
  "local counterterms" etc.

* Sometimes the lack of invariance CANNOT be removed in this way.
  "True anomaly"
\[ \exp(i k \mathbf{A}) \]
descends to a function on \( A/G \) for any \( k \).

If our gauge group is \( G = \mathbb{R} \),
this term is not anomalous.

If our gauge group is \( G = U(1) \),
it does not descend to a function on \( A/G \) unless \( k \in \mathbb{Z} \).
If \( k \not\in \mathbb{Z} \), we say this physical
quantity is anomalous.
Look at these ideas in the context of our gauged QM.

There won't be any interesting anomalies for \( \mathcal{L}_0 \).

For simplicity use \( \mathcal{L}_0 \) to put
\[
A^{(e)}(t) = \frac{\mu}{\beta} \quad \text{constant}
\]

**Equations of motion are not** changed

\[
\phi(t) = \frac{2\pi \omega t}{\beta}
\]

\[
\mathcal{L}(\mu; \beta) = \sum_n \sum_{w \in \mathbb{Z}} e^{2\pi i \beta (w + \frac{\mu}{2\pi})^2} e^{ik_\mu z_0} e^{-2\pi i \beta (w + \frac{\mu}{2\pi})}
\]

\[
\phi \rightarrow \phi + A^{(e)} = \frac{2\pi \omega}{\beta} + \frac{\mu}{\beta} \quad \omega \rightarrow \omega + \frac{\mu}{2\pi}
\]
\[
\text{P.S.E.} \Rightarrow \quad Z(\mu; B) = e^{i(k - B)\mu} \sum_{m \in \mathbb{Z}} e^{-\frac{\beta}{2\pi} (m - 8)^2} \quad \begin{array}{c} \Rightarrow \\ \Rightarrow \end{array} \\
= e^{i k \mu} \text{Tr} \left[ e^{-\beta H_{B}} e^{i \mu \mathcal{Q}} \right]
\]

\[
Q \cdot \Psi_m = m \cdot \Psi_m
\]

If \( k \in \mathbb{Z} \) periodic in \( \mu \rightarrow \mu + 2\pi \),

No anomaly for \( G = SU(2) \)

\( Z(\mu; B) \) descends to a function on \( A/G \).

Also \( Z(\mu; B) = Z(-\mu; B) \)

if \( k = 8 \in \mathbb{Z} \). No anomaly under \( \mathbb{Z}_2 \)
\[
\mathcal{G}/\mathcal{G}_0 \cong \text{D}_\infty = \langle \sigma, \delta \rangle
\]

\[
\sigma : \mu \rightarrow -\mu \\
\delta : \mu \rightarrow \mu + 2\pi
\]

\[
k = \mathbb{Z} \subset \mathbb{Z}/2 \\
k \in \mathbb{Z}.
\]

2B even ok.

2B odd. clash.

\[
k = \mathbb{Z} + 1/2 \\
\text{but we also needed } k \in \mathbb{Z}.
\]

\[
\Rightarrow \quad \text{Anomaly for } O(2).
\]

\[
\Theta = \pi
\]
Actually, by changing the problem again, we can make sense of a $\frac{1}{2}$-integer C.S. term.

\[ B = \frac{1}{2} \]

essential point already clear by looking at leading terms in $\beta \to \infty$

\[
Z \to \frac{e^{-\beta E_{\text{local}}}}{2\pi i} e^{i(k-\frac{1}{2})\mu} (e^{i\mu/2} e^{-i\mu/2}) + \ldots \]

If $k = 0$

\[ (1 + e^{-i\mu}) \]

periodic in $\mu \to \mu + 2\pi$

not invertible in $\mu \to -\mu$.

If $k = \frac{1}{2}$:

\[ \text{invertible in } \mu \to -\mu \]

not invertible under $\mu \to \mu + 2\pi$.
So we can have symmetry under s but not 5 or under 5 but not s depending on choice of k.

In general in the theory of anomalies if there is one definition of Z that is int. under \( \mathcal{H}_1 \), \( C \), and another definition so that \( Z \) int. under \( \mathcal{H}_2 \) but \( \text{No def. int. under both } \mathcal{H}_1 \) and \( \mathcal{H}_2 \) we say there is a "mixed anomaly."
Making sense of $k \notin \mathbb{Z}$ even when $G = U(1)$

$m = S^1$

"edge" $\triangleright$ "bulk" $\Sigma$

$1+1$ manifold.

$$\exp(ik \oint A) = \exp(ik \int F)$$

$\Sigma$

$F = dA$ is gauge invariant.

