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Lecture 8: Lie Algebras from Lie Groups

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1. Introduction

Quite generally, when one is confronted with a nonlinear object or phenomenon it is often useful to reduce the problem to a linear problem, at the cost of restricting the domain of applicability.

Lie groups are not linear – they are curved manifolds. If one chooses coordinates then the group multiplication law is, in general, given by a complicated power series in the coordinates.

Nevertheless, Lie's theorem reduces many questions about Lie groups to questions about *Lie algebras*. Questions about curved manifolds turn out to be equivalent to questions about linear algebra. This is a profound simplification, and it leads to a very rich theory.

2. Geometrical approach to the Lie algebra associated to a Lie group

2.1 Lie's approach

A good way to approach the subject is the way Sophus Lie did himself. A Lie group is a group with continuous (or smooth) parameters. We convert the associativity of the group law into a differential equation, and study the integrability of that differential equation.

Suppose $x^1, \dots, x^n$ are coordinates in a neighborhood $\mathcal{U}$ of 1_G, where $n = \dim G$. We take $x = 0$ to correspond to 1_G. Thus, we have a smooth parametrization of group elements $g(x)$. The product of two group elements near the identity will be another group element near the identity. Thus, there is a neighborhood $\mathcal{U}' \subset \mathcal{U}$ so that if $g(x), g(y) \in \mathcal{U}'$ then we can write the group law as:

$$g(x)g(y) = g(\phi(x,y)) \tag{2.1}$$ eq:grplaw

for some smooth functions $\phi^i(x,y)$, $i = 1, \dots n$.

The group laws can be expressed in terms of ϕ:

1. Associativity:

$$\phi(x, \phi(y,z)) = \phi(\phi(x,y), z) \tag{2.2}$$ eq:associa

2. Identity: $\phi(0,x) = \phi(x,0) = x$

3. Inverse: $\phi(x, x_0) = \phi(x_0, x) = 0$ is solvable for a unique x in terms of x_0.

Now, let us differentiate the associativity condition (2.2) with respect to the z^k:

$$\frac{\partial \phi^i(x, \phi(y,z))}{\partial \phi^j(y,z)} \frac{\partial \phi^j(y,z)}{\partial z^k} = \frac{\partial \phi^i}{\partial z^k}(\phi(x,y), z) \tag{2.3}$$ eq:slieone

where repeated indices are summed.

Now define an $n \times n$ matrix function of one variable:

$$u^i{}_k(x) := \frac{\partial \phi^i(x,y)}{\partial y^k}\bigg|_{y=0} \tag{2.4}$$ eq:defnsu

Then, setting $z = 0$ in (2.3) gives

$$\frac{\partial \phi^i(x,y)}{\partial y^j} u^j{}_k(y) = u^i{}_k(\phi(x,y)) \tag{2.5}$$ eq:lieseq

Note that (2.5) is equivalent to the equality of first order differential operators:

$$u^j{}_k(y) \frac{\partial}{\partial y^j} = u^i{}_k(\phi(x,y)) \frac{\partial}{\partial \phi^i} \tag{2.6}$$ eq:livfone

Here we are holding x fixed and viewing $y^i \to \phi^i(x,y)$ as a nonlinear change of coordinates. Nevertheless the equation is true for any x and hence we can write (2.6) as

$$u^j{}_k(y) \frac{\partial}{\partial y^j} = u^i{}_k(\phi) \frac{\partial}{\partial \phi^i} \tag{2.7}$$ eq:livfonep

Now, (2.7) is a remarkable equation because the LHS depends only on y and the RHS depends only on ϕ. We will apply this observation in one moment.

A second observation is that if we denote

$$I_k(y) := u^j{}_k(y) \frac{\partial}{\partial y^j} \tag{2.8}$$ eq:invtvf

then we can compute

$$
\begin{aligned}
[I_k(y), I_m(y)] &= \left(u^j{}_k \partial_j u^l{}_m - u^j{}_m \partial_j u^l{}_k\right) \frac{\partial}{\partial y^l} \\
&= \left(u^j{}_k \partial_j u^l{}_m - u^j{}_m \partial_j u^l{}_k\right) (u^{tr,-1})_l{}^p I_p(y) \\
&:= f^p_{km}(y) I_p(y)
\end{aligned} \tag{2.9}
$$

eq:commis

Now, by (2.7) $I_k(y) = I_k(\phi)$ so it follows that $f^p_{mk}(y) = f^p_{mk}(\phi)$, and since $\phi = \phi(x, y)$ with arbitrary x, it follows that $f^p_{mk}(y)$ is *constant*! Let us consider the finite-dimensional vector space spanned by the first order differential operators) $I_k(y)$ in (2.8). That is, we consider the linear combinations of the $I_k(y)$ with constant (real) coefficients. Note that this vector space is *closed under commutator*. We denote this vector space $L(G)$ or sometimes $\mathfrak{g}$. Note that

$$
\mathrm{dim} G = \mathrm{dim} L(G) \tag{2.10}
$$

On the LHS we have the dimension of a manifold, and on the RHS the dimension of a vector space. Moreover, it follows from general properties of differential operators that if X_1, X_2, X_3 are any three vector fields in $L(G)$ then

$$
[[X_1, X_2], X_3] + [[X_3, X_1], X_2] + [[X_2, X_3], X_1] = 0 \tag{2.11}
$$

eq:jacobi

This crucial identity is known as the *Jacobi identity*.

These are the crucial properties which are abstracted into the general definition:

Definition : An abstract Lie algebra $\underline{\mathbf{g}}$ over a field k is a vector space over k together with a product $(v_1, v_2) \to [v_1, v_2] \in \mathfrak{g}$, such that for all $v_1, v_2, v_3, \in \mathfrak{g} \quad \alpha, \beta \in k$.

1.) $[v_1, v_2] = -[v_2, v_1]$
2.) $[\alpha v_1 + \beta v_2, v_3] = \alpha[v_1, v_3] + \beta[v_2, v_3]$
3.) $[[v_1, v_2], v_3] + [[v_3, v_1], v_2] + [[v_2, v_3], v_1] = 0.$

2.2 Left-invariant vector fields and the Lie algebra

We will now rephrase Lie's argument in the language of modern differential geometry.

2.2.1 Review of some definitions from differential geometry

Tangent vectors are directional derivatives along paths. If we imagine $M \subset \mathbb{R}^N$ then we literally take a tangent plane. In general if $p \in M$ let $C^1(p)$ be the functions defined in some neighborhood of $p \in M$, which are differentiable at p. A directional derivative along a curve $\gamma(t)$ such that $\gamma(0) = p$ is a linear functional on this space defined by

$$
\dot{\gamma}(f) := \frac{d}{dt}|_{t=0} f(\gamma(t)) \tag{2.12}
$$

Then *the space of tangent vectors at p*, T_pM, is the linear span of these linear functionals $C^1(p) \to \mathbb{R}$. A *vector field* is a continuous system of tangent vectors.

Now suppose that $\phi : M_1 \to M_2$ is a map between manifolds. Let us study how geometric objects behave with respect to such maps.

First of all, functions are contravariant. That is we define the *pullback* on functions

$$\phi^* : Fun(M_2) \to Fun(M_1) \tag{2.13}$$

by $\phi^*(f) = f \circ \phi$. Here $Fun(M)$ is the space of all (say, continuous) functions from M to (say) the real numbers.

By contrast, vectorfields *push forward*:

$$\phi_* : T_p M_1 \to T_{\phi(p)} M_2 \tag{2.14}$$

the definition is that $\phi_*(\dot{\gamma})$ is the directional derivative along the curve $\phi(\gamma(t))$.

Two facts we will need below are:

• The commutator of two first-order differential operators is a first order differential operator. This defines the commutator of two vector fields:

$$[\xi_1, \xi_2] \tag{2.15}$$

Thus, $Vect(M)$ is a Lie algebra, for any manifold M. It is infinite-dimensional. It is not difficult to show that

$$\phi_*[\xi_1, \xi_2] = [\phi_*(\xi_1), \phi_*(\xi_2)] \tag{2.16}$$

•

Vector fields act on functions to produce new functions. From the above definitions it follows that:

$$\phi_*(V)(f) = V(\phi^*(f)) \tag{2.17}$$

Exercise

In general, functions do *not* push forward. They only pull back. However, if ϕ is an invertible map between manifolds show that it makes sense to define $\phi_*(f) = f \circ \phi^{-1}$. This is used when pushing forward the product of a function and a vector field, fV in the next section.

2.2.2 The geometrical definition of a Lie algebra

Definition. A *left translation*, or *right translation* by an element $g \in G$ is the diffeomorphism:

$$\begin{aligned} L_g : G \to G \qquad & h \mapsto g \cdot h \\ R_g : G \to G \qquad & h \mapsto h \cdot g^{-1} \end{aligned} \tag{2.18}$$

eq:leftrighttr

Exercise
Show that L_g, R_g define injections of groups $G \hookrightarrow Diff(G)$.

Now we introduce the important idea of left and right invariance: A left-invariant function satisfies:

$$L_g^*(f) = f \tag{2.19}$$ eq:lftinvfun

for all $g \in G$. Of course, this just means that f is constant, because if we evaluate (2.19) at $g = 1$ then

$$L_g^*(f)|_{h=1} = f|_{h=1} \Rightarrow f(g \cdot 1) = f(1) \tag{2.20}$$ eq:lftinvfunp

However, the notion of left- or right- invariant tensors still leaves room for very interesting examples.

Definition. A vector field $\xi \in Vect(G)$ is left (or right) invariant if

$$\begin{aligned}(L_{g_0})_*(\xi_g) &= \xi_{g_0 \cdot g} \\ (R_{g_0})_*(\xi_g) &= \xi_{g g_0^{-1}}\end{aligned} \tag{2.21}$$ eq:livr

is satisfied for all $g, g_0 \in G$, respectively.

The picture is: For a general vector field ξ_g is the directional derivative at g to some curve $\gamma_g(t)$ going through g, while $\xi_{g_0 g}$ is the directional derivative of some -*a priori* unrelated - curve $\tilde{\gamma}_{g_0 g}(t)$ going through $g_0 g$. Then the condition of left-invariance:

$$(L_g)_*(\xi_g) = \xi_{g_0 g} \tag{2.22}$$

means $\tilde{\gamma}_{g_0 g}(t)$ has the same directional derivative at $t = 0$ as the curve $g_0 \cdot \gamma_g(t)$.

We now recognize our first order operators $I_k(y)$ defined in (2.8) as vector fields on the group G. The equation (2.7) is the statement that these are left-invariant vector fields.

Now, let us return to the general situation. Note that if ξ_1, ξ_2 are two LIVF's then

$$(L_g)_*[\xi_1, \xi_2] = [(L_g)_*(\xi_1), (L_g)_*(\xi_2)] = [\xi_1, \xi_2] \tag{2.23}$$

This leads to the

Geometric definition of the Lie algebra: We define the Lie algebra $L(G)$ of a Lie group G to be the Lie algebra of left invariant vector fields on G.

Examples

- $G = U(1)$. The general vector field on $U(1)$ is

$$\xi = f(\theta) \frac{d}{d\theta} \tag{2.24}$$

Left invariant (and right-invariant) vector fields satisfy $f(\theta+\theta_0)=f(\theta)$ for all θ_0. That is, $f(\theta)$ must be constant.

• $G=GL(n,k)$, $k=\mathbb{R},\mathbb{C}$. We can choose as global coordinates on the manifold the matrix elements g_{ij} of $g\in G$. Introduce an $n\times n$ matrix of vector fields $\frac{\partial}{\partial g}$ whose components are:

$$(\frac{\partial}{\partial g})_{ij} := \frac{\partial}{\partial g_{ij}} \tag{2.25}$$

$(\frac{\partial}{\partial g_{ij}})_{g_0}$ is the directional derivative along the curve g_0+te_{ij} where e_{ij} is a matrix unit. Note that if g_0 is invertible then this curve is indeed in $GL(n,k)$ for sufficiently small $|t|$. One computes

$$(L_{g_0})_*(\frac{\partial}{\partial g}) = g_0^{tr}\cdot\frac{\partial}{\partial g} \tag{2.26}$$

where matrix multiplication on the RHS is understood. To do this, let $f_{kl}(g)=g_{kl}$ be the function which picks out a matrix element. Then

$$(L_{g_0})_*(\frac{\partial}{\partial g_{ij}})(f_{kl}) = \frac{\partial}{\partial g_{ij}} f_{kl}\circ L_{g_0} = \frac{\partial}{\partial g_{ij}}\sum_s (g_0)_{ks} f_{sl} = (g_0)_{ki}\delta_{j,l} \tag{2.27}$$

Therefore,

$$(L_{g_0})_*(\frac{\partial}{\partial g_{ij}}|_g) = \sum_k (g_0)_{ki}\frac{\partial}{\partial g_{kj}}|_{g_0 g} \tag{2.28}$$

Therefore

$$\xi_{ij} := (g^{tr}\frac{\partial}{\partial g})_{ij} = \sum_k g_{ki}(\frac{\partial}{\partial g})_{kj} \tag{2.29}$$

is a matrix of left-invariant vector fields on $GL(n)$. Proof:

$$\begin{aligned}(L_{g_0})_*(g^{tr}\frac{\partial}{\partial g}) &= (g_0^{-1}g)^{tr}g_0^{tr}\cdot\frac{\partial}{\partial g}\\ &= g^{tr}\cdot\frac{\partial}{\partial g}\end{aligned} \tag{2.30}$$

The n^2 vector fields ξ_{ij} are linearly independent (simply consider their values at $g_0=1$) and form a basis for all LIVF's on $GL(n)$, by an argument given below. Finally, note that by direct computation we find

$$[\xi_{ij},\xi_{kl}] = \delta_{jk}\xi_{il} - \delta_{li}\xi_{kj} \tag{2.31}$$ eq:glsc

and the structure constants are indeed constant.

Returning to the general case, let us now suppose ξ is a LIVF. Then $\xi_g=(L_g)_*(\xi_1)$, so a LIVF is completely determined by its value at $g=1$, and hence we can identify $L(G)$ with the tangent space at $g=1$: $L(G)\cong T_1(G)$, at least, as a vector space. Conversely, given a tangent vector $X\in T_1(G)$. Then to $X\in T_1(G)$ we can associate a curve $\gamma_X(t)$ through $g=1$ whose tangent vector at $g=1$ is X. (As we will see below for a matrix group we can take $\gamma_X(t)$ to be the curve $\exp[tX]$ where we literally exponentiate the matrix.) If

G is a Lie group, then the *right* action of G on itself defines a global system of *left-invariant* vector fields on G as follows: If $X \in T_1(G)$ is the directional derivative along a curve $\gamma_X(t)$ passing through $g = 1$ then then the curves $\gamma_{X,g}(t) := g\gamma_X(t)$ through g have directional derivatives defining a vector field $\xi(X) \in T_g G$. This vector field is left-invariant, and in this way we define the map from $T_1 G$ to the left-invariant vector fields on G.

The left-invariant vector fields $\xi(X)$ are called the *fundamental vector fields* on G. Every LIVF of G is of the form $\xi(X)$ for some unique X (indeed $X = \xi_1$) so we can define $[X, Y] \in T_1 G$ by

$$[\xi(X), \xi(Y)] = \xi([X, Y]) \tag{2.32}$$ eq:LieTone

This defines $T_1 G$ as a Lie algebra, and then ξ defines a homomorphism of Lie algebras $T_1(G) \to Vect(G)$.

To get a better feel for these vector fields consider again the example of $G = GL(n, \mathbb{F})$. The Lie algebra is $\mathfrak{g} = Mat_n(\mathbb{F})$. Let e_{ij} be the matrix unit with a 1 in the i^{th} row and j^{th} column, and zero in all other matrix elements. Then one easily computes the vector field by considering the curve $g\exp[te_{ij}]$:

$$\xi(e_{ij}) = \sum_{k=1}^{n} g_{ki} \frac{\partial}{\partial g_{kj}} \tag{2.33}$$ eq:vertcli

where we are regarding the matrix elements g_{ij} as coordinates on the group. Thus the fundamental vector fields on G are just the vector fields ξ_{ij} we examined above.

Then, combining (2.31) and (2.32) we have

$$\xi\left([e_{ij}, e_{i'j'}]\right) = [\xi(e_{ij}), \xi(e_{i'j'})] = \delta_{ji'}\xi(e_{ij'}) - \delta_{ij'}\xi(e_{i'j}) \tag{2.34}$$

and we conclude that $[e_{ij}, e_{i'j'}]$ coincides with matrix commutator, as it should.

If $G \subset GL(n, \mathbb{F})$ is a subgroup of a matrix group then the matrix elements g_{ij} are not all independent, so some vectors $\frac{\partial}{\partial g_{kj}}$ will be expressed in terms of others. Put differently, a path such at $1 + te_{ij}$ will not be a path within the matrix subgroup in general. These linear dependences are most easily computed using the dual cotangent space $T_1^* G$ and the Maurer-Cartan form, as discussed below.

Exercise

Write out the right-invariant vector fields on $GL(n, \mathbb{F})$.

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3. The exponential map

We have discussed in chapter 2 how to exponentiate matrices. We would now like to generalize this notion to define a map

$$\exp : L(G) \to G \tag{3.1}$$

Since not all Lie groups are matrix groups it will be a little more abstract.

Consider a homomorphism $f : \mathbb{R} \to GL(n, \mathbb{R})$. Such a map must satisfy

$$\begin{aligned} \frac{d}{dt} f(t) &= \lim_{h \to 0} h^{-1} [f(t+h) - f(t)] \\ &= \lim_{h \to 0} h^{-1} [f(h) - 1] f(t) \\ &= A f(t) \end{aligned} \tag{3.2}$$

so $f(t)$ is the unique solution to this differential equation with $f(0) = 1$. We write

$$f(t) = e^{tA} \tag{3.3}$$

Recall that for $A \in M_n(k) \quad k = \mathbb{R}$ or $\mathbb{C}$, we defined

$$\exp A := \sum_{j \geq 0} \frac{A^j}{j!} \in M_n(k) \tag{3.4}$$

defines the *exponential of a matrix.* In general any function defined by a power series can be evaluated for matrix arguments, as above.

In a sufficiently small neighborhood $\mathcal{U}$ of 0 the map $\exp : M_n(\mathbb{R}) \to GL(n, \mathbb{R})$ is invertible with

$$\log g = -\sum_{k=1}^{\infty} (1-g)^k / k \tag{3.5}$$

as its unique inverse.

As we have said, not all Lie groups are matrix groups. So we define exp more generally (and more intrinisically) as follows:

Theorem: If G is a Lie group there is a 1-1 correspondence between the tangent space $T_1 G$ and group homomorphisms $f : \mathbb{R} \to G$ (aka "1-parameter subgroups").

Proof: A homomorphism clearly determines a tangent vector $X \in T_1 G$. Conversely, given a tangent vector $X \in T_1 G$ we consider the left-invariant vector field $\xi(X)$. We study the tangent curves to $\xi(X)$: these are curves $\gamma_X(t)$ such that

$$\dot{\gamma}_X(t) = \xi(X)|_{\gamma_X(t)}. \tag{3.6}$$

eq:tangentcurv

We would like to show that the tangent curve through $g = 1$ defines a group homomorphism of $\mathbb{R}$ into G.

We can take the tangent curve to satisfy $\gamma_X(0) = 1_G$. Equation (3.6) is just an ODE. By the theory of ODE's we know that we can find $\gamma_X(t)$ on a sufficiently small neighborhood $t \in (-\epsilon, \epsilon)$. Now consider a t_0 in this interval. Similarly, we can study this differential equation near $\gamma_X(t_0)$. Now $\gamma_X(t+t_0)$ and $\gamma_X(t_0)\gamma_X(t)$ are both solutions, since $\xi(X)$ is left-invariant. By the uniqueness of solutions to ODE's

$$\gamma_X(t + t_0) = \gamma_X(t_0) \gamma_X(t) \tag{3.7}$$

for $t, t_0, t+t_0$ in the interval, so $\gamma_X(t)$ is a local homomorphism. To extend the definition of $\gamma_X(t)$ to large values of t we take

$$\gamma_X(t) := (\gamma_X(t/n))^n \tag{3.8}$$

eq:largetee

for sufficiently large n. (Exercise: Show that it doesn't matter what n you pick.) ♠

Evaluation of $\gamma_X(t)$ at $t=1$ defines the map

$$\exp : T_1 G \to G \tag{3.9}$$

whose derivative at 1_G is the identity. In this way we define exp, even if G is not a matrix group.

Remarks:

• $\exp : \mathbb{R} \to U(1)$ given by $x \to e^{2\pi i x}$ shows that exp is in general not 1-1.

•

A Lie group can be given the structure of a Riemannian manifold in a canonical way. The geodesics through 1_G are precisely the 1-parameter subgroups. For a compact Lie group, we get a complete Riemannian manifold, and a general theorem of differential geometry shows that any two points on a complete Riemannian manifold can be joined by a geodesic. Therefore, for a compact Lie group, exp is onto.

• For a noncompact Lie group exp need not be onto. We will give counterexamples when we survey the Lie algebras of matrix groups below.

• As we will see, finite dimensional Lie algebras can be exponentiated to form Lie groups. However, there exist examples of Lie algebras which cannot be "exponentiated" to form Lie groups.

Exercise

Show that the choice of n does not matter in (3.8).

Exercise *Exponentials and inverses*

a.) Show that for the abstract exp map we have:

$$e^A e^{-A} = 1 \tag{3.10}$$

for any $A \in \mathfrak{g}$.

b.) Define $Ad(G)$ on $\mathfrak{g}$ ..... *** NEED SOME DETAILS *** Show that

$$\exp[BAB^{-1}] = B\exp(A)B^{-1} \tag{3.11}$$

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4. Baker-Campbell-Hausdorff formula

sec:sBCH

The exponential map is neither one-one nor onto in general. However, for a finite dimensional Lie group, in a sufficiently small neighborhood of the identity $1_G \in \mathcal{U}_2 \subset G$ and $0 \in \mathcal{U}_1 \subset T_1G$, $\exp : \mathcal{U}_1 \to \mathcal{U}_2$ is 1-1 and onto.

It is therefore natural to ask what the full group law looks like in terms of the linear space $\mathfrak{g} := T_1G$. Thus we would like to express

$$C(A,B) = \log e^A e^B \tag{4.1}$$

in terms of A, B.

If we imagine A, B are sufficiently small (we will say they are "order one") then we can expand in Taylor series

$$C(A,B) = A + B + \frac{1}{2}b(A,B) + \cdots \tag{4.2}$$

where $b(A,B) \in \mathfrak{g}$ are the terms of order 2 and $\cdots$ are the terms of order ≥ 3. The factor of $\frac{1}{2}$ is for later convenience. [1]

Since $C(A,0) = A$ and $C(0,B) = B$, $b(A,B)$ has no terms of order A^2 or B^2 and hence is a bilinear map:

$$b : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \tag{4.3}$$

Moreover, since $\exp(-A) = (\exp(A))^{-1}$ we have $C(-B,-A) = -C(A,B)$ and therefore $b(A,B) = -b(B,A)$, that is, b is skew-symmetric. Thus, working infinitesimally, we derive the Lie product on T_1G.

Now, the amazing thing about Lie groups is that

the entire Taylor series $C(A,B)$ can be expressed solely in terms of b.

The formula that does this is known as the BCH formula, and we will now derive it for the case of matrix groups in the next section.

4.1 Statement and derivation

The BCH formula allows us to understand the Lie group operation in terms of the matrices in the exponential. It also provides a convenient way to understand the relationship between Lie groups and Lie algebras. [2]

The problem we want to solve in this section is:

Let A, B be $n \times n$ matrices. Find a matrix C such that $e^C = e^A e^B$ where C is expressed as a (possibly infinite) series in A, B

[1] To be more precise, we can introduce parameters t_1, t_2 which are real and small and consider the multiplication of $\exp[t_1A]\exp[t_2B]$. The "degree" is the total degree of t_1 and t_2 in the Taylor expansion.

[2] The proof in this section follows Miller, *Symmetry Groups and their applications.*

The series we will obtain is convergent for "small" matrices. We can make this precise by, for example defining $\parallel A \parallel = \max|a_{ij}|$ and demanding that $\parallel A \parallel$, $\parallel B \parallel$ be sufficiently small. We comment on the radius of convergence below.

Lemma 1: Let A be a constant matrix in t. Then

$$\frac{d}{dt}e^{tA} = Ae^{tA} = e^{tA}A \tag{4.4}$$

Proof: Series expansion. ♠

To state the next lemma we introduce some notation:

Definition 4.1: For $A \in M_n(k)$ we denote by $\mathrm{Ad}(A)$ the linear transformation $M_n(k) \to M_n(k)$ defined by

$$\mathrm{Ad}(A) : B \mapsto [A, B] \tag{4.5}$$

We also denote:

$$(\mathrm{Ad}(A))^m B = \overbrace{[A, [A, \cdots [A, B] \cdots]}^{m \text{ times}} \tag{4.6}$$

where there are m commutators on the RHS.

Lemma 2:

$$\begin{aligned} e^A B e^{-A} &= e^{\mathrm{Ad}A}(B) = \Sigma \frac{(\mathrm{Ad}A)^j}{j!} B \\ &= B + [A, B] + \frac{1}{2}[A, [A, B]] + \cdots \end{aligned} \tag{4.7}$$

Proof: Let $B(t) = e^{tA} B e^{-tA}$.

Note $B(0) = B$

$\dot{B}(t) = Ae^{tA}Be^{-tA} - e^{tA}Be^{-tA}A = (\mathrm{Ad}A)(B(t))$

Now, by induction

$$(\frac{d}{dt})^j B(t) = (\mathrm{Ad}A)^j B(t) \tag{4.8}$$

$B(t)$ is analytic in t, so we can write the Taylor series around zero. ♠

Now we come to the much more nontrivial: **Lemma 3**: Let

$$f(z) = \frac{e^z - 1}{z} = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots \tag{4.9}$$

Then

$$-(\frac{d}{dt}e^{A(t)})e^{-A(t)} = e^{A(t)}\frac{d}{dt}e^{-A(t)} = -f(\mathrm{Ad}(A(t))) \cdot \dot{A}(t) \tag{4.10}$$

where $A(t)$ is any differentiable matrix function of t.

Proof:
This is nontrivial because $\dot{A}(t)$ does not commute with $A(t)$ in general!

$$\begin{aligned} B(s,t) &\equiv e^{sA(t)}\frac{d}{dt}e^{-sA(t)} \\ \frac{\partial B}{\partial s} &= A(t)e^{sA(t)}\frac{d}{dt}e^{-sA(t)} - e^{sA(t)}\frac{d}{dt}\left[e^{-sA(t)}A(t)\right] \\ &= \mathrm{Ad}(A(t))B(s,t) - \dot{A}(t) \\ \frac{\partial^j B}{\partial s_j} &= (\mathrm{Ad}(A(t)))^j B(s,t) - (\mathrm{Ad}A(t))^{j-1}\dot{A}(t) \end{aligned} \tag{4.11}$$

$B(0,t) = 0$ therefore again by Taylor:

$$\frac{1}{j!}\frac{\partial^j}{\partial s^j}B(s,t)\mid_{s=0} = -\mathrm{Ad}(A(t))^{j-1}\dot{A}(t) \qquad j \geq 1 \tag{4.12}$$

So

$$e^{sA(t)}\frac{d}{dt}(e^{-sA(t)}) = -\sum_{j=1}^{\infty}\frac{s^j(\mathrm{Ad}(A(t)))^{j-1}}{j!}\dot{A}(t) \tag{4.13}$$

Now set s=1. ♠
Note: you can rewrite this lemma as the statement:

$$\frac{d}{dt}e^{-A(t)} = -\int_0^1 e^{-(1-s)A(t)}\dot{A}(t)e^{-sA(t)}ds \tag{4.14}$$ eq:intfrm

because:

$$\begin{aligned} \frac{d}{dt}e^{-A(t)} &= -\int_0^1 e^{-(1-s)A(t)}\dot{A}(t)e^{-sA(t)}ds \\ &= -e^{-A(t)}\int_0^1 e^{sAd(A(t))}ds\dot{A}(t) \\ &= -e^{-A(t)}\left(\frac{e^{Ad(A(t))}-1}{Ad(A(t))}\right)\dot{A}(t) \end{aligned} \tag{4.15}$$ eq:intfrmi

Note that (4.14) is an intuitively appealing formula. For a finite product we have:

$$\begin{aligned} \frac{d}{dt}\Big[M_1(t)M_2(t)M_3(t)\cdots M_n(t)\Big] &= (\frac{d}{dt}M_1(t))M_2(t)M_3(t)\cdots \\ &+ (M_1(t))(\frac{d}{dt}M_2(t))M_3(t)\cdots \\ &+ (M_1(t))(M_2(t))(\frac{d}{dt}M_3(t))\cdots \\ &+ \cdots + M_1(t)M_2(t)\cdots(\frac{d}{dt}M_n(t)) \end{aligned} \tag{4.16}$$

Now regard

$$e^{A(t)} = \prod_{i=1}^{N} [e^{A(t)\Delta s}] \tag{4.17}$$

where $\Delta s = 1/N$. Now take $N \to \infty$.

Now we are finally ready to state the main theorem:

Theorem: (Baker-Campbell-Hausdorff formula)
Let:

$$g(z) = \frac{\log z}{z-1} = \sum_{j=0}^{\infty} \frac{(1-z)^j}{j+1} \tag{4.18}$$

be a power series about 1. Then when A, B are $n \times n$ matrices with $\parallel A \parallel, \parallel B \parallel$ sufficiently small, the matrix C given by the expansion:

$$\boxed{C = B + \int_0^1 g(e^{t\mathrm{Ad}A} e^{\mathrm{Ad}B})(A) dt} \tag{4.19}$$

satisfies $C = \log(e^A e^B)$.

Explicitly the first few terms are: [3]

$$\boxed{C = A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] + \frac{1}{12}[B,[B,A]] + \frac{1}{24}[A,[B,[A,B]]] + \cdots} \tag{4.20}$$

where the next terms are order ϵ^5 if we scale A, B by ϵ.[4]

Proof:

Introduce:

$$e^{C(t)} = e^{tA} e^B \tag{4.21}$$

eq:defcee

$C(0) = B$, and $C(1)$ is the matrix we want. We derive a differential equation for $C(t)$. By Lemma 4 we have:

$$e^{C(t)} \frac{d}{dt} e^{-C(t)} = -f(\mathrm{Ad}C(t))\dot{C}(t) \tag{4.22}$$

with

$$f(z) = \frac{e^z - 1}{z} \tag{4.23}$$

On the other hand, plugging in the definition (4.21),

$$e^{C(t)} \frac{d}{dt} e^{-C(t)} = e^{tA} \frac{d}{dt} e^{-tA} = -A \tag{4.24}$$

[3]It is useful to note that $[A,[B,[A,B]]] = -[B,[A,[B,A]]] = B^2A^2 - A^2B^2$

[4]One can find an algorithm for generating the higher order terms in Varadarajan's book on group theory.

by using Lemma 2. Therefore:

$$f(\mathrm{Ad}C(t))\dot{C}(t) = A \tag{4.25}$$

Now, f is a power series about 1 so it immediately follows that

$$\dot{C}(t) = f(Ad(C(t)))^{-1}A \tag{4.26}$$ eq:invtser

Let us make this more explicit.
Introduce the power series about 1

$$g(w) = \frac{\log w}{w-1} = \sum_{j=0}^{\infty} \frac{(1-w)^j}{j+1} \tag{4.27}$$

which satisfies the equation:

$$f(z)g(e^z) = \frac{e^z - 1}{z} \cdot \frac{z}{e^z - 1} = 1 \tag{4.28}$$

regarded as an identity of power series in z. Now we can substitute for z any operator $\mathcal{O}$, and use

$$g(e^{\mathcal{O}}) = f(\mathcal{O})^{-1}, \tag{4.29}$$ eq:invrl

and therefore we can solve for $\dot{C}$:

$$\begin{aligned} \dot{C}(t) &= f(\mathrm{Ad}(C(t)))^{-1} \cdot A \\ &= g(\exp(\mathrm{Ad}(C(t))) \cdot A \end{aligned} \tag{4.30}$$ eq:first

where we applied (4.29) with $\mathcal{O} = (Ad(C(t))$. This hardly seems useful, since we still don't know $C(t)$, but now since we have power series we can say

$$e^{\mathcal{O}} = e^{Ad(C(t))} = e^{\mathrm{Ad}(tA)}e^{\mathrm{Ad}(B)} \tag{4.31}$$ eq:formeff

Proof of claim:
Note that for all H:

$$\begin{aligned} e^{\mathrm{Ad}C(t)}H &= e^{C(t)}He^{-C(t)} && \text{by lemma 2} \\ &= e^{tA}e^{B}He^{-B}e^{-tA} && \text{def.ofC} \\ &= e^{\mathrm{Ad}(tA)}e^{\mathrm{Ad}(B)}H && \text{by lemma 2} \\ \Rightarrow e^{\mathrm{Ad}(C(t))} &= e^{\mathrm{Ad}(tA)}e^{\mathrm{Ad}(B)} \end{aligned} \tag{4.32}$$

Therefore:

$$\dot{C}(t) = g(e^{\mathrm{Ad}(tA)}e^{\mathrm{Ad}(B)}) \cdot A \tag{4.33}$$ eq:dffq

Now we integrate equation (4.33)

$$C(t) = C(0) + \int_0^t g(e^{\mathrm{Ad}(sA)} e^{\mathrm{Ad}(B)}) A ds \tag{4.34}$$

but $C(0) = B$, so

$$C = C(1) = \log(e^A e^B) = B + \int_0^1 g(e^{\mathrm{Ad}(s\ A)} e^{\mathrm{Ad}(B)}) A\ ds \tag{4.35}$$

which is what we wanted to show. ♠.

Remarks For suitable operators A, B on Hilbert space the BCH formula continues to hold.

Exercise
Use the BCH theorem to show that if

$$g_1 = e^{t_1 A_1}, \quad g_2 = e^{t_2 A_2} \tag{4.36}$$

the *group commutator*, $g_1 g_2 g_1^{-1} g_2^{-1}$ corresponds to the Lie algebra commutator:

$$g_1 \cdot g_2 \cdot g_1^{-1} \cdot g_2^{-1} = 1 + t_1 t_2 [A_1, A_2] + \mathcal{O}(t_1^2, t_2^2) \tag{4.37}$$

Thus we say a Lie algebra is "abelian" if $[A_1, A_2] = 0$ for all $A_1, A_2 \in \mathfrak{g}$. Otherwise it is "nonabelian."

Moreover, if $A_i = \frac{d}{dt}|_0 g_i(t)$ then $[A_1, A_2]$ is the Lie algebra element associated to the curve

$$g_{12}(t) = g_1(\sqrt{t}) \cdot g_2(\sqrt{t}) \cdot g_1^{-1}(\sqrt{t}) \cdot g_2^{-1}(\sqrt{t}) \tag{4.38}$$

Exercise
Work out the BCH series to order 5 in A, B.

Exercise
Show that we can also write:

$$C = \log(e^A e^B) = A + \int_0^1 g(e^{-\mathrm{Ad}(s\ B)} e^{-\mathrm{Ad}(A)}) B\ ds \tag{4.39}$$

4.2 Two Important Special Cases

4.2.1 The Heisenberg algebra

Suppose $[A, B] = z \cdot 1$ where z is a *scalar*. Then all higher commutators vanish and

$$e^A e^B = e^{A+B+\frac{1}{2}z\cdot 1} \tag{4.40}$$

hence:

$$e^A e^B = e^z \cdot e^B \cdot e^A \tag{4.41}$$

One very common application of this formua is in *quantum mechanics*. For example, if we consider the quantum mechanics of a single particle on a line then A, B can be scalar multiples of $\hat{q}, \hat{p}$, respectively where

$$[\hat{p}, \hat{q}] = -i\hbar 1 \tag{4.42}$$

The above formula says:

$$e^{\alpha\hat{p}} e^{\beta\hat{q}} = e^{-i\hbar\alpha\beta} e^{\beta\hat{q}} e^{\alpha\hat{p}} \tag{4.43}$$ eq:heisgroup

More generally,

$$e^{\alpha_1\hat{p}+\beta_1\hat{q}} e^{\alpha_2\hat{p}+\beta_2\hat{q}} = e^{-\frac{i}{2}(\alpha_1\beta_2-\alpha_2\beta_1)} e^{(\alpha_1+\alpha_2)\hat{p}+(\beta_1+\beta_2)\hat{q}} \tag{4.44}$$ eq:heisgroupi

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Exercise *The magnetic translation group*

Consider the problem of an electron confined to a two-dimensional plane x_1, x_2 Usually in quantum mechanics translations in x^1, x^2 by a^1, a^2 are represented by

$$U_1 = e^{ia^1 p_1} \qquad U_2 = e^{ia^2 p_2} \tag{4.45}$$ eq:transl

with $U_1 U_2 = U_2 U_1$, because $[p_1, p_2] = 0$.

Now suppose that there is a constant magnetic field B perpendicular to the x_1, x_2 space, as in the quantum Hall effect.

The Hamiltonian for a free electron in the presence of the magnetic field is:

$$H = \frac{1}{2m}\left((p_1 - eA_1)^2 + (p_2 - eA_2)^2\right) \tag{4.46}$$ eq:hamil

In a uniform magnetic field we can choose a gauge so that this can be put in the form:

$$H = \frac{1}{2m}\left((p_1 - \frac{eBx_2}{2})^2 + (p_2 + \frac{eBx_1}{2})^2\right) \tag{4.47}$$ eq:hamila

Now p_1, p_2 *do not commute with* H. This is hardly surprising since H is no longer translation invariant. Moreover the gauge invariant momenta $\tilde{p}_i := p_i - eA_i$ also do not commute with the Hamiltonian.

a.) Show that $[\tilde{p}_i, H] \neq 0$.

b.) Show that if we define the *magnetic translation operators*:

$$\pi_1 := p_1 + \frac{eBx_2}{2} \qquad \pi_2 := p_2 - \frac{eBx_1}{2} \tag{4.48}$$ eq:pies

then these satisfy $[\pi_i, H] = 0$. In fact, $[\pi_i, \tilde{p}_j] = 0$.

c.) The price we pay is that:

$$[\pi_1, \pi_2] = i\hbar eB \tag{4.49}$$ eq:picomm

The "magnetic translation group" is the obtained from the operators

$$\begin{aligned} U_1 &= \exp[ia^1\pi_1] \\ U_2 &= \exp[ia^2\pi_2] \end{aligned} \tag{4.50}$$

d.) Show that U_1, U_2 satisfy the relations:

$$U_1 U_2 = \exp[-i\hbar eBa^1a^2] U_2 U_1 \tag{4.51}$$ eq:heisgrp

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Comment on the "noncommutative plane"

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4.2.2 All orders in B, first order in A

Consider the following problem: Find $e^A e^B = e^C$, when $A = \epsilon$ is infinitesimal, to all orders in B but to first order in ϵ.

$$\begin{aligned} BCH \Rightarrow C &= B + \int_0^1 g(e^{t\mathrm{Ad}\epsilon} e^{\mathrm{Ad}B})(\epsilon) dt \\ &= B + \int_0^1 g\big(e^{\mathrm{Ad}B}\big)(\epsilon) dt + \mathcal{O}(\epsilon^2) \\ &= B + \frac{\mathrm{Ad}B}{e^{\mathrm{Ad}B} - 1}(\epsilon) \\ = B + \epsilon - \frac{1}{2}[B, \epsilon] &+ \frac{1}{12}[B, [B, \epsilon]] \\ &- \frac{1}{720}[B, [B, [B, [B, \epsilon]]]] + \cdots \end{aligned} \tag{4.52}$$

Note:

$$\begin{aligned} \frac{x}{e^x - 1} &= 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{x^4}{720} + \frac{x^6}{30240} - \frac{x^8}{1209600} + \frac{x^{10}}{47900160} \\ &- \frac{691}{1307674368000}x^{12} + \cdots \\ &\equiv \sum_{n=0}^{\infty} \frac{B_n x^n}{n!} \end{aligned} \tag{4.53}$$ eq:bernnum

is an important expansion in classical function theory -the numbers B_n are known as the Bernoulli numbers

There are many applications of this formula. One in particle physics is to spontaneous symmetry breaking where the formula above gives the chiral transformation law of the pion field. Here $B = \pi(x)$ is the pion field and ϵ is the chiral transformation parameter.

Exercise

Show that $e^C = e^A e^\epsilon$ is given to first order in ϵ by

$$C = A - \frac{\mathrm{Ad}(A)}{e^{-\mathrm{Ad}(A)} - 1}\epsilon \tag{4.54}$$

4.3 Region of convergence

The BCH formula has a finite radius of convergence. This is clear from the finite radius of convergence of the series expansion for $\log z$ around $z = 1$. Therefore it suffices to check that $e^{t\mathrm{Ad}A} e^{\mathrm{Ad}B}$ has characteristic values sufficiently close to 1.

In general, if A is diagonalizable with eigenvalues λ_i, then the eigenvalues of $\mathrm{Ad}A$ are $\lambda_i - \lambda_j$.

Exercise

a.) Let B be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that $Ad(B)$ acting on the space of all $n \times n$ complex matrices has the n^2 eigenvalues $\lambda_i - \lambda_j$ $1 \leq i, j \leq n$.

b.) Show that the Baker-Campbell-Hausdorff series can fail to converge if the eigenvalues of A differ by $2\pi i\mathbb{Z}$.

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5. Abstract Lie Algebras

ubsec:ssGenDef

5.1 Basic Definitions

Our discussion of vector fields on Lie groups led to the general notion of a Lie algebra. We repeat the definition:

Definition 5.1.1: An abstract Lie algebra $\underline{\mathbf{g}}$ over k is a vector space over k together with a product $(v_1, v_2) \to [v_1, v_2] \in \mathfrak{g}$, such that for all $v_1, v_2, v_3, \in \mathfrak{g}$ $\alpha, \beta \in k$.

1.) $[v_1, v_2] = -[v_2, v_1]$
2.) $[\alpha v_1 + \beta v_2, v_3] = \alpha[v_1, v_3] + \beta[v_2, v_3]$
3.) $[[v_1, v_2], v_3] + [[v_3, v_1], v_2] + [[v_2, v_3], v_1] = 0.$

Remarks

• You might be tempted to write

$$[v_1, v_2] \stackrel{?}{=} v_1 \cdot v_2 - v_2 \cdot v_1 \tag{5.1}$$ eq:tempted

but in the abstract definition (and in some examples) there is no sense in which we have defined a second product $v_1 \cdot v_2$ on $\underline{\mathbf{g}}$. Of course, this is true in some examples as in the Lie algebra $M_n(\mathbb{R})$ of all $n \times n$ matrices, but is certainly not true in general. For example, as we will see, the Lie algebra of $U(n)$ is the real vector space of $n \times n$ antihermitian matrcies. However, the product of antihermitian matrices is not antihermitian. $M_n(\mathbb{R})$ has more structure. Another example is the Lie algebra of vector fields. The product of vector fields is a second order differential operator, and is certainly not a vector field.

•

Any set of matrices that is closed under commutator satisfies the Jacobi identity:

$$[[X_1, X_2], X_3] + [[X_3, X_1], X_2] + [[X_2, X_3], X_1] = 0 \tag{5.2}$$

Simply because the product $X_1 X_2$ *is* defined for matrices. More generally, *any* associative algebra has a Lie algebra product by $[a_1, a_2] = a_1 a_2 - a_2 a_1$. Note that associativity is crucial. Moreover, if $\mathcal{A}$ is an associative algebra then the matrix commutator defines a Lie algebra structure on $M_n(\mathcal{A})$.

Exercise
Verify the statement of Remark 2.

Exercise
Considering Lie algebras from the viewpoint of algebras in general show that
a.) Lie algebras are nonassociative. Indeed, show that the associator is

$$[x_1, x_2, x_3] = [x_2, [x_3, x_1]] \tag{5.3}$$

b.) If a Lie algebra has a unit then it is trivial (over a field of characteristic $\neq 2$).

Just as for groups we can define a notion of homomorphism or isomorphism of Lie algebras, etc.

Definition 1 A *homomorphism* of two Lie algebras $\mu : \underline{\mathbf{g}}_1 \to \underline{\mathbf{g}}_2$ is
1.) a linear transformation (remember the $\underline{\mathbf{g}}_i$ are vector spaces)
2.) which preserves the Lie bracket:

$$\mu([X, Y]) = [\mu(X), \mu(Y)] \tag{5.4}$$

A one-one onto homomorphism is an isomorphism.

Some of the operations of linear algebra generalize nicely to Lie algebras. For example, a sub-Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ is a vector subspace such that

$$[X, Y] \in \mathfrak{h} \tag{5.5}$$

for all $X, Y \in \mathfrak{h}$. Furthermore, a *direct sum* of two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, written $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, is the direct sum as a vector space, but also has the property that

$$[X_1, X_2] = 0 \tag{5.6}$$ eq:dirctsum

for all $X_1 \in \mathfrak{g}_1$ and $X_2 \in \mathfrak{g}_2$.

We can also make a *semidirect sum* of Lie algebras, analogous to the semidirect product of Lie groups. In the exercise below we introduce the notion of a derivation of an algebra. If there is a homomorphism of Lie algebras from $\mathfrak{g}$ to the Lie algebra of derivations of $\mathfrak{h}$, say $Y \in \mathfrak{g} \to D_Y \in \mathrm{Der}(\mathfrak{h})$, (note that D_Y need not be an inner derivation), then we can define a Lie bracket on $\mathfrak{h} \oplus \mathfrak{g}$ by:

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2] + D_{Y_1}(X_2) - D_{Y_2}(X_1), [Y_1, Y_2]) \tag{5.7}$$ eq:semidrctsum

In the exercises you check that this indeed defines a Lie algebra structure.

In general, if $\mathfrak{h} \subset \mathfrak{g}$ is a sub-Lie algebra it is not true that the vector space quotient $\mathfrak{g}/\mathfrak{h}$ is another Lie algebra. However, if $\mathfrak{h}$ is an *ideal*, then $\mathfrak{g}/\mathfrak{h}$ can be given the structure of a Lie algebra.

Definition 2 An *ideal*, or, *invariant subalgebra* in $\mathfrak{g}$ is a sub Lie algebra $\mathfrak{h}$ such that for all $X \in \mathfrak{h}$, and all $Y \in \mathfrak{g}$,

$$[X, Y] \in \mathfrak{h} \tag{5.8}$$

Note that there is no distinction between left- and right-ideals in a Lie algebra.

Some of the operations of linear algebra *do not* generalize to Lie algebras. For example, the tensor product of Lie algebras $\mathfrak{g}_1 \otimes \mathfrak{g}_2$ makes sense as a vector space, but does not have any natural Lie bracket.

Finally, suppose $\mu : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie-algebra homomorphism. Then it is easy to show that $\ker\mu \subset \mathfrak{g}_1$ is not only a Lie subalgebra but an ideal and, just as for groups, if μ is surjective then

$$\mathfrak{g}_1/\ker\mu \cong \mathfrak{g}_2 \tag{5.9}$$

Exercise *Ideals*

Show that if $\mathfrak{h} \subset \mathfrak{g}$ is an ideal then $\mathfrak{g}/\mathfrak{h}$ is a well-defined Lie algebra.

Give a counterexample when $\mathfrak{h}$ is not an ideal.

Exercise *Tensor products*
Is the tensor product of two Lie algebras a Lie algebra?

Exercise *Derivations*
A *derivation* of an algebra $\mathcal{A}$ over a field k is a k-linear map $D : \mathcal{A} \to \mathcal{A}$ which satisfies the "Leibniz rule"

$$D(ab) = D(a)b + aD(b) \tag{5.10}$$

eq:leibniz

a.) Show that if D_1, D_2 are two derivations of $\mathcal{A}$ then

$$[D_1, D_2] = D_1D_2 - D_2D_1 \tag{5.11}$$

is a derivation (although D_1D_2 is not a derivation). Conclude that the set of derivations $Der(\mathcal{A})$ itself forms a Lie algebra.

b.) If $a \in \mathcal{A}$ and $\mathcal{A}$ is associative, then the *inner derivation* D_a defined by

$$D_a(b) := ab - ba \tag{5.12}$$

is indeed a derivation, and that

$$[D_{a_1}, D_{a_2}] = D_{[a_1,a_2]} \tag{5.13}$$

c.) If $\mathcal{A}$ is a Lie algebra then show that the inner derivation

$$D_y(x) := [y, x] \tag{5.14}$$

is indeed a derivation:

$$D_y([x_1, x_2]) = [D_y(x_1), x_2] + [x_1, D_y(x_2)] \tag{5.15}$$

and moreover

$$[D_{y_1}, D_{y_2}] = D_{[y_1,y_2]} \tag{5.16}$$

d.) Show that if D is any derivation then

$$[D, D_a] = D_{D(a)} \tag{5.17}$$

and conclude that the inner derivations form an ideal in the Lie algebra of derivations.

Exercise *Semidirect sums*

a.) Show that (5.7) is a valid Lie bracket: It is antisymmetric and satisfies the Jacobi relation.

b.) Going back to chapter one, let us consider the semidirect product $H \ltimes G$ of Lie groups where $\alpha : G \to Aut(H)$ is a continuous homomorphism. Show that if we write $\alpha_{1+\epsilon Y+\cdots} = 1 + \epsilon D_Y + \cdots$ then the Lie algebra of $H \ltimes G$ is indeed the semidirect sum

$$\mathfrak{h}\tilde{\oplus}\mathfrak{g} \tag{5.18}$$

5.2 Examples: Lie algebras of dimensions 1, 2, 3

1.) One dimensional real Lie algebras $= \mathbb{R}$. Let $\vec{e}_1$ be the a basis vector. Then $[\vec{e}_1, \vec{e}_1] = 0$. The corresponding Lie group is $(\mathbb{R}^*, \times) \cong (\mathbb{R}, +)$.

2.) Two dimensional real Lie algebras are $\mathfrak{g} = \mathbb{R}^2$, as vector spaces. Let $\vec{e}_1, \vec{e}_2$ be basis vectors. We can define:

a.) The abelian Lie algebra:

$$[\vec{e}_1, \vec{e}_2] = 0. \tag{5.19}$$

The corresponding Lie group is $\mathbb{R}^* \times \mathbb{R}^*$.

b.) The nonabelian Lie algebra:

$$[\vec{e}_1, \vec{e}_2] = \vec{e}_1 \tag{5.20}$$

Proposition. Any 2-dimensional Lie algebra is isomorphic to (a) or (b), and these are not isomorphic.

Proof: (We follow Talman and Wigner):

Suppose $\mathfrak{g} =$ 2-dimensional nonabelian Lie algebra. Let $\vec{a}, \vec{b} \in \mathfrak{g}$ be linearly independent vectors. Then $[\vec{a}, \vec{b}] = \vec{e}_1 \neq 0$. *Any* two vectors in $\mathfrak{g}$ can be written

$$\vec{w}_1 = \alpha_1 \vec{a} + \beta_1 \vec{b} \tag{5.21}$$

$$\vec{w}_2 = \alpha_2 \vec{a} + \beta_2 \vec{b} \tag{5.22}$$

$$[\vec{w}_1, \vec{w}_2] = (\alpha_1 \beta_2 - \beta_1 \alpha_2)\vec{e}_1. \tag{5.23}$$

Choose a *new* basis $\vec{e}_1, \vec{k}$ where $\vec{k}$ is linearly independent of $\vec{e}_1$.

$$[\vec{e}_1, \vec{k}] = c\vec{e}_1 \qquad c \neq 0 \tag{5.24}$$

put $\vec{k} = c\vec{e}_2$ then $[\vec{e}_1, \vec{e}_2] = \vec{e}_1$. ♠

Exercise

Consider the Lie algebra spanned by: $\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\}$.

Show

$$\vec{e}_1 \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{5.25}$$

$$\vec{e}_2 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.26}$$

is a Lie algebra isomorphism with the Lie algebra in example 2b above.

Describe the corresponding Lie group.

Example 3 The three-dimensional Lie algebras. By similar arguments one can list the 6 inequivalent 3D real Lie Algebras.

a.) $[\vec{e}_1, \vec{e}_2] = [\vec{e}_1, \vec{e}_3] = [\vec{e}_2, \vec{e}_3] = 0$ b.) $[\vec{e}_1, \vec{e}_2] = [\vec{e}_1, \vec{e}_3] = 0 \quad [\vec{e}_2, \vec{e}_3] = \vec{e}_1$ c.) $[\vec{e}_1, \vec{e}_2] = 0 \quad [\vec{e}_1, \vec{e}_3] = \vec{e}_1 \quad [\vec{e}_2, \vec{e}_3] = \lambda\vec{e}_2$ d.) $[\vec{e}_1, \vec{e}_2] = 0 \quad [\vec{e}_1, \vec{e}_3] = \vec{e}_1 \quad [\vec{e}_2, \vec{e}_3] = \vec{e}_1 + \vec{e}_2$ e.) $[\vec{e}_1, \vec{e}_2] = 0 \quad [\vec{e}_1, \vec{e}_3] = \lambda\vec{e}_1 - \vec{e}_2 \quad [\vec{e}_2, \vec{e}_3] = \vec{e}_1 + \lambda\vec{e}_2$

f.) $[\vec{e}_1, \vec{e}_2] = \vec{e}_3 \quad [\vec{e}_2, \vec{e}_3] = \vec{e}_1 \quad [\vec{e}_3, \vec{e}_1] = \vec{e}_2$

Remarks

• The examples c,e exhibit an interesting phenomenon. Note that there is a continuous parameter λ so there are infinitely many inequivalent 3d Lie algebras, parametrized by λ. Parameter spaces labelling inequivalent mathematical objects are often called moduli spaces.

• (b) is called the *Heisenberg algebra*

• (f) is the Lie algebra of $sl(2)$, or of $su(2)$ depending on whether we work over $\mathbb{C}$ or over $\mathbb{R}$.

For a proof of the above classification, together with matrix representations, see the book of Wigner and Talman.

Exercise

Show that the matrices e_{13}, e_{23} and $-(e_{11} + \lambda e_{22})$ give a matrix representation of the Lie algebra of type (c) above.

5.3 Structure constants

Suppose we choose a basis $\{\vec{e}_i\}$ for $\underline{\mathbf{g}}$ as a vector space. Then there exist constants

$$[\vec{e}_i, \vec{e}_j] = \sum_k f_{ij}^k \vec{e}_k \tag{5.27}$$

eq:structurecc

Definition f_{ij}^k are called the *structure constants of* $\underline{\mathbf{g}}$ *wrt the basis* $\{\vec{e}_i\}$.

Remark: It sometimes happens that Lie algebras are defined in terms of a special basis together with relations (5.27). One must always be sure to check that such a presentation in fact satisfies the Jacobi identity. There have been cases where someone proposes a new Lie algebra, but the Jacobi identity fails (and hence no such Lie algebra exists).

Exercise *Properties of structure constants*

Show that

(a) $f_{jk}^i = -f_{kj}^i$

(b)

$$\sum_\ell \left[f_{ij}^\ell f_{\ell k}^m + f_{jk}^\ell f_{\ell i}^m + f_{ki}^\ell f_{\ell j}^m \right] = 0 \qquad \forall i, j, k, m \tag{5.28}$$

In (a) there is in general no symmetry property relating i to j, k. (b) is equivalent to the "Jacobi identity."

Exercise

A symmetric bilinear form $h : \mathfrak{g} \times \mathfrak{g} \to k$ is said to be an *invariant form* if

$$h(Ad(X)Y, Z) + h(Y, Ad(X)Z) = 0 \tag{5.29}$$

for all X, Y, Z.

Choosing a basis e_i for $\mathfrak{g}$ let $h_{ij} = h(e_i, e_j)$ and define

$$f_{ijk} := h_{i\ell} f_{jk}^\ell \tag{5.30}$$

Show that if h is an invariant form then f_{ijk} is totally antisymmetric on all three indices.

5.4 Representations of Lie algebras and Ado's Theorem

In order to discuss the main theorem of this chapter, Lie's theorem, we need a few definitions and an auxiliary result, Ado's theorem.

The space End(V) of endomorphisms of a vector space V is a Lie algebra. Thus we can define:

Definition: A *representation* of a Lie algebra $\mathfrak{g}$ with representation space V is a Lie algebra homomorphism

$$T : \mathfrak{g} \to End(V) \tag{5.31}$$

where $End(V)$ is the space of all linear transformations on V. We also say V *is a* $\mathfrak{g}$*-module.* A matrix representation is a homomorphism $T : \mathfrak{g} \to M_n(k)$ for some n.

Put differently, we have a vector space V together with a multiplication

$$\begin{aligned} \mathfrak{g} \times V &\to V \\ (X, v) &\to X \cdot v \end{aligned} \tag{5.32}$$

which is bilinear in X, v and such that

$$[X, Y] \cdot v = X \cdot (Y \cdot v) - Y \cdot (X \cdot v) \tag{5.33}$$

Upon choosing a basis for V a $\mathfrak{g}$-module provides a matrix representation of $\mathfrak{g}$.

Two representations

$$T_i : \mathfrak{g} \to End(V_i) \tag{5.34}$$

are said to be *equivalent* if there is an isomorphism $S : V_1 \to V_2$ such that $S^{-1}T_2(X)S = T_1(X)$ for all $X \in \mathfrak{g}$. If the representation defines an isomorphism of $\mathfrak{g}$ with its image, i.e. if ker$T = \{0\}$, then the representation is said to be *faithful.*

Remarks

• Given any matrix representation of a Lie group G we automatically get a matrix representation of the Lie algebra $L(G)$: If $X = \frac{d}{dt} g(t)|_0 \in \mathfrak{g}$ then

$$T(X) := \frac{d}{dt} T(g(t))|_0 \tag{5.35}$$ eq:diffgrp

Exercise

Show that (5.35) defines a Lie algebra representation $T([X, Y]) = [T(X), T(Y)]$ by using the BCH formula.

• Any Lie algebra has a canonical representation: $V = \mathfrak{g}$,

$$X \in \mathfrak{g} \to T(X) = Ad(X) \in End(\mathfrak{g}) \tag{5.36}$$

where

$$Ad(X) : \mathfrak{g} \to \mathfrak{g} \tag{5.37}$$

is the linear operator:

$$Ad(X)(Y) := [X, Y] \tag{5.38}$$ eq:adrep

This is called the *adjoint representation*.

Exercise *The adjoint representation*
a.) Show that (5.38) *is* a representation. To do this you must show:

$$[Ad(X), Ad(Y)] = Ad([X, Y]) \tag{5.39}$$ eq:adrepii

b.) Show that the matrix elements of the adjoint representation wrt a basis X_i for $\mathfrak{g}$ are the given by the *structure constants* relative to that basis:

$$(Ad(X_i))_j{}^k = f_{ij}^k \tag{5.40}$$ eq:strccnt

c.) Show that the kernel of the homomorphism $X \to Ad(X)$ is the *center* of the Lie algebra:

$$Z(\mathfrak{g}) := \{X | Ad(X)Y = 0 \qquad \forall Y \in \mathfrak{g}\} \tag{5.41}$$

Thus the adjoint representation is in general not a faithful representation of $\mathfrak{g}$.
d.) Suppose $\mathfrak{g}$ is abelian. Describe the adjoint representation.
e.) Define the symmetric bilinear form

$$h(X, Y) = \mathrm{Tr}_{\mathfrak{g}} Ad(X) Ad(Y) \tag{5.42}$$

Show that $h(X, Y)$ is an invariant bilinear form. This is known as the Cartan-Killing metric, and is very important in determining the structure of semisimple Lie algebras.

Evidently, although every Lie algebra has a canonical representation - the adjoint representation - this need not be a faithful representation.

Theorem[Ado's Theorem]. Every finite dimensional Lie algebra has a faithful matrix representation.

Proof: The proof is somewhat involved. See Jacobsen, *Lie Algebras*, p. 202. The main idea is this: The kernel of the adjoint representation is the center $Z(\mathfrak{g})$. Therefore, if one can find some representation V of $\mathfrak{g}$ which is faithful on $Z(\mathfrak{g})$ then we can take a direct sum with the adjoint representation: $V \oplus \mathfrak{g}$ to produce a faithful representation of $\mathfrak{g}$.

One begins by constructing a faithful representation of $Z(\mathfrak{g})$. If it is c-dimensional then in a $c + 1$-dimensional vector space we choose a nilpotent operator N such that $N^c \neq 0$. Then $Z(\mathfrak{g})$ is represented by the commutative Lie algebra generated by $N, N^2, \ldots, N^c$. Indeed, we can choose any basis for $Z(\mathfrak{g})$ and map the basis elements to the powers of N. Such a representation is faithful.

Thus we have produced a faithful representation, $V \oplus \mathfrak{g}$, of $Z(\mathfrak{g}) \oplus (\mathfrak{g}/Z(\mathfrak{g}))$. Next, one needs to upgrade this into a representation of $\mathfrak{g}$ itself. For the details, see Jacobsen. ♠

************ should there be a discussion of irreps and schur's lemma here? *************

6. Lie's theorem

We have now seen how Lie algebras emerge from Lie groups. What about going the other way, i.e. from Lie algebras to Lie groups?

Consider the case of matrix Lie algebras. If *commutators* are closed under some property then exponentiating matrices will (locally) form a group, by the BCH formula. For example, if A is anti-hermitian, $A = -A^{\dagger}$, then $U = \exp(A)$ is unitary:

$$UU^{\dagger} = (e^{A})(e^{A^{\dagger}}) = e^{A}e^{-A} = 1 \tag{6.1}$$

Conversely, consider the collection of unitary matrices of the form

$$\tilde{U}(N) = \{e^{A} : A^{\dagger} = -A\} \tag{6.2}$$

Claim: $\tilde{U}(N)$ is locally closed under multiplication, i.e., it is closed as long as the BCH formula converges. The main thing we must check is that if we write

$$e^{A}e^{B} = e^{C} \tag{6.3}$$

with C given by the BCH formula then C is antihermitian. Suppose A, B antihermitian $A^{\dagger} = -A, B^{\dagger} = -B$. Then

$$[A, B]^{\dagger} = -[A, B] \tag{6.4}$$

is antihermitian. Now, for the same reason, all the terms in the BCH series are antihermitian since the expansion coefficients in the Taylor series expansion are *real*. Thus, just using BCH we would be tempted to conclude that $\tilde{U}(N)$ forms a group. However, we should worry about three problems: The BCH series does not always converge, the exponential map can fail to be surjective, and the exponential map is generally not 1-1. In fact, it turns out that the exponential map for $U(N)$ is actually surjective, and hence $\tilde{U}(N)$ above is a group after all, but we need extra information to reach this conclusion, and our discussion would also fail for certain noncompact groups.

Thus, for the general case we are going to have to work harder.

The central theorem of the subject, Lie's theorem, is the following: [5]

Theorem:

a.) Every finite dimensional Lie algebra $\mathfrak{g}$ arises from a unique (up to isomorphism) connected and simply connected Lie group G.

b.) Under this correspondence, Lie group homomorphisms $f : G_1 \to G_2$ are in $1-1$ correspondence with Lie algebra homomorphisms $\mu : T_1G_1 \to T_1G_2$.

[5] We follow closely G. Segal's elegant exposition in *Lectures on Lie Groups and Lie Algebras*, by R. Carter, G. Segal, and I. MacDonald.

Remarks

• Locally isomorphic groups have the same Lie algebra, so we need to use the simply connected cover to get a 1-1 correspondence.

• **Warning**: It follows from the theorem that if $\mathfrak{h}$ is a sub Lie algebra of $\mathfrak{g} = T_1 G$ then there is a Lie group H such that $\mathfrak{h} \cong T_1 H$, and moreover there is a group homomorphism $H \to G$. However, H need not be homomorphic to any topological subgroup of G. For example, take $G = SU(n)$, $n > 2$ and consider a $U(1) \times U(1)$ subgroup K of the maximal torus, say

$$Diag\{\lambda, \mu, (\lambda\mu)^{-1}, 1, \dots, 1\} \qquad \lambda, \mu \in U(1) \tag{6.5}$$

eq:embedding

The group K has as Lie algebra the abelian Lie algebra $\mathbb{R} \oplus \mathbb{R}$. The corresponding Lie algebra homomorphism is

$$\nu : (x, y) \to Diag\{2\pi i x, 2\pi i y, -2\pi i(x + y), 0, \dots, 0\} \tag{6.6}$$

Now, for $\mathfrak{h}$ take the abelian Lie algebra $\mathbb{R}$. Its corresponding connected and simply connected Lie group is $(\mathbb{R}, +)$. Choose the homomorphism μ to be obtained by embedding $\mathbb{R}$ into $\mathbb{R} \oplus \mathbb{R}$ via $1 \to (1, \alpha) \in \mathbb{R} \oplus \mathbb{R}$. Considering $\mathbb{R} \oplus \mathbb{R}$ to be a subalgebra of $L(SU(n))$ using ν we have

$$\mu : x \to Diag\{2\pi i x, 2\pi i \alpha x, -2\pi i(1 + \alpha)x, 0, \dots, 0\} \tag{6.7}$$

The corresponding Lie group homomorphism $(\mathbb{R}, +) \to SU(n)$ guaranteed by the theorem takes

$$x \to Diag\{e^{2\pi i x}, e^{2\pi i \alpha x}, e^{-2\pi i(1+\alpha)x}, 1, \dots, 1\} \tag{6.8}$$

If α is rational, the image is a topological subgroup of $SU(n)$. Now consider the case where α is irrational. The image is a copy of the Lie group $\mathbb{R}$ embedded in $SU(n)$, but this image is not a topological subgroup of $SU(n)$ since it is not a submanifold. Indeed, it densely fills the $U(1) \times U(1)$ subgroup K.

• Lie's theorem has an elegant and concise statement in the mathematical language of categories. There is a category $\mathcal{C}_1$ whose objects are finite dimensional connected and simply connected Lie groups, and whose morphisms are group homomorphisms. There is a category $\mathcal{C}_2$ whose objects are finite dimensional Lie algebras, and whose morphisms are Lie algebra homomorphisms. In categorical language, Lie's theorem is simply the statement that the functor $G \to L(G)$ is an equivalence of the categories $\mathcal{C}_1$ and $\mathcal{C}_2$. Indeed, the point of the theorem is to construct the functor $L(G) \to G$ such that the composition of the two functors is the identity. [6]

Proof: Let us now sketch the proof of Lie's theorem.

The hard part is showing that you can always exponentiate a finite dimensional Lie algebra to form a Lie group.

[6] In general to prove an equivalence of categories one only needs a natural transformation of FG and GF to the identity.

We use Ado's theorem to assume that $\mathfrak{g} \subset M_n(\mathbb{R})$ is a sub Lie-algebra of $M_n(\mathbb{R})$ for some n. [7]

Our Lie group will be built by thinking about the (infinite dimensional!) space of smooth paths

$$\gamma : [0,1] \to \mathfrak{g} \tag{6.9}$$

with $\gamma(0) = \gamma(1) = 0$. To such a path we can associate a group element $g_\gamma \in GL(n, \mathbb{R})$ by the "holonomy"

$$g_\gamma = P\exp \int_0^1 \gamma(t')dt' \tag{6.10}$$

This can be defined using the path ordered product or as the solution $g_\gamma(1)$ to the first order ODE:

$$\frac{d}{dt} g_\gamma(t) = \gamma(t) g_\gamma(t) \tag{6.11}$$

eq:dffqa

For bounded paths the path ordered exponential always converges, so $g_\gamma(1)$ is well-defined.

Figure 1: A closed loop in the Lie algebra maps to an open path in $GL(n, \mathbb{R})$.

fig:looplie

We would like the set of elements g_γ to form a subgroup. To do this we can add the technical condition that all derivatives $\frac{d^n}{dt^n}\gamma$, $n \geq 1$, vanish at $t = 0, 1$.

Define concatenation of paths:

$$\gamma_1 * \gamma_2 = \begin{cases} 2\gamma_2(2t), & 0 \leq t \leq \frac{1}{2} \\ 2\gamma_1(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases} \tag{6.12}$$

eq:compone

The solution of (4.33) for this concatenation is

$$g_{\gamma_1 * \gamma_2}(t) = \begin{cases} g_{\gamma_2}(2t), & 0 \leq t \leq \frac{1}{2} \\ g_{\gamma_1}(2t-1) g_{\gamma_2}(1), & \frac{1}{2} \leq t \leq 1 \end{cases} \tag{6.13}$$

eq:componea

and hence:

$$g_{\gamma_1 * \gamma_2} = g_{\gamma_1} g_{\gamma_2} \tag{6.14}$$

eq:comptwo

[7] This can be dispensed with, but the proof is harder. See below.

Thus, group elements of the form g_γ are closed under multiplication. Define $G' \subset GL(n,\mathbb{R})$ to be the set of matrices g_γ corresponding to paths of the above type. From (6.14) we see that G' is closed under group multiplication. Moreover, running γ backwards gives the inverse element, so G' is indeed a subgroup of $GL(n,\mathbb{R})$. Unfortunately, G' is not quite the group we are after, because it might not be a topological subgroup of $GL(n,\mathbb{R})$, as the above example with irrational α shows. Therefore, we cannot conclude it is a Lie group.

To take care of this we define $\mathcal{P}$ to be the space of all closed paths $\gamma : [0,1] \to \mathfrak{g}$ with all derivatives vanishing at $t = 0, 1$:

$$\mathcal{P} = \{\gamma : [0,1] \to \mathfrak{g} | \left(\frac{d}{dt}\right)^n \gamma|_{t=0,1} = 0, n \geq 0\} \tag{6.15}$$ eq:curlype

This is an infinite-dimensional manifold. To compute the tangent space consider a path of elements in $\mathcal{P}$ going through some $\gamma_*(t)$. That is, a path of paths $\gamma(s,t)$ with $\gamma(0,t) = \gamma_*(t)$. Differentiating a path of such paths gives

$$T_{\gamma_*(t)}\mathcal{P} \cong \mathcal{P} \tag{6.16}$$ eq:curlypea

Moreover $\mathcal{P}$ has a multiplication $\gamma_1 * \gamma_2$. But it is not quite an infinite-dimensional Lie group - there is no identity or inverse. The natural identity would be the zero path $\gamma_0(t) = 0$, but $\gamma_0 * \gamma$ is only homotopic to γ. ($\mathcal{P}$ is in fact an infinite-dimensional *homotopy-*Lie group.)

Now, equation (6.14) shows that $\mathcal{P}$ has a map to G' preserving multiplication. The group we are after is obtained from $\mathcal{P}$ by imposing an equivalence relation:

$$\gamma_1 \sim \gamma_2 \qquad \Leftrightarrow \qquad g_{\gamma_1} = g_{\gamma_2} \tag{6.17}$$ eq:equivrel

We claim that $G := \mathcal{P}/\sim$ is a finite dimensional manifold, locally modeled on $\mathfrak{g}$, and moreover it is a Lie group. We will compute its tangent space and Lie algebra in a moment. Note that the quotient relation allows us to show that G really is a group. Moreover, it is clearly connected and simply-connected, because we can shrink paths.

Thus, G will be a nice smooth Lie group, although its image $G' \subset GL(n,\mathbb{R})$ might be very bad.

Now we would like to compute the tangent space to G, that is, the Lie algebra of G. We work in the neighborhood of the zero path $\gamma_0(t) = 0$ in $\mathcal{P}$.

Recall that there is a neighborhood $\mathcal{U}$ of the identity in $GL(n,\mathbb{R})$ so that log is well-defined. By definition of the topology of $\mathcal{P}$, there is an open neighborhood $\tilde{\mathcal{U}}$ of the zero path in $\mathcal{P}$ obtained by considering those paths so that $g_\gamma(t)$ lies in $\mathcal{U}$ for all t and hence $\log g_\gamma(t)$ is well-defined. Define $\eta_\gamma(t)$ for such a path by

$$g_\gamma(t) = e^{\eta_\gamma(t)}. \tag{6.18}$$ eq:defeta

At this point we only know that $\eta_\gamma : [0,1] \to M_n(\mathbb{R})$. Note that although $\gamma(t)$ are closed paths in $\mathfrak{g}$, $\eta_\gamma(t)$ in general will be an open path. We can take $\eta_\gamma(0) = 0$, but in general $g_\gamma \neq 1$ and in this case $\eta_\gamma(1)$ cannot vanish.

Now we show that $\eta_\gamma(t)$ is in fact valued in $\mathfrak{g}$, not just $M_n(\mathbb{R})$. To do this, note that

$$\gamma(t) = (\frac{d}{dt}g_\gamma(t))g_\gamma(t)^{-1} = \frac{e^{\mathrm{Ad}(\eta_\gamma(t))}-1}{\mathrm{Ad}(\eta_\gamma(t))}(\dot{\eta}_\gamma(t)) \tag{6.19}$$ eq:diffleta

Now suppose $\gamma(t)$ is scaled by ϵ. Then there a solution to (6.19) given by a power series in ϵ: $\eta_{\epsilon\gamma(t)} = \epsilon\eta_1(t) + \epsilon^2\eta_2(t) + \cdots$ where we solve

$$\begin{aligned}
\dot{\eta}_1 &= \gamma(t) \\
\dot{\eta}_2 + \frac{1}{2}[\eta_1, \dot{\eta}_1] &= 0 \\
\dot{\eta}_3 + \frac{1}{2}[\eta_1, \dot{\eta}_2] + \frac{1}{2}[\eta_2, \dot{\eta}_1] + \frac{1}{6}[\eta_1, [\eta_1, \dot{\eta}_1]] &= 0 \\
\cdots &= 0
\end{aligned} \tag{6.20}$$

By induction we see that $\eta_\gamma(t) \in \mathfrak{g}$ for all t.

Now we remark that by (6.18) and the fact that exp is $1-1$ in our neighborhood, the equivalence relation (6.17) is the same as $\eta_{\gamma_1}(1) = \eta_{\gamma_2}(1)$. The local neighborhood $\mathcal{K} = \pi(\tilde{\mathcal{U}})$ of the identity in G is precisely

$$\mathcal{K} \cong \{\eta : [0,1] \to \mathfrak{g} | \eta(0) = 0\}/\sim \tag{6.21}$$ eq:newlocnb

where

$$\eta_1 \sim \eta_2 \qquad \Leftrightarrow \qquad \eta_1(1) = \eta_2(1) \tag{6.22}$$

But this means that an equivalence class is uniquely parametrized by the element of $\mathfrak{g}$ given by the endpoint of the path. Moreover, the space $\mathcal{P}$ is trivially a Lie algebra: The Lie bracket is just

$$[\eta_1, \eta_2](t) := [\eta_1(t), \eta_2(t)] \tag{6.23}$$ eq:liebrckt

and this Lie bracket descends to the equivalence relation. Thus, the space (6.21) is isomorphic to $\mathfrak{g}$ as a Lie algebra.

This establishes part (a) of the theorem.

To show part (b), first note that a homomorphism of Lie groups easily determines a homomorphism of Lie algebras by considering one-parameter subgroups through 1. Now for the converse, suppose we are given $\mu : \mathfrak{g}_1 \to \mathfrak{g}_2$. Using part (a) we can identify $\mathfrak{g}_1 \cong T_1G_1$ and $\mathfrak{g}_2 \cong T_1G_2$ where G_1, G_2 are connected and simply connected.

We would like to define $f(g)$ by writing $g = e^X$ and taking

$$f(e^X) := e^{\mu(X)} \tag{6.24}$$ eq:triy

Certainly, by BCH, if (6.24) makes sense then given that μ is a homomorphism of Lie algebras, f is a homomorphism of Lie groups. The problem is that X need not exist, and even if it does, it need not be unique. So, if $e^X = e^{X'}$ then it might not be true that $\mu(X) = \mu(X')$ and hence the attempt to define f by (6.24) in fact fails.

Instead we define $f(g)$ as follows. To begin, we choose a path $g(t)$, now in G_1, not $\mathfrak{g}_1$, with $g(0) = 1$ and $g(1) = g$. We get a corresponding path in $\mathfrak{g}_1$:

$$\gamma(t) := \dot{g}(t)g^{-1}(t). \tag{6.25}$$

Let $\tilde{\gamma}(t) := \mu(\gamma(t))$ be the corresponding path in $\mathfrak{g}_2$. We can solve

$$\frac{d}{dt}\tilde{g}(t) = \tilde{\gamma}(t)\tilde{g}(t) \tag{6.26}$$

with boundary condition $\tilde{g}(0) = 1_{G_2}$, and attempt to define

$$f(g) := \tilde{g}(1). \tag{6.27}$$

eq:trydefii

Note that if we have two group elements g_1, g_2 then, as we have seen $g_1 \cdot g_2$ corresponds to $\gamma_1 * \gamma_2$, hence to $\tilde{\gamma}_1 * \tilde{\gamma}_2$ hence to $\tilde{g}_1\tilde{g}_2$, so we do get a homomorphism.

The problem is, we need to show that $f(g)$ in (6.27) is well-defined since we made a choice of path $g(t)$ to define it. It is exactly at this point we make use of the fact that G_1 is simply connected. Suppose that $g_0(t), g_1(t)$ are two paths with $g_0(1) = g_1(1)$. Since G_1 is simply connected we can write a homotopy $g(s,t)$ between these two paths.

That is, there is a smooth map $g(s,t)$ mapping $I \times I \to G$ such that $g(0,t) = g_0(t)$, $g(1,t) = g_1(t)$, $g(s,0) = 1$ and $g(s,1) = g$:

Then we define:

$$\begin{aligned} \gamma(s,t) &:= \frac{\partial}{\partial t}g(s,t)g(s,t)^{-1} \\ \eta(s,t) &:= \frac{\partial}{\partial s}g(s,t)g(s,t)^{-1} \end{aligned} \tag{6.28}$$

eq:twodervs

Note that these satisfy the Maurer-Cartan equation

$$\frac{\partial}{\partial s}\gamma - \frac{\partial}{\partial t}\eta = [\eta, \gamma] \tag{6.29}$$

eq:mceq

An important point is that the converse also holds. Namely, (6.29) is the integrability condition for the existence of a solution $g(s,t)$ to (6.28) for a given γ, η.

Therefore, we apply the Lie algebra homomorphism μ to (6.29) to get

$$\frac{\partial}{\partial s}\tilde{\gamma} - \frac{\partial}{\partial t}\tilde{\eta} = [\tilde{\eta}, \tilde{\gamma}] \tag{6.30}$$

eq:mceqa

and, using the converse result, this is the integrability condition for the existence of a solution $\tilde{g}(s,t)$ to

$$\begin{aligned} \frac{\partial}{\partial t}\tilde{g}(s,t) &:= \tilde{\gamma}(s,t)\tilde{g}(s,t) \\ \frac{\partial}{\partial s}\tilde{g}(s,t) &:= \tilde{\eta}(s,t)\tilde{g}(s,t) \end{aligned} \tag{6.31}$$ eq:twodervsa

Now, $g(s,1) = g$ is independent of s. Therefore $\eta(s,1) = 0$. Therefore $\tilde{\eta}(s,1) = 0$. Therefore $\frac{\partial}{\partial s}\tilde{g}(s,1) = 0$, so (6.27) is well-defined. ♠

Remarks

• As we mentioned, it is possible to avoid Ado's theorem, and proceed directly from $\mathfrak{g}$ to the definition of the simply connected Lie group. To do this, we return to $\mathcal{P}$, but now we introduce a more subtle equivalence relation. Now we say that $\gamma_0 \sim \gamma_1$ if there exists a homotopy $\gamma(s,t)$ of paths in $\mathcal{P}$ such that there is a partner $\eta(s,t)$ satisfying the Maurer-Cartan equation:

$$\frac{\partial}{\partial s}\gamma - \frac{\partial}{\partial t}\eta = [\eta, \gamma] \tag{6.32}$$

for some $\eta(s,t) \in \mathfrak{g}$. Then we claim that $\mathcal{P}/\sim := G$ is the desired Lie group. The hard part is to show that the Lie algebra of G is indeed $\mathfrak{g}$. To do this we solve

$$f(Ad\eta_\gamma(t))\dot{\eta}_\gamma(t) = \gamma(t) \tag{6.33}$$

and show that $\gamma_1 \sim \gamma_2$ iff $\eta_{\gamma_1}(1) = \eta_{\gamma_2}(1)$. Then

$$L(G) = \{\eta : [0,1] \to \mathfrak{g}\}/\sim \tag{6.34}$$

is isomorphic to $\mathfrak{g}$.

• We are really using here the theory of flat connections. The Maurer-Cartan equation is the statement that the gauge field $A = \gamma dt + \eta ds$ is a flat gauge field: $F = dA + A^2 = 0$.

• One of the advantages of the above presentation is that it can be generalized to infinite-dimensional Lie groups. See

J. Milnor, "Remarks On Infinite Dimensional Lie Groups," *In *Les Houches 1983, Proceedings, Relativity, Groups and Topology, Ii*, 1007-1057*

7. Lie Algebras for the Classical Groups

Now we can review the Lie algebras of some matrix groups, including the classical groups. As we have seen, if G is a matrix group, that is, a Lie subgroup of $GL(n,k)$ then the Lie algebra associated to the Lie group G is:

$$L(G) \equiv \{A = \left.\frac{d}{dt}\right|_0 g(t) : g(t) = any \quad curve \quad g(t) \quad with \quad g(0) = 1\} \tag{7.1}$$

One way to think about Lie algebras related to the classical groups is as "infinitesimal group elements." Since Lie groups are manifolds we can consider a neighborhood of the identity element 1_G. In this neighborhood the manifold looks like a linear space, namely, the tangent space T_1G. We now want to get a better understanding of how the structure of the group translates into an algebraic structure on T_1G in the classical matrix groups.

For example, if G is a matrix group then matrices $g \in G$ close to the identity can be written as

$$g \cong 1 + \epsilon A \tag{7.2}$$

where ϵ is small. Thus infinitesimal group elements defines a matrix A in the Lie algebra. Thus, for example, if A is anti-hermitian then $1 + \epsilon A$ is unitary to order ϵ^2:

$$(1 + \epsilon A)(1 + \epsilon A)^\dagger = 1 + \mathcal{O}(\epsilon^2) \tag{7.3}$$

We will now list the relevant properties for the classical matrix groups. Our discussion will parallel the survey we did of the matrix groups:

1. Definition and dimension.

2. Properties of the exponential map.

3. A natural choice of a *Cartan subalgebra* $\mathfrak{t} \subset \mathfrak{g}$. By definition, a *Cartan subalgebra* is a maximal abelian subalgebra of $\mathfrak{g}$. It is of supreme importance in the theory of roots and weights and representations. A Cartan subalgebra is the Lie algebra of a corresponding Cartan torus.

7.1 A useful identity

The following identity is extremely useful: Given *any* $n \times n$ matrix $A \in M_n(k)$, the determinant and the trace may be related as follows:

$$\boxed{\det(\exp(A)) = \exp(\mathrm{Tr}A)} \tag{7.4}$$ eq:trcelog

This is sometimes written

$$\mathrm{logdet}C = \mathrm{Trlog}C \tag{7.5}$$

but the latter identity only makes sense when the log is well-defined.

We have already proven (7.4) above as an application of Jordan canonical form.

7.2 $GL(n,k)$ and $SL(n,k)$

The Lie algebra of $GL(n,k)$ is just the Lie algebra of all $n \times n$ matrices:

$$\boxed{gl(n,k) \equiv L(GL(n,k)) = M_n(k)} \tag{7.6}$$

Let us check this by producing a one-parameter family. Let $A \in M_n(k)$ and consider $g(t) := \exp(tA)$. These matrices are all invertible, and

$$\frac{d}{dt}|_{t=0}g(t) = A \tag{7.7}$$

Thus we get the full Lie algebra of $n \times n$ matrices.

Again by $\mathrm{det}e^A = \exp(\mathrm{Tr}(A))$ we get

$$\boxed{sl(n,k) \equiv L(SL(n,k)) = \{A \in M_n(k) \mid \mathrm{tr}\, A = 0\}} \tag{7.8}$$

We have a direct sum of Lie algebras:

$$gl(n,k) = k \oplus sl(n,k) \tag{7.9}$$

A natural choice of Cartan subalgebra is the subalgebra of diagonal matrices.

Remarks

• The exponential map is *not* onto for $SL(n,\mathbb{R})$. Let us consider $SL(2,\mathbb{R})$. If $g = e^A$ then g lies on some one-parameter subgroup $g(t) = e^{tA}$. Any such can be conjugated into a maximal abelian subgroup. From our classification of the conjugacy classes of $SL(2,\mathbb{R})$ it follows from $hgh^{-1} = e^{hAh^{-1}}$ that A can be conjugated to one of three possible forms:

$$A = x\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.10}$$ eq:hyperbol

$$A = x\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{7.11}$$ eq:elliptic

$$A = x\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{7.12}$$ eq:parabol

It is easy to check that the trace of e^A for A conjugate to (7.10)(7.11)(7.12) is ≥ -2, and the trace is $= -2$ only for $g = -1 = \exp[i\pi\sigma^2]$. In particular *The parabolic conjugacy classes with $Tr(g) = -2$ and the hyperbolic conjugacy classes with $Tr(g) < -2$ are not in the image of the exponential map.*

Exercise *Structure constants for $GL(n,k)$.*

The Lie algebra has a natural basis given by the matrix units: e_{kj} = The matrix with zero entries except = 1 at $(kj)^{th}$ matrix element.

Calculate the structure constants

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj} \tag{7.13}$$

Exercise *Structure constants for $sl(2,\mathbb{C})$ and $sl(2,\mathbb{R})$*

Show that we may choose a basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad E^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad E^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{7.14}$$ eq:spc

with structure constants

$$
\begin{aligned}
[E^+, E^-] &= H \\
[H, E^\pm] &= \pm 2E^\pm
\end{aligned}
\tag{7.15}
$$

eq:sltwo

We will see a lot more of this later.

Exercise *The exponential map is onto for $GL(n,\mathbb{C})$*
Show that

$$
\exp : gl(n,\mathbb{C}) \to GL(n,\mathbb{C}) \tag{7.16}
$$

eq:expmapon

is *surjective.*

Answer: It suffices to check this statement for Jordan form. Therefore, try to write $\lambda + N$, where $N = e_{12} + e_{23} + \cdots$ is a $d \times d$ Jordan block in the image of the exponential map. Recall $N^d = 0$ but $N^{d-1} \neq 0$. Note that

$$
\exp[x1 + \alpha_1 N + \alpha_2 N^2 + \cdots + \alpha_{d-1} N^{d-1}] = \tag{7.17}
$$

$$
= e^x \left(1 + \alpha_1 N + (\alpha_2 + \frac{1}{2}\alpha_1^2) N^2 + \cdots + (\alpha_{d-1} + \cdots + \frac{\alpha_1^{d-1}}{(d-1)!} N^{d-1} \right) \tag{7.18}
$$

and note that we can set $e^x = \lambda$ and then solve the upper-triangular system of equations for the α_i.

Exercise

a.) Show that matrices of the form

$$
\begin{pmatrix} -x & 0 \\ 0 & -1/x \end{pmatrix} \tag{7.19}
$$

for $x > 0$ are in $SL(2,\mathbb{R})$, are in the image of the exponential map for $SL(2,\mathbb{C})$, but not in the image of the exponential map for $SL(2,\mathbb{R})$.

b.) Show that

$$
\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \tag{7.20}
$$

is in $SL(2,\mathbb{C})$, is in the image of the exponential map for $GL(2,\mathbb{C})$, but not for $SL(2,\mathbb{C})$.

7.3 $O(n,k)$

Suppose $g(t) = e^{tA}$ is a family of orthogonal matrices. Then

$$g(t)g(t)^{tr} = 1 \tag{7.21}$$ eq:orthone

since they are orthogonal for each t. Differentiate (7.21) wrt t to get:

$$\frac{d}{dt}|_{t=0} e^{tA} e^{tA^{tr}} = A + A^{tr} = 0 \tag{7.22}$$ eq:orthlie

and we can conclude from (7.22) that the Lie algebra of $O(n,k)$ is the Lie algebra of $n \times n$ skew matrices:

$$\boxed{o(n,k) \equiv L(O(n,k)) = \{A \in M_n(k) \mid A^{tr} = -A\}} \tag{7.23}$$

Note that $o(n,k) = so(n,k)$ because the groups $O(n,k)$ and $SO(n,k)$ have the same neighborhood of the identity. In general locally isomorphic Lie groups have the same Lie algebra.

Note that the exponential map is *not* one to one. For example:

$$\exp[2\pi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}] = 1 \tag{7.24}$$

Let us give a basis for the Lie algebra. Introduce the matrices:

$$T_{ij} := e_{ij} - e_{ji} \tag{7.25}$$

so that $T_{ij} = -T_{ji}$. Then T_{ij} for $1 \le i < j \le n$ form a basis for $o(n,k)$. We compute the structure constants:

$$[T_{ij}, T_{kl}] = \delta_{jk} T_{il} - \delta_{li} T_{kj} - \delta_{ik} T_{jl} + \delta_{lj} T_{ki} \tag{7.26}$$ eq:soscs

N.B. T_{ij} are elements of a Lie algebra - they are NOT matrix elements!

For $o(2r,k)$, or $o(2r+1,k)$, a natural basis for a Cartan subalgebra is

$$T_{1,2}, T_{3,4}, \dots, T_{2r-1,2r}. \tag{7.27}$$

7.4 More general orthogonal groups

Recall that for any symmetric bilinear form D we can define an orthogonal group $O(D)$. If we choose a basis and represent D as a symmetric matrix D_{ij} then $O(D)$ becomes the set of matrices with

$$gDg^{tr} = D \tag{7.28}$$

Following the procedure above we find the Lie algebra is the set of matrices A such that

$$AD + DA^{tr} = 0. \tag{7.29}$$

In particular, specializing to $D = \eta = ((+1)^p, (-1)^q)$ we can rewrite this condition as:

$$(\eta A)^{tr} = -(\eta A) \tag{7.30}$$

Thus, the matrices are skew-symmetric *after* multiplication by η. Thus we could choose a basis $\hat{T}_{ij} \equiv \eta T_{ij}$, for $o(p,q)$ where T_{ij} are a basis for $o(p+q)$. Thus a basis for the Lie algebra is:

$$\hat{T}_{ij} = \eta_{ii} e_{ij} - \eta_{jj} e_{ji}, \qquad 1 \le i < j \le n \tag{7.31}$$

We can read off immediately the structure constants for $o(p,q)$:

$$[\hat{T}_{ij}, \hat{T}_{kl}] = \eta_{jk}\hat{T}_{il} - \eta_{li}\hat{T}_{kj} - \eta_{ik}\hat{T}_{jl} + \eta_{lj}\hat{T}_{ki} \tag{7.32}$$ eq:soscsp

Again, we caution that the subcripts do not label matrix elements but elements in a basis for the Lie algebra. In particular it is always true that:

$$\hat{T}_{ij} = -\hat{T}_{ji}. \tag{7.33}$$

However, note that if $\eta_{ii} = \eta_{jj}$ then $\hat{T}_{ij}$ is an antisymmetric matrix, but if $\eta_{ii} = -\eta_{jj}$ then $\hat{T}_{ij}$ is a *symmetric* matrix.

Exercise
Compute the structure constants for the Lorentz group

7.4.1 Lie algebra of $SO^*(2n)$

There is one other "real form" of $SO(2n, \mathbb{C})$ given by matrices of the type:

$$\begin{pmatrix} A_+ + iA_- & B_+ + iB_- \\ -B_+ + iB_- & A_+ - iA_- \end{pmatrix} \tag{7.34}$$

where $A_\pm, B_\pm$ are real $n \times n$ matrices with

$$(A_\pm)^{tr} = -A_\pm \qquad\qquad (B_\pm)^{tr} = \pm B_\pm \tag{7.35}$$

For more discussion about this consult Gilmore, p.343.

7.5 $U(n)$

Proceeding to unitary matrices $g(t) = e^{tA}$ is unitary if:

$$e^{tA^\dagger} e^{tA} = 1 \tag{7.36}$$

Differentiating at $t = 0$ we conclude that:

$$A^\dagger = -A \tag{7.37}$$

So:

$$\boxed{\begin{aligned} u(n) = L(U(n)) = \{A \in M_n(\mathbb{C}) \mid A^\dagger = -A\} \\ su(n) = L(SU(n)) = \{A \in L(U(n)) : \mathrm{tr}\, A = 0\} \\ u(n) = \mathbb{R} \oplus su(n) \end{aligned}} \tag{7.38}$$

A Cartan subalgebra is generated by matrices ie_{aa}, $a = 1, \dots n$ for $u(n)$, but for $su(n)$ we need to have traceless matrices. One choice of basis is $i(e_{aa} - e_{a+1,a+1})$, $1 \leq a \leq n-1$.

Remarks

• Physicists usually separate out a factor of $i = \sqrt{-1}$ and write

$$U = e^{iH} \tag{7.39}$$

where H is <u>*hermitian*</u>.

•

Note that the condition $A^\dagger = -A$ is only preserved by multiplication of A by real numbers so we must consider these as Lie algebras over $\mathbb{R}$.

Exercise

Let us consider $SU(2)$ in particular. Show that the Lie algebra can be thought of as the real span of three anti-hermitian matrices (iJ_k), $k = 1, 2, 3$, with J_k Hermitian, such that

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{7.40}$$

where ϵ_{ijk} is the antisymmetric tensor on three indices. Take $J_i = +\frac{1}{2}\sigma_i$.

Note that the structure constants $-\epsilon_{ijk}$ of this Lie algebra are *real* if we use the real basis $\frac{i}{2}\sigma^i$.

Exercise *Relating* $o(4)$ *and* $su(2)$

We saw in the previous lecture that at the level of groups:

$$SO(4) = \frac{SU(2) \times SU(2)}{\mathbb{Z}_2} \tag{7.41}$$ eq:sutwotwo

This must be reflected in the structure of the Lie algebras.

The Lie algebra version of (7.41) is that

$$o(4) \cong su(2) \oplus su(2) \tag{7.42}$$

Do this by defining

$$\tilde{A}_{\mu\nu} := \frac{1}{2}\epsilon_{\mu\nu\lambda\rho} A_{\lambda\rho} \tag{7.43}$$

for any antisymmetric tensor $A_{\mu\nu} = -A_{\nu\mu}$. In particular, for the generators of $so(4)$ we can define:

$$\tilde{T}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}T_{\lambda\rho} \tag{7.44}$$

Now form the "self-dual" and "anti-self-dual" combinations of generators:

$$A^{\pm}_{\mu\nu} = \frac{1}{2}[T_{\mu\nu} \pm \tilde{T}_{\mu\nu}] \tag{7.45}$$

a.) Show that

$$\tilde{A}^{\pm}_{\mu\nu} = \pm A^{\pm}_{\mu\nu} \tag{7.46}$$

b.) Show that $[A^+, A^-] = 0$ and that

$$\begin{aligned}
J_1 \to J_1^+ &:= A^+_{23} = A^+_{14} = \frac{1}{2}(T_{23} + T_{14}) \\
J_2 \to J_2^+ &:= A^+_{31} = A^+_{24} = \frac{1}{2}(T_{31} + T_{24}) \\
J_3 \to J_3^+ &:= A^+_{12} = -A^+_{34} = \frac{1}{2}(T_{12} + T_{34})
\end{aligned} \tag{7.47}$$

defines an isomorphism of the Lie subalgebra of the $A^+_{\mu\nu}$ and $su(2)$. Show that the corresponding statement for the $A^-_{\mu\nu}$ is

$$\begin{aligned}
J_1 \to J_1^- &:= \frac{1}{2}(T_{23} - T_{14}) \\
J_2 \to J_2^- &:= \frac{1}{2}(T_{31} - T_{24}) \\
J_3 \to J_3^- &:= \frac{1}{2}(T_{12} - T_{34})
\end{aligned} \tag{7.48}$$

Exercise *The 't Hooft symbols*

When working in Euclidean 4-dimensional physics the 't Hooft symbol is often of great use. This is defined by

$$\alpha^{\pm,i}_{\mu\nu} := \frac{1}{2}(\pm\delta_{i\mu}\delta_{\nu 4} \mp \delta_{i\nu}\delta_{\mu 4} + \epsilon_{i\mu\nu}) \tag{7.49}$$ eq:thooft

where $1 \le \mu, \nu \le 4$, $1 \le i \le 3$ and it is understood that $\epsilon_{i\mu\nu}$ is zero unless $1 \le \mu, \nu \le 3$.

a.) Show that in the previous exercise we can take:

$$J^{\pm,i} = \frac{1}{2}\alpha^{\pm,i}_{\mu\nu}T_{\mu\nu} \tag{7.50}$$

b.) Show that if we regard $\alpha^{\pm,i}$ as 4×4 matrices with matrix elements $\alpha^{\pm,i}_{\mu\nu}$ then

$$[\alpha^{\pm,i}, \alpha^{\pm,j}] = -\epsilon^{ijk}\alpha^{\pm,k} \tag{7.51}$$ eq:tfhti

$$[\alpha^{\pm,i}, \alpha^{\mp,j}] = 0 \tag{7.52}$$ eq:tfhtii

$$\{\alpha^{\pm,i}, \alpha^{\pm,j}\} = -\frac{1}{2}\delta^{ij} \tag{7.53}$$ eq:tfhtiia

See Belitsky, VanDoren, VanNieuwenhuizen, hep-th/0004186, for many more identities of this nature. A closely related tensor are the so-called "'t Hooft symbols" $\eta^a_{\mu\nu}$. Definitions differ, but roughly $\eta = 2\alpha$.

Exercise *Spinor indices and 't Hooft symbols*

7.5.1 $U(p,q)$

The discussion here parallels that for $O(p,q)$. Introduce η as before. Now a one-parameter subgroup through 1 in $U(p,q)$ satisfies

$$e^{tA^\dagger}\eta e^{tA} = \eta \tag{7.54}$$

for $A \in u(p,q)$. Thus,

$$(\eta A)^\dagger = -(\eta A) \tag{7.55}$$

and ηA is antihermitian.

Etc.

7.5.2 Lie algebra of $SU^*(2n)$

7.6 $Sp(2n)$

Following the same procedure as before we take a family of symplectic matrices $g(t) = e^{tA}$. They must satisfy:

$$e^{tA^{tr}} J e^{tA} = J \tag{7.56}$$

Differentiating at $t = 0$ gives:

$$A^{tr}J + JA = 0 \tag{7.57}$$ eq:sympcond

Conversely, (7.57) and $J^2 = -1$ imply:

$$-Je^{A^{tr}}J = e^{-JA^{tr}J} = e^{-A} \quad \Rightarrow \quad e^{A^{tr}}J = Je^{-A} \quad \Rightarrow \quad e^A \in Sp(2n,k) \tag{7.58}$$

by pulling J to left and using (7.57) successively. so:

$$\boxed{sp(2n,k) \equiv L(Sp(2n,k)) = \{A \in M_{2n}(k) \mid (JA)^{tr} = +(JA)\}} \tag{7.59}$$

Now we can easily count the dimension of $Sp(2n,\mathbb{R})$. We need only take a real symmetric $2n \times 2n$ matrix and identify it with JA. Thus the dimension is:

$$\dim_{\mathbb{R}} Sp(2n,\mathbb{R}) = \frac{1}{2}2n(2n+1) \tag{7.60}$$

eq:dimen

in accord with a direct analysis at the group level.

Recall that the Cartan torus of $USp(2n)$ is given by diagonal matrices of the form

$$\begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{pmatrix} \tag{7.61}$$

where Λ is a diagonal matrix of phases. The corresponding Cartan subalgebra is spanned by

$$i(e_{aa} - e_{a+n,a+n}) \qquad a = 1,\dots,n \tag{7.62}$$

Exercise

Define a basis $m_{\alpha\beta}$ for the Lie algebra from

$$Jm_{\alpha\beta} = -(e_{\alpha\beta} + e_{\beta\alpha}) \tag{7.63}$$

i.e.

$$m_{\alpha\beta} = J(e_{\alpha\beta} + e_{\beta\alpha}) \tag{7.64}$$

Show that

$$[m_{\alpha\beta}, m_{\gamma\delta}] = J_{\beta\gamma}m_{\alpha\delta} + J_{\alpha\gamma}m_{\beta\delta} + J_{\beta\delta}m_{\alpha\gamma} + J_{\alpha\delta}m_{\beta\gamma} \tag{7.65}$$

One way to remember this is that the result must be *symmetric* on (α,β) as well as (γ,δ) and *antisymmetric* on $(\alpha,\beta) \leftrightarrow (\gamma,\delta)$.

Exercise

It is also worthwhile to examine the condition (7.57) in terms of the block decomposition. Represent the matrix A in block form:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{7.66}$$

Show that (7.57) implies:

$$A_{11}^{tr} = -A_{22} \tag{7.67}$$

$$A_{12}^{tr} = A_{12} \qquad A_{21}^{tr} = A_{21} \tag{7.68}$$

Recall that if $g \in Sp(2n)$ then $g^{tr} \in Sp(2n)$. Prove this using the Lie algebra.

Exercise $su(1,1) \cong sl(2,\mathbb{R}) \cong sp(2,\mathbb{R})$

a.) Show that a basis for the *real* Lie algebra $su(1,1)$ can be taken to be

$$E^{\pm} = \frac{1}{2}(\sigma^1 \mp i\sigma^3) \tag{7.69}$$

$$H = \sigma^2 \tag{7.70}$$

and these satisfy the above standard commutation relations of $sl(2,\mathbb{R})$.

b.) Note that $sl(2,\mathbb{R}) \cong sp(2,\mathbb{R})$ trivially from the above characterization of block diagonal form.

Exercise $su(2) \cong usp(2,\mathbb{R})$

Exercise *Solving the general quadratic Hamiltonian in quantum mechanics*

Show that symplectic transformations allow us to solve the general quadratic Hamiltonian in quantum mechanics for a finite number of degrees of freedom:

Consider the general quadratic Hamiltonian for N degrees of freedom $r = 1, \dots, N$.

$$H = M_{rs}P_rP_s + A_{rs}P_rX_s + A^*_{rs}X_sP_r + G_{rs}X_rX_s \tag{7.71}$$ eq:ghen

where, after quantization $[P_i, Q_j] = -i\delta_{ij}\hbar$, $[P_i, P_j] = [X_r, X_s] = 0$. We restrict to Hermitian Hamiltonians so that M_{rs} and G_{rs} are real symmetric. A_{rs} is an arbitrary complex matrix.

a.) By adding a c-number we can assume that A_{rs} is real.

b.) Write the Hamiltonian as

$$H = \begin{pmatrix} X_r & P_r \end{pmatrix} J \begin{pmatrix} -A_{rs} & -M_{rs} \\ G_{rs} & A^{tr}_{rs} \end{pmatrix} \begin{pmatrix} X_s \\ P_s \end{pmatrix} \tag{7.72}$$ eq:symphamil

and observe that it is of the form

$$H = \begin{pmatrix} X_r & P_r \end{pmatrix} Jh \begin{pmatrix} X_s \\ P_s \end{pmatrix} \tag{7.73}$$ eq:sntne

where $h \in sp(2N,\mathbb{R})$.

Conclude that by a complex symplectic transformation

$$\begin{pmatrix} X_s \\ P_s \end{pmatrix} \to g \begin{pmatrix} X_s \\ P_s \end{pmatrix} \tag{7.74}$$

we can bring $h \to g^{-1}hg$ into the Lie algebra of the Cartan torus. That is, we can bring it to the form

$$\begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix} \tag{7.75}$$

where ω is diagonal and real. In this basis the Hamiltonian takes the form

$$H = \sum_{i=1}^{N} \omega_i \{A_i^\dagger, A_i\} \tag{7.76}$$

c.) Finding the relevant transformation explicitly can be hard. The Hamiltonian can be simplified and the problem can be reduced to the diagonalization of an $N \times N$ Hermitian matrix as follows:

By a symplectic transformation $X \to TX, P \to T^{tr,-1}P$ we can bring M_{rs} to the form δ_{rs}. Then, by conjugating with $\exp[i\frac{1}{2}S_{nm}X_nX_m]$ we can make A_{rs} antisymmetric. Thus we can bring H to the form

$$H = P_rP_r + A_{rs}(P_rX_s + X_sP_r) + G_{rs}X_rX_s = (P_r + A_{rs}X_s)^2 + (G + A^tA)_{rs}X_rX_s \tag{7.77}$$ eq:ghens

with A_{rs} real antisymmetric. (This may be interpreted as charged particles in a constant magnetic field in a quadratic potential.)

The Heisenberg equations of motion

$$-i\hbar\frac{\partial}{\partial t}\mathcal{O}(t) = [H, \mathcal{O}] \tag{7.78}$$

become:

$$-i\partial_t \begin{pmatrix} X_r \\ P_r \end{pmatrix} = K \begin{pmatrix} X_s \\ P_s \end{pmatrix} \tag{7.79}$$ eq:heisenb

where

$$K = -2i \begin{pmatrix} A & 1 \\ -G & A \end{pmatrix} \tag{7.80}$$ eq:kay

1. K is in the symplectic Lie algebra $sp(2N)$ and hence by a symplectic transformation can be brought to the Cartan subalgebra:

$$U^{-1}KU = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \tag{7.81}$$ eq:cartfrm

where ω is $N \times N$ diagonal. This is the transformation to creation/annihilation operators.

To find U note that eigenvectors of K are of the form

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{7.82}$$

with

$$L(E)v_1 = \frac{E^2}{4}v_1 \qquad v_2 = -(A - \frac{i}{2}E)v_1 \tag{7.83}$$

where $L(E) = G + A^2 - iEA$. This is an Hermitian matrix and can be diagonalized with eigenvalues $f_i(E)$. We then solve the quadratic equations $f_i(E) = E^2/4$.

---

8. Central extensions of Lie algebras and Lie algebra cohomology

The ideas here parallel those for central extensions of groups:

A central extension of a Lie algebra $\mathfrak{g}$ by an abelian Lie algebra $\mathbf{z}$ is a Lie algebra $\widetilde{\mathfrak{g}}$ such that we have an exact sequence of Lie algebras:

$$0 \to \mathbf{z} \to \widetilde{\mathfrak{g}} \to \mathfrak{g} \to 0 \tag{8.1}$$

with $\mathbf{z}$ mapping into the center of $\widetilde{\mathfrak{g}}$. As a *vector space* (but not necessarily as a Lie algebra) $\widetilde{\mathfrak{g}} = \mathbf{z} \oplus \mathfrak{g}$ so we can denote elements by (z, X) and the Lie bracket has the form

$$[(z_1, X_1), (z_2, X_2)] = (c(X_1, X_2), [X_1, X_2]) \tag{8.2}$$

where $c : \Lambda^2\mathfrak{g} \to \mathbf{z}$ is known as a *two-cocycle* on the Lie algebra. That is $c(X, Y)$ is bilinear, it satisfies

$$c(X, Y) = -c(Y, X) \tag{8.3}$$ eq:twococyci

and the Jacobi relation requires

$$c([X_1, X_2], X_3) + c([X_3, X_1], X_2) + c([X_2, X_3], X_1) = 0. \tag{8.4}$$ eq:twococycia

Two different cocycles can define isomorphic Lie algebras. If there is a *linear* function $f : \mathfrak{g} \to \mathbf{z}$ such that

$$c(X, Y) = b_f(X, Y) = f([X, Y]) \tag{8.5}$$ eq:cobodn

then the cocycle is said to be trivial, and the central extension is isomorphic to $\mathbf{z} \oplus \mathfrak{g}$ as a Lie algebra. Indeed,

$$X \to (f(X), X) \tag{8.6}$$

is an explicit Lie algebra homomorphism splitting the sequence. (Check!)

More generally, if two cocycles differ by a cocycle of the form b_f then they define isomorphic Lie algebras. Thus, again, classifying isomorphism classes of central extensions is a cohomology problem, in this case, Lie algebra cohomology.

Remark: Suppose that $\widetilde{G}$ is a Lie group central extension of a Lie group G. Then a 2-cochain $c(X,Y)$ defines a left-invariant 2-form on G, and a 2-cocycle defines a closed left-invariant two-form ω on G. A central extension of the Lie group, defined by the group cocycle $f(g_1, g_2)$, also defines a corresponding 2-cocycle on the Lie algebra by

$$c(X_1, X_2) = \frac{1}{2\pi i}\frac{d}{dt_1}|_0\frac{d}{dt_2}|_0 \log\left[\frac{f(e^{t_1X_1}, e^{t_2X_2})}{f(e^{t_2X_2}, e^{t_1X_1})}\right] \quad (8.7)$$ eq:twoconscy

8.1 Example: The Heisenberg Lie algebra and the Lie group associated to a symplectic vector space

In general, given any real vector space V and a skew bilinear form $\omega : V \times V \to \mathbb{R}$ we can view V as an abelian Lie algebra and use ω to define a central extension

$$0 \to \mathbb{R} \to \mathrm{heis}(V,\omega) \to V \to 0 \quad (8.8)$$ eq:heisliealg

We simply use the cocycle $c(v_1, v_2) = \omega(v_1, v_2)$.

When ω is a symplectic form, *i.e.* ω is nondegenerate, then the corresponding Lie algebra is called a *Heisenberg Lie algebra*.

The corresponding Lie group $\mathrm{Heis}(V,\omega)$ is $U(1) \times V$ as a set with multiplication

$$(\lambda_1, v_1)(\lambda_2, v_2) = (\lambda_1\lambda_2 e^{i\omega(v_1,v_2)}, v_1 + v_2) \quad (8.9)$$

Let us return to the corollary (4.44) of the BCH formula:

$$e^{\alpha_1\hat{p}+\beta_1\hat{q}}e^{\alpha_2\hat{p}+\beta_2\hat{q}} = e^{-\frac{i}{2}(\alpha_1\beta_2-\alpha_2\beta_1)}e^{(\alpha_1+\alpha_2)\hat{p}+(\beta_1+\beta_2)\hat{q}} \quad (8.10)$$ eq:heisgroupii

We can take $V = \mathbb{R}^2 = \{(\alpha, \beta)\}$ and we recognize

$$\omega((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = -\frac{1}{2}(\alpha_1\beta_2 - \alpha_2\beta_1) \quad (8.11)$$

as the symplectic form in this case.

(Of course, we could take $V = \mathbb{C}^2$ and consider it as a 4-dimensional real vector space. If we take α, β pure imaginary then the operators are unitary in the usual representation on $L^2(\mathbb{R})$.)

Exercise
Check that $e^{i\omega(v_1,v_2)}$ is a 2-cocycle on the abelian group V.

Exercise

Referring to the exercise in section **** on the magnetic translation group

Show that the magnetic translation group generated by the U_i for all $a^i \in \mathbb{R}$ is a Heisenberg group, extending the abelian group of translations in the plane. Show that the skew bilinear form is

$$\omega(\vec{a}, \vec{b}) = -eB(\vec{a} \times \vec{b}) \tag{8.12}$$ eq:skewfrm

---

8.2 Lie algebra cohomology

The notion of a two-cocycle above fits into a more general theory of Lie algebra cohomology. So to put it into proper context let us briefly consider the topic more generally.

We consider the simplest case of central extensions by a one-dimensional abelian Lie algebra k, $\mathbf{z} = k$.

We define a complex

$$d : \Lambda^k \mathfrak{g}^* \to \Lambda^{k+1} \mathfrak{g}^* \tag{8.13}$$ eq:lacoh

by taking as differential:

$$d\omega(X_1, \dots, X_{k+1}) := \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], \dots \hat{X}_i, \dots, \hat{X}_j \dots) \tag{8.14}$$ eq:extderi

The resulting differential (8.13) may also be expressed by identifying $\Lambda^* \mathfrak{g}^* = \Lambda^*[\theta^a]$ where θ^a are of degree 1 and a runs over a basis for the Lie algebra. We then *define* the differential to be:

$$d\theta^a := -\frac{1}{2} f_{bc}{}^a \theta^b \theta^c \tag{8.15}$$ eq:lacohii

---

Exercise

Check that this is a differential, that is, that $d^2 = 0$.

The cohomology of the complex $(\Lambda^* \mathfrak{g}^*, d)$ is known as *Lie algebra cohomology* and denoted $H^*(\mathfrak{g})$. Note that it can be formulated purely algebraically. The differential defined by (8.14) or, equivalently, (8.15) is sometimes called the *Chevalley-Eilenberg differential.*

It is useful to translate this into the physicist's language:

Suppose we have a Lie algebra $\mathfrak{g}$ with basis T_a, a is an index running over the generators. Let us introduce the Clifford algebra:

$$\begin{aligned} \{c^a, c^{a'}\} &= 0 \\ \{b_a, b_{a'}\} &= 0 \\ \{c^a, b_{a'}\} &= \delta^a{}_{a'} \end{aligned} \tag{8.16}$$ eq:ghosti

where $c^a, b_{a'}$ are referred to as ghosts and antighosts, respectively.

We can quantize the Clifford algebra by choosing a Clifford vacuum

$$b_{a'}|0\rangle = 0 \tag{8.17}$$ eq:ghostii

and the resulting Hilbert space is spanned by $|0\rangle, c^a|0\rangle, \dots$.

The Hilbert space is graded by the "ghost number operator" $N = \sum_a c^a b_a$, and we have an isomorphism of the vector space of states of ghost number k with $\Lambda^k \mathfrak{g}^*$:

$$\omega \leftrightarrow \frac{1}{k!}\omega_{a_1 \cdots a_k} c^{a_1} \cdots c^{a_k}|0\rangle \tag{8.18}$$ eq:ghostiii

Under the isomorphism (8.18) the Chevalley-Eilenberg differential becomes what is known as the *BRST operator*:

$$Q := -\frac{1}{2} f_{a_2 a_3}{}^{a_1} c^{a_2} c^{a_3} b_{a_1} \tag{8.19}$$ eq:ghostiv

The BRST cohomology is the cohomology of Q, and is graded by ghost number.

BRST cohomology enters physics in the quantization of theories with local gauge symmetry. For the moment let us note a very natural generalization. Suppose we have a representation ρ of the Lie algebra $\mathfrak{g}$. We can then consider the complex

$$\Lambda^* \mathfrak{g}^* \otimes V \tag{8.20}$$

and introduce a differential

$$Q = c^a t_a - \frac{1}{2} f_{a_2 a_3}{}^{a_1} c^{a_2} c^{a_3} b_{a_1} \tag{8.21}$$ eq:ghostv

where $t_a = \rho(T_a)$ are the representation matrices of the rep. V.

Geometrically, the cohomology $H^*_Q(\Lambda^* \mathfrak{g}^* \otimes V)$ can be identified, for G compact and connected, with the cohomology $H^*_{DR}(G; \mathcal{V})$ of a homogeneous vector bundle over the group G.

Exercise
Check that $Q^2 = 0$

Exercise
Compute the degree 2 Lie algebra cohomology of $su(2)$ and show that it is trivial.

---

9. Left-invariant differential forms

We introduced the Lie algebra of a Lie group by thinking about the left-invariant vector fields on the group. It is quite useful to think about the dual picture of left-invariant differential forms:

Definition: A differential form $\omega \in \Omega^*(G)$ is *left invariant* if

$$(L_g)^*(\omega) = \omega \tag{9.1}$$

for all $g \in G$.

Suppose first that $G = GL(n, \mathbb{F})$. Then we can construct an explicit basis of left-invariant 1-forms as follows. The basis of one-forms dual to $\xi(e_{ij})$ is

$$\Theta_{i'j'} = (g^{-1})_{i'k} dg_{kj'} \tag{9.2}$$

eq:dualbasis

One can check explicitly that $\Theta_{i'j'}$ is a dual basis to $\xi(e_{ij})$ by computing the contraction:

$$\iota(\xi(e_{ij}))(\Theta_{i'j'}) = \langle \Theta_{i'j'}, \xi(e_{ij}) \rangle = \delta_{ii'}\delta_{jj'} \tag{9.3}$$

eq:contraction

It is convenient to assemble the 1-forms Θ_{ij} into a matrix of one-forms

$$\Theta := \Theta_{ij} e_{ij} = g^{-1} dg. \tag{9.4}$$

eq:maurercarta

Let us check the left-invariance directly in matrix notation:

$$(L_{g_0})^*(g^{-1}dg) = (g_0 g)^{-1} d(g_0 g) = g^{-1} g_0^{-1} g_0 dg = g^{-1} dg \tag{9.5}$$

The matrix of 1-forms (9.4) is known as the *Maurer-Cartan form.* Note that, if $X \in gl(n, \mathbb{F})$ then we have

$$\boxed{\iota(\xi(X))(\Theta) = X} \tag{9.6}$$

eq:contr

Suppose now that $G \subset GL(n, \mathbb{F})$ is a Lie subgroup. Under restriction to G the matrix elements g_{ij} are not all independent, but satisfy polynomial relations. Nevertheless, we can pull back the expression $\Theta = g^{-1}dg$ to G. Formally, we should introduce the inclusion map $j : G \hookrightarrow GL(n, \mathbb{F})$ and speak of $\Theta_G = j^*(\Theta)$. But to avoid cluttered notation we drop this.

Now, for $g \in G$, the matrix of 1-forms will be valued in the lie algebra $\mathfrak{g}$ of G considered as a subalgebra of $Mat_n(\mathbb{F})$. Indeed, if

$$\gamma_X(t) = g_0 e^{tX} \tag{9.7}$$

is a curve through g_0 whose directional derivative is $\xi(X)$ then

$$\gamma_X^*(g^{-1}dg) = e^{-tX} g_0^{-1} d(g_0 e^{tX}) = X dt. \tag{9.8}$$

This shows that

$$\langle \xi(X), \Theta \rangle = X \in \mathfrak{g} \tag{9.9}$$

In this way $\Theta = g^{-1}dg$ defines the Maurer-Cartan forms on any Lie subgroup of a matrix group.

Example $G = SO(2)$ If

$$g = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{9.10}$$

so

$$g^{-1}dg = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} d\theta \tag{9.11}$$

so

$$\langle \frac{\partial}{\partial\theta}, g^{-1}dg \rangle = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{9.12}$$

10. The Maurer-Cartan equation

A very important property of the left-invariant 1-forms is the *Maurer-Cartan equation.* Differentiating, using $d^2 = 0$ and the Leibniz rule gives:

$$d\Theta = -g^{-1}dg \wedge g^{-1}dg = -\Theta \wedge \Theta \tag{10.1}$$

eq:derivmc

We thus have the *Maurer-Cartan equation*:

$$\boxed{d\Theta + \Theta \wedge \Theta = 0} \tag{10.2}$$

eq:mcequations

Since in general the matrix product of two Lie algebra elements is not in the Lie algebra, it might not be immediately obvious to the reader that $\Theta \wedge \Theta$ is valued in $\mathfrak{g}$. To make this obvious we proceed as follows:

It is often useful to choose a basis T_a for $\mathfrak{g}$. Then we can define *structure constants*:

$$[T_a, T_b] = f_{ab}{}^c T_c \tag{10.3}$$

Now, since $g^{-1}dg$ is valued in the Lie algebra $\mathfrak{g}$ we can expand:

$$g^{-1}dg = e^a T_a \tag{10.4}$$

and this defines a system of 1-forms e^a on G which span $T_g^*(G)$ at any point g. Then we compute

$$\begin{aligned} 0 &= de^a T_a + e^a T_a \wedge e^b T_b \\ &= de^a T_a + \frac{1}{2} e^a \wedge e^b [T_a, T_b] \end{aligned} \tag{10.5}$$

In terms of the structure constants we have:

$$\boxed{de^a + \frac{1}{2} f_{bc}{}^a e^b \wedge e^c = 0} \tag{10.6}$$

eq:structure

If follows from the above that the dual basis of vectorfields e_a is precisely the system of fundamental vector fields described above

$$e_a = \xi(T_a) \tag{10.7}$$ eq:funda

Remarks:

• Although we have taken a rather concrete approach. The equations (10.6) and (10.7) hold for all Lie groups. (Not all finite dimensional Lie groups are matrix groups. Heisenberg groups give a counterexample. However, by the Peter-Weyl theorem, all compact Lie groups are subgroups of some unitary group, and hence are matrix groups.)

• Later on, we will interpret this as a zero-fieldstrength condition for a pure gauge field.

• Note that we can write:

$$\Theta \wedge \Theta = \frac{1}{2}[\Theta, \Theta] \tag{10.8}$$

where it is useful to introduce the notation

$$[\alpha, \beta] := \alpha \wedge \beta - (-1)^{pq} \beta \wedge \alpha \tag{10.9}$$ eq:gradcomm

when α is a matrix-valued p-form and β is a matrix-valued q-form. Thus, the ordinary commutator would read $\Theta\Theta - \Theta\Theta = 0$, but since we are discussing matrices whose elements are in a *graded* algebra the natural notion of commutator is the "graded commutative" (10.9).

• For future use in our discussion of connections on principal bundles, note that (10.2) implies

$$\mathcal{L}_{\xi(X)}\Theta = (d\iota(X) + \iota(X)d)\Theta = -\iota(X)\Theta^2 = -Ad(X)(\Theta) \tag{10.10}$$ eq:liederv

where in the last equality we consider Θ to be a Lie-algebra valued one-form, valued in the adjoint representation.

Exercise

One way of defining the exterior derivative of a differential k-form field is via the formula:

$$\begin{aligned} d\omega(X_1, \dots, X_{k+1}) &:= \sum_{i=1}^{k+1} (-1)^{i+1} X_i\big(\omega(\dots, \hat{X}_i, \dots)\big) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], \dots \hat{X}_i, \dots, \hat{X}_j \dots) \end{aligned} \tag{10.11}$$ eq:extderiv

where $\hat{X}$ means that entry is deleted.

a.) Using this formula, show that the Maurer-Cartan equation is equivalent to the fact that the left-invariant vector fields form a closed Lie algebra.

b.) In particular, show that (10.6) is compatible with

$$[e_a, e_b] = +f_{ab}^c e_c \tag{10.12}$$

c.) Since left invariant forms are determined by their value at $g = 1$ we can identify $\Omega_L^k(G) \cong \Lambda^k \mathfrak{g}^*$, as vector spaces. This gives us a new perspective on Lie algebra cohomology defined above.

(Hint: If ω is left-invariant and X_i are left-invariant then $\omega(X_1, \dots, X_k)$ is left-invariant, and hence constant.)

Exercise *Right-invariance*

a.) Show that

$$\tilde{\Theta} = dgg^{-1} \tag{10.13}$$

is a matrix of right-invariant 1-forms.

b.) Show that

$$d\tilde{\Theta} - \tilde{\Theta} \wedge \tilde{\Theta} = 0 \tag{10.14}$$

c.) Show that if we expand

$$\Theta = \omega_L^a T_a \tag{10.15}$$

$$\tilde{\Theta} = \omega_R^a T_a \tag{10.16}$$

then

$$\omega_L^a = (Ad(g))^a{}_b \omega_R^b \tag{10.17}$$

so the bases of left- and right-invariant forms are related by the adjoint representation.

---

11. Examples

11.1 $SU(2)$

Introduce angular coordinates:

$$\begin{aligned} g &= e^{\frac{i}{2}\sigma^3\phi} e^{\frac{i}{2}\sigma^2\theta} e^{\frac{i}{2}\sigma^3\psi} \\ &= e^{i\frac{1}{2}\phi\sigma_3} \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix} e^{i\frac{1}{2}\psi\sigma_3} \end{aligned} \tag{11.1}$$

eq:angl

$$0 \le \theta \le \pi, \quad \phi \sim \phi + 2\pi \quad \psi \sim \psi + 4\pi$$

Then

$$\Theta = g^{-1}dg = \frac{i}{2}\sigma^a e^a \tag{11.2}$$

$$
\begin{aligned}
e^1 &= \cos\psi \sin\theta d\phi - \sin\psi d\theta \\
e^2 &= \sin\psi \sin\theta d\phi + \cos\psi d\theta \\
e^3 &= d\psi + \cos\theta d\phi
\end{aligned}
\tag{11.3}
$$

eq:leftforms

which is conveniently written as

$$
g^{-1}dg = \frac{i}{2}\sigma^3(d\psi + \cos\theta d\phi) + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} e^{-i\psi}(d\theta + i\sin\theta d\phi) - \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} e^{i\psi}(d\theta - i\sin\theta d\phi)
\tag{11.4}
$$

With the basis $T^a = \frac{i}{2}\sigma^a$ the structure constants are $f_{ab}^{\ \ c} = -\epsilon_{abc}$, and one easily verifies the Maurer-Cartan equations directly:

$$
de^3 = e^1 \wedge e^2 \qquad + cyclic
\tag{11.5}
$$

eq:sutmc

The dual basis of vector fields is

$$
\begin{aligned}
e_3 &= \frac{\partial}{\partial\psi} \\
e_2 &= \cos\psi\frac{\partial}{\partial\theta} + \frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\phi} - \sin\psi \cot\theta \frac{\partial}{\partial\psi} \\
e_1 &= -\sin\psi\frac{\partial}{\partial\theta} + \frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\phi} - \cos\psi \cot\theta \frac{\partial}{\partial\psi}
\end{aligned}
\tag{11.6}
$$

eq:leftvfs

and one checks with an explicit computation that

$$
[e_a, e_b] = -\epsilon_{abc} e_c
\tag{11.7}
$$

eq:explicit

Remark: Note that $d\psi$ is *not* a globally well-defined form on $SU(2)$, nor is $\cos\theta d\phi$. However, the e^a *are* globally well-defined. Similarly, the e_a are globally well-defined vector fields.

Exercise

It is often useful to introduce coordinates on $\mathbb{C}^2$:

$$
\begin{aligned}
z^1 &= re^{i(\psi+\phi)/2}\cos\theta/2 \\
z^2 &= re^{i(\psi-\phi)/2}\sin\theta/2
\end{aligned}
\tag{11.8}
$$

which satisfy $|z^1|^2 + |z^2|^2 = r^2$. Setting $r = 1$ gives a coordinate system the covers $SU(2)$ (but degenerates at the poles).

Write out the forms in terms of these variables.

Exercise

Show that under $g \to g^{-1}$ we have

$$e^a_L(g) = -e^a_R(g^{-1}) \tag{11.9}$$

and that in coordinates this is $\phi \leftrightarrow -\psi, \theta \to -\theta$

Deduce that

$$\begin{aligned} e^1_R &= -\cos\phi \sin\theta d\psi + \sin\phi d\theta \\ e^2_R &= \sin\phi \sin\theta d\psi + \cos\phi d\theta \\ e^3_R &= d\phi + \cos\theta d\psi \end{aligned} \tag{11.10}$$

eq:rightforms

with dual basis of vector fields

$$\begin{aligned} e_3^R &= \frac{\partial}{\partial\phi} \\ e_2^R &= \cos\phi \frac{\partial}{\partial\theta} + \frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\psi} - \sin\phi \cot\theta \frac{\partial}{\partial\phi} \\ e_1^R &= -\sin\phi \frac{\partial}{\partial\theta} - \frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\psi} + \cos\phi \cot\theta \frac{\partial}{\partial\phi} \end{aligned} \tag{11.11}$$

eq:leftvfsa

11.1.1 The Heisenberg group

An interesting example which is not a compact Lie group are the Heisenberg groups.

The three-dimensional Heisenberg group is the group of matrices

$$g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \tag{11.12}$$

with $x, y, z \in \mathbb{R}$. The multiplication law is

$$(x, y, z) \cdot (x', y'z') = (x + x', y + y', z + z' + xy') \tag{11.13}$$

(note it is not abelian.)

An easy computation shows that

$$g^{-1}dg = \begin{pmatrix} 0 & dx & dz - xdy \\ 0 & 0 & dy \\ 0 & 0 & 0 \end{pmatrix} \tag{11.14}$$

12. Metrics on Lie groups

A very natural class of metrics on Lie groups are those which are left or right invariant.

As with forms and vector fields (and indeed all tensors), invariant metrics are uniquely determined by their value at $g = 1$, where they are bilinear forms on the Lie algebra. A natural way to make such a metric is to choose a representation ρ of the Lie group and define:

$$\begin{aligned} ds^2 &= \mathrm{Tr}_\rho(g^{-1}dg) \otimes (g^{-1}dg) \\ &= \mathrm{Tr}_\rho(dgg^{-1}) \otimes (dgg^{-1}) \end{aligned} \qquad (12.1)$$ eq:natmetric

Using equation (4.4)(??) above we see that this corresponds to the metric

$$(X_1, X_2) = \mathrm{Tr}_\rho(\rho(X_1)\rho(X_2)) \qquad (12.2)$$ eq:natmetricii

on the Lie algebra.

Warning: While it is natural, the left-right invariant metric (12.1) might be zero. For example, consider the group of upper triangular matrices with 1 on the diagonal. Similarly, consider the Heisenberg group above. Also, for infinite-dimensional groups, the cyclicity of the trace used in (12.1) might be invalid.

Metrics which are simultaneously left- and right- invariant correspond to Ad-invariant metrics on the Lie algebra. Every Lie algebra has a canonical representation, namely, the Adjoint representation. This defines the Ad-invariant Cartan-Killing metric:

$$g_{CK}(X_1, X_2) := \mathrm{Tr}_{\mathfrak{g}}(Ad(X_1)Ad(X_2)) \qquad (12.3)$$ eq:cartkill

Example 1: $O(n)$ has a basis for the Lie algebra $T_{ij} = e_{ij} - e_{ji}$, $1 \le i, j \le n$. Then

$$g_{CK}(T_{ij}, T_{kl}) = -2(n-2)\delta_{i,k}\delta_{j,l} \qquad (12.4)$$ eq:onck

Example 2: For $SU(2)$, if we regard $SU(2) = S^3$ as the unit sphere in Euclidean $\mathbb{R}^4$ then the metric is expressed in terms of the left-invariant forms is $1/2$ of the metric in the 2-dimensional representation:

$$\begin{aligned} d\Omega_3^2 &= \frac{1}{4}\sum_{a=1}^{3} e^a \otimes e^a \\ &= \frac{1}{4}\Big[(d\psi + \cos\theta d\phi)^2 + (d\theta)^2 + \sin^2\theta(d\phi)^2\Big] \\ &= \frac{1}{4}\Big[(d\psi)^2 + (d\theta)^2 + (d\phi)^2 + 2\cos\theta d\psi d\phi\Big] \end{aligned} \qquad (12.5)$$ eq:sumtrc

---

Exercise

Using the round metric of $SU(2)$ compute the inner products of the left and right invariant vector fields and show that

$$g(e^L_a, e^L_b) = g(e^R_a, e^R_b) = \delta_{ab}$$
$$g(e^L_a, e^R_b) = R_{ab}(\psi, \theta, \phi) \quad (12.6)$$

eq:lrips

where R_{ab} is the $SO(3)$ rotation matrix associated with the Euler angles.

Exercise

Show that if we write the metric on a Lie algebra as

$$(T_a, T_b) := I_{ab} \quad (12.7)$$

then Ad-invariance is equivalent to:

$$f_{ac}{}^d I_{db} = I_{ad} f_{cb}{}^d \quad (12.8)$$

Exercise

Show that if we choose a basis T_a for $\mathfrak{g}$ then the metric components of g_{CK} are given by

$$(g_{CK})_{ab} = f_{ac}{}^d f_{bd}{}^c \quad (12.9)$$

eq:carkillii

Exercise

Compute the CK form for $U(n)$, $SU(n)$, $USp(2n)$.

Exercise

Show that the volume of $SU(N)$ in the metric

$$ds^2 = -Tr_N[(g^{-1}dg) \otimes (g^{-1}dg)] \quad (12.10)$$

eq:metric

is

$$\mathrm{vol}\,(SU(N)) = \frac{\sqrt{N}}{2\pi}(2\pi)^{\frac{1}{2}N(N+1)}\frac{1}{1!2!3!\cdots(N-1)!} \tag{12.11}$$

Hint:

Consider $SU(N)/SU(N-1) = S^{2N-1}$ and relate the local coordinates of S^{2N-1} and $SU(N)$ via

$$g = 1 + iy^N Diag\{\epsilon, \dots, \epsilon, 1\} + \sum_i (z^i e_{iN} - \bar{z}^i e_{Ni}) + \cdots \tag{12.12}$$ eq:localcoord

at $g = 1$. (Here $\epsilon = -1/(N-1)$.). Here we are thinking of S^{2N-1} as the solutions to $\sum_{i=1}^N |z^i|^2 = 1$ and $z^N = x^N + iy^N$. In these coordinates, the group metric (12.10) becomes

$$ds^2 = 2\sum_{i=1}^{N-1}[(dx^i)^2 + (dy^i)^2] + \frac{N}{N-1}(dy)^2 \tag{12.13}$$ eq:metrspher

Volumes of G and G/K for compact symmetric spaces (in the Killing metric) were computed in A. Kojun and Y. Ichiro, "Volumes of compact symmetric spaces," Tokyo J. Math. **20** 1997)p.87.

12.1 Simple Lie groups and the index of a representation

If we define a simple Lie algebra as one which has no nontrivial invariant subalgebras then we have the following key result:

Exercise *Uniqueness of the Cartan-Killing form*

a.) Show that the Ad-invariant bilinear forms on a finite-dimensional simple Lie algebra are unique up to a constant.

b.) Show that if T_a is an ON basis in the CK form, then the most general invariant metric on a simple Lie group is

$$ds^2 = \Omega^2 \sum_{a=1}^{\mathrm{dim}G} e^a \otimes e^a \tag{12.14}$$

where Ω^2 is a constant.

Given the uniqueness of the CK form up to scale, for any representation ρ of $\mathfrak{g}$ set

$$g_\rho(X_1, X_2) = \mathrm{Tr}_\rho(\rho(X_1)\rho(X_2)) = \ell(\rho) g_{CK}(X_1, X_2) \tag{12.15}$$ eq:liemetric

where $\ell(\rho)$ is a constant called the index of the representation. This is often useful to have handy when working with instanton effects in gauge theory.

Example 1: For $SU(2)$ we can easily compute the index of the irreducible representations by simply computing the norm-squared of J_3. For the adjoint representation (spin =1) we have

$$g_{CK}(J_3, J_3) = 1^2 + 0^2 + (-1)^2 = 2 \tag{12.16}$$

eq:cksut

and for spin j we have

$$g_{\mathbf{j}}(J_3, J_3) = \sum_{m=-j}^{+j} m^2 = \sum_{\ell=0}^{2j} (-j+\ell)^2 = \frac{1}{3} j(j+1)(2j+1) \tag{12.17}$$

eq:cksutii

and therefore,

$$f(\mathbf{j}) = \frac{1}{6} j(j+1)(2j+1) \tag{12.18}$$

Example 2 For $SU(N)$, $\mathfrak{g} = su(N)$ is the algebra of $N \times N$ traceless antihermitian matrices.

$$\mathrm{Tr}_{adj} = 2N \mathrm{Tr}_N \tag{12.19}$$

eq:indxemb

Remarks

• Tables of in the indexes of representations are given in books by Patera.

Exercise

Show that

$$\frac{\ell(\rho_1)}{\ell(\rho_2)} = \frac{\mathrm{dim}\rho_1 \cdot C_2(\rho_1)}{\mathrm{dim}\rho_2 \cdot C_2(\rho_2)} \tag{12.20}$$

where $C_2(\rho)$ is the value of the Casimir in the representation ρ.

Exercise

Suppose that the decomposition of a tensor product of representations is given by

$$\rho_1 \otimes \rho_2 = \sum_\lambda N^\lambda \rho_\lambda \tag{12.21}$$

Show that

$$\ell(\rho_1) D(\rho_2) + \ell(\rho_2) D(\rho_1) = \sum_\lambda N^\lambda \ell(\rho_\lambda) \tag{12.22}$$

eq:relateindx

where $D(\rho)$ is the dimension of ρ.

Exercise

Show that for a semisimple Lie algebra the degree two Lie algebra cohomology vanishes: There are no nontrivial central extensions.

12.2 Metrics on abelian groups

In contrast to the simple groups, where the invariant metrics are unique up to an overall constant, the metrics on abelian groups are far from unique. Indeed, the torus $U(1)^n$ has a moduli space of invariant metrics $GL(n,\mathbb{R})/O(n)$.

12.3 Geodesics

Geodesics through the origin are of the form $\gamma(t) = \exp[tX]$ for $X \in \mathfrak{g}$.

********** EXPLAIN WHY **************

12.4 Compact and noncompact real forms: The signature of the CK metric

For a good discussion on this topic see Gilmore's book.

By the KAN decomposition (Gram-Schmidt decomposition) we see that every connected Lie group is (topologically) a product of its maximal compact subgroup and a Euclidean space.

A general result identifies the compact subalgebra with the negative definite generators and the noncompact generators with the positive definite subspace.

Key example: Compact generator

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus 0 \oplus 0 \oplus \cdots \tag{12.23}$$

This is compact in the sense that its 1-parameter subgroup forms a compact orbit:

$$\exp[tX] = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \oplus 1 \oplus 1 \oplus \cdots \tag{12.24}$$

Note that:

$$\mathrm{Tr}(X)^2 = -2 \tag{12.25}$$

while a noncompact generator

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 0 \oplus 0 \oplus \cdots \tag{12.26}$$

has

$$\exp[tX] = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \oplus 1 \oplus 1 \oplus \cdots \tag{12.27}$$

and

$$\mathrm{Tr}(X)^2 = +2 \tag{12.28}$$

12.5 Example: $SL(2,R)$

The Lie algebra $sl(2,\mathbb{R})$ is the Lie algebra of traces 2×2 real matrices.

One natural basis of the Lie algebra has commutation relations:

$$[K_0, K_\pm] = \pm K_\pm, \qquad [K_+, K_-] = -2K_0 \ . \tag{12.29}$$

eq:sucom

In the 2-dimensional representation we could take, for example:

$$K_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad K_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad K_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \tag{12.30}$$

Maurer-Cartan forms:

$$\begin{aligned} U &:= e^{C_+K_+}e^{C_0K_0}e^{C_-K_-} \\ dUU^{-1} &= e^{-C_0}dC_-K_- + (dC_0 - 2e^{-C_0}C_+dC_-)K_0 + (dC_+ + e^{-C_0}C_+^2dC_- - C_+dC_0)K_+ \\ U^{-1}dU &= dC_+e^{-C_0}K_+ + (dC_0 - 2C_-e^{-C_0}dC_+)K_0 + (dC_- - C_-dC_0 + C_-^2e^{-C_0}dC_+)K_- \end{aligned} \tag{12.31}$$

eq:mcforms

The Killing metric is:

$$(aK_0 + bK_+ + cK_-, aK_0 + bK_+ + cK_-) = 2(a^2 - 4bc) \tag{12.32}$$

Note that this is a Lorentzian signature metric.

Casimir:

$$4K_0^2 - 2(K_+K_- + K_-K_+) \tag{12.33}$$

The compact generator is $K_+ + K_-$ and the KAN decomposition is:

$$SO(2)\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \tag{12.34}$$

Exercise

a.) Compute the Maurer-Cartan forms in the global coordinates of the KAN decomposition.

b.) Suppose we write the Gauss decomposition:

$$g = \begin{pmatrix} 1 & 0 \\ \chi & 0 \end{pmatrix}\begin{pmatrix} e^\phi & 0 \\ 0 & e^{-\phi} \end{pmatrix}\begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix} \tag{12.35}$$

(warning: These are not globally valid coordinates on $SL(2,\mathbb{R})$.) Show that

$$\mathrm{Tr}(g^{-1}dg)^3 = 2e^{2\phi}d\phi\wedge d\psi\wedge d\chi \tag{12.36}$$

References:

1. A comprehensive reference on the geometry (but not the topology) of Lie groups is Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, 1978.

2. A less systematic, but more readable account for physicists is R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications*, John Wiley, 1974.

13. Some Remarks on Infinite Dimensional Lie Algebras

13.1 Generalities

The study of infinite dimensional Lie algebras and their representations is an active subject in modern mathematical physics. Infinite dimensional Lie algebras arise in conformal field theory, string theory, gauge theory of elementary particles, and gravitational theory classical and quantum mechanical. They have also play a role in some field theoretical applications of condensed matter physics such as the quantum Hall effect, and the Kondo problem.

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Banach Lie groups. Comments from Milnor.

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13.2 Groups of operators on Hilbert space

A very important infinite-dimensional group is the group $U(\mathcal{H})$ of unitary operators on Hilbert space. The topology of this group depends strongly on how we define the group precisely. The inductive limit $\lim_{n\to\infty} U(n)$ has lots of topology (it is a classifying space for K^1).

On the other hand, there is a very deep and remarkable theorem, known as Kuiper's theorem $U(\mathcal{H})$ which states that in the norm topology $U(\mathcal{H})$ is *contractible*!

13.3 Gauge Groups

For another set of examples, consider a Lie group G, and a manifold M. The space of all smooth maps $M \to G$ forms an infinite-dimensional group with Lie algebra the space of all maps $M \to \mathfrak{g}$. Except for the case where M is one-dimensional, very little is known about these groups. The one-dimensional case is of central importance in string theory and low-dimensional field theory.

13.3.1 Loop algebras

Suppose $\mathfrak{g}$ is a finite dimensional simple Lie algebra. We can associate to it an infinite dimensional Lie algebra whose elements are maps

$$f : S^1 \to \mathfrak{g} \tag{13.1}$$

We will be vague about the precise class of maps - it should be some completion of the space of Laurent polynomials in $z = e^{i\theta}$. The set of all such maps is itself a Lie algebra for we can define

$$[f_1, f_2](z) := [f_1(z), f_2(z)] \tag{13.2}$$

This infinite dimensional Lie algebra is known as the loop algebra $L\mathfrak{g}$.

It is often useful to choose a basis T^a for $\mathfrak{g}$, for this induces an obvious basis for $L\mathfrak{g}$:

$$T^a_n := T^a z^n \tag{13.3}$$

with commutation relations

$$[T^a_n, T^b_m] = f^{ab}_c T^c_{n+m} \tag{13.4}$$

Loop algebras admit a very interesting central extension: [8]

$$0 \to \mathbb{R} \to \widetilde{L\mathfrak{g}} \to L\mathfrak{g} \to 0 \tag{13.5}$$

Elements of $\widetilde{L\mathfrak{g}}$ are pairs (f, ξ) where f is a map, and $\xi \in \mathbb{C}$. The bracket of the central extension is

$$[(f_1, \xi_1), (f_2, \xi_2)] := ([f_1, f_2], \omega(f_1, f_2)) \tag{13.6}$$

where

$$\omega(f_1, f_2) := \frac{1}{2\pi i} \oint (f_1'(z), f_2(z))_{\mathfrak{g}} dz \tag{13.7}$$

where in the integral we use an ad-invariant bilinear form $(\cdot, \cdot)_{\mathfrak{g}}$, such as the Cartan-Killing form on $\mathfrak{g}$. (Recall that for G simple all such forms are the same up to scale, consistent with the uniqueness of the central extension.)

Again, it is useful to write this out in terms of a basis. Using the bilinear form on $\mathfrak{g}$ we define

$$(T^a, T^b)_{\mathfrak{g}} = g^{ab} \tag{13.8}$$

and we define $K := (0, 1) \in \widetilde{L\mathfrak{g}}$. Then

$$\begin{aligned} [T^a_n, T^b_m] &= f^{ab}_c T^c_{n+m} + n g^{ab} \delta_{n+m,0} K \\ [K, T^a_n] &= 0 \end{aligned} \tag{13.9}$$

eq:kmalg

This is the way the algebras are often written in the physics literature.

13.4 Diffeomorphism Groups

A nice example is given by the space of vector fields on a manifold and the related group of diffeomorphisms of a manifold.

Already, the simplest special case is highly nontrivial. Consider the group diffeomorphisms of the circle.

A natural basis of vector fields on the circle is given by Fourier decomposition:

$$\ell_n = i e^{in\theta} \frac{d}{d\theta} \tag{13.10}$$

[8]It is an easy result that, up to isomorphism, this is the unique nontrivial central extension when $\mathfrak{g}$ is simple.

They satisfy the Lie algebra:

$$[\ell_n, \ell_m] = (n - m)\ell_{n+m} \tag{13.11}$$

One can show that there is a one-dimensional cohomology space and a representative cocycle is

$$\omega(\ell_n, \ell_m) = \delta_{n+m,0} n^3 \tag{13.12}$$

eq:repcocycl

Note this is cohomologous to

$$\omega(\ell_n, \ell_m) = \delta_{n+m,0}(n^3 - n) \tag{13.13}$$

eq:repcocycla

The unique central extension of the Lie algebra of vector fields on S^1 is known as the Virasoro algebra.

The Virasoro algebra can be defined in terms of a basis L_n, $n \in \mathbb{Z}$ and a central element c, with structure constants:

$$\begin{aligned} [c, c] &= 0 \\ [c, L_n] &= 0 \\ [L_n, L_m] &= (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \end{aligned} \tag{13.14}$$

eq:virasoro

The normalization of the central element c is convenient in physical applications. The choice of cocycle is made so that the Lie algebra generated by $L_{\pm 1}, L_0$ forms a copy of $sl(2)$ and is not extended.

Exercise

a.) Check the Jacobi relations of the Virasoro algebra.

b.) Show that $\omega(\ell_n, \ell_m) = n\delta_{n+m,0}$ is a coboundary.

c.) Show that the Lie algebra cohomology of the conformal group is one-dimensional and hence show that the above is the most general central extension.

13.4.1 Gravity in 1+1 dimensions

Consider quantum gravity in $1 + 1$ dimensions. If space is compact and connected it must be S^1, we are therefore interested in invariant states under $G = Diff^+(S^1)$. A natural basis for the corresponding (complexified) Lie algebra is given by $\ell_n = e^{in\theta}\frac{d}{d\theta}$, $n \in \mathbb{Z}$. Actually, because this is an infinite-dimensional group we should allow for anomalies and work with a central extension. The corresponding Lie algebra is the *Virasoro algebra* given above. Let us see how the central extension comes about physically.

Let us consider the BRST operator. We introduce (diffeomorphism) ghosts and antighosts c_n, b_n, $n \in \mathbb{Z}$ with $\{c_n, b_m\} = \delta_{n+m,0}$.

Because we are working with an infinite-dimensional Clifford algebra there are many inequivalent quantizations. The one relevant to radial quantization in the complex plane of string theory is [9]

$$\begin{aligned} c_n|0\rangle &= 0 \qquad n > 1 \\ b_n|0\rangle &= 0 \qquad n > -2 \end{aligned} \tag{13.15}$$

eq:ghostvac

The idea here is that the "Dirac sea" is filled by $c_2 \wedge c_3 \wedge c_4 \wedge \cdots$. For this reason, elements of the ghost Hilbert space are referred to as "semi-infinite differential forms" in the math literature.

Following the formula (8.19) above we have in this case:

$$Q_{\text{ghost}} = -\frac{1}{2} \sum_{m,n\in\mathbb{Z}} (m-n) : c_{-m} c_{-n} b_{m+n} : \tag{13.16}$$

eq:virq

The normal ordering is necessary so that Q has a well-defined action on the ghost Hilbert space.

Now one finds a big surprise: $Q^2 \neq 0$ due to the need to regularize the infinite sums! Using ζ-function regularization one computes

$$Q^2_{\text{ghost}} = \sum_{m,n} \frac{(m-13m^3)}{12} \delta_{m+n,0} c_m c_n \tag{13.17}$$

eq:virqiii

If we consider the Lie algebra cohomology in a nontrivial Virasoro module with central charge $\rho(c)$ then the BRST operator is:

$$Q = \sum c_n \rho(L_n) - \frac{1}{2} \sum_{m,n\in\mathbb{Z}} (m-n) : c_{-m} c_{-n} b_{m+n} : -a c_0 \tag{13.18}$$

eq:virqiv

where we have allowed for a normal-ordering shift $\rho(L_0) \to \rho(L_0) - a$. Then

$$Q^2 = \frac{1}{2} \sum_{m,n\in\mathbb{Z}} \left[[\rho(L_n), \rho(L_m)] - (n-m)\rho(L_{n+m}) + \frac{((1+12a)m - 13m^3)}{6} \delta_{m+n,0} \right] c_m c_n \tag{13.19}$$

eq:virqiiia

so we get $Q^2 = 0$ for $a = 1$ and $\rho(c) = 26$.

In the worldsheet approach to string theory we work with $1+1$ dimensional quantum gravity coupled to matter and this is one way of understanding the restriction to matter with $c = 26$. Only in this case is there a well-defined spectrum of string states from BRST cohomology.

[9]In general, a primary field of dimension Δ has an expansion $\Phi = \sum_n \Phi_n z^{-n-\Delta}$ and hence $\Phi_n|0\rangle = 0$ for $n > -\Delta$, so that $\lim_{z\to 0} \Phi(z)|0\rangle$ is well-defined. The ghost field $c(z)$ has $\Delta = -1$ and the antighost $b(z)$ has dimension $\Delta = +2$.

13.4.2 The exponential map for $Diff(S^1)$

• For $Diff(S^1)$ the Exp map is neither 1-1 nor onto, even in an arbitrarily small neighborhood of the origin!

• For $Vir_{\mathbb{C}}$ there is no corresponding Lie group.

For a discussion of (1) above see Pressley+Segal, Loop Groups, Sections 3.3.1 and the article by Milnor. Briefly, to show exp is not $1-1$ consider rotation $R_{2\pi/n}$. Consider the centralizer $Z(R_{2\pi/n})$. This consists of diffeomorhisms which are periodic in the sense that $\phi(\theta+2\pi/n)=\phi(\theta)+2\pi/n \mathrm{mod} 2\pi$. Now for any ϕ