$ik \int F$ makes sense as a gauge invariant real number for any $k$.

(If $\partial \Sigma = \phi$ then $\left[ \frac{F}{2\pi} \right] = \text{image of } c_1(\mu) \text{ in } H_{dR}$.)
Fractional C-S. levels appear
• topological insulators
• fractional quantum Hall effect ("spin Chern-Simons theory")
• susy + sugra + string theory

Heisenberg Extensions

\[ 1 \rightarrow A \rightarrow \tilde{G} \rightarrow G \rightarrow 1 \]

Class of extensions  \( A = \text{Abelian} \)
 Central  \( G = \text{Abelian} \)

but \( \tilde{G} \) is — in a sense — "maximally non-abelian"

We’ve met Heis \((\mathbb{Z}_n \times \mathbb{Z}_n)\)
QM. of particle on disc. appxt to a circle.
Preliminary: Some useful identities for manipulating exponentials of operators.

\[ A \in M_n(k) \]

or a suitable operator on \( \mathcal{H} \).

('want all powers \( A^n \) to exist')

\[ \exp(A) = 1 + \sum_{n=1}^{\infty} \frac{A^n}{n!} \]

\[ \exp(\alpha A) \exp(\beta A) = \exp((\alpha + \beta)A) \]

\[ \frac{d}{dt} \left( e^{tA} \right) = Ae^{tA} = e^{tA}A \]

\[ e^A e^B e^{-A} = e^{(e^A B e^{-A})} \]
\[ \text{Def: } A \in M_n(k) \]

\[ \text{Ad} (A) \subseteq \text{End} (M_n(k)) \]

\[ \text{Ad} (A) \text{ linear map on vector space of matrices.} \]

\[ \text{Ad}(A) : B \rightarrow [A, B] \]

\[ (\text{Ad}(A))^m : B \rightarrow [A, [A, \ldots [A, B] \ldots ]] \]

\[ \text{Claim: } e^{A B} e^{-A} = \exp(\text{Ad}(A))(B) \]

\[ \text{pf: } B(t) = e^{tA} B e^{-tA} \]

\[ B(0) = B \quad B(1) = \text{what we want.} \]
\[
\frac{d}{dt}(e^{tA}B e^{-tA})
= A \cdot B(t) - B(t) \cdot A
= \text{Ad}(A)(B(t))
\]

\[
\left(\frac{d}{dt}\right)^n B(t) = \text{Ad}(A)^n(B(t))\bigg|_{t=0}
= \text{Ad}(A)^n(B)
\]

\[
B(t) = \sum \frac{t^n}{n!} \left(\frac{d}{dt}\right)^n B(t)\bigg|_{t=0}
= \exp(t \text{Ad}(A))(B)\bigg|_{t=1}
\]
Suppose now \( A(t) \)

matrix/operator-valued function of \( t \)

In general:

\[
\frac{d}{dt} e^{A(t)} \neq \dot{A}(t) e^{A(t)}
\]

\[
\neq e^{A(t)} \dot{A}(t)
\]

Note that, in general, \( \dot{A}(t) \)
and \( \dot{A}(t) \) don't commute.

Next time: We'll give a useful formula for \( \frac{d}{dt} e^{A(t)} \)

Also

\[ e^A e^B \neq e^{A+B} \]

When \( A \) and \( B \) do not commute.

We'll give a formula for \( e^{A\cdot B} = e^C \)

\( C = \) function of \( A \) and \( B \).
Q.M. of a particle moving on a general Riemannian manifold $(\mathcal{M}, g_{\mu\nu})$

\[ S = \int \frac{1}{2} g_{\mu\nu}(x(t)) \dot{x}^\mu(t) \dot{x}^\nu(t) \]

\[ \int [dx(t)] \ e^{-S} = Z[g_{\mu\nu}] \]

Map $(\mathcal{S} \rightarrow \mathcal{M})$

In general the S.C. appxt. will **NOT** be exact!

S.C. Appxt. $\delta S = 0 \quad \iff \text{closed geodesics}$

\[ \sum_{\text{closed geodesics } x_c(t)} Z[x_c(t)] e^{-S[x_c]} \]
For special case
\[ \mathcal{E} = \frac{\mathcal{H}}{P} \]

\[ g_{\mu\nu} = \text{hyperbolic metric} \]

S.C. appx. \( \Rightarrow \) Selberg trace formula

Surprising: This is exact!

Related: (not rigorous) Gutzwiller trace formula - generalizes the idea to classically chaotic systems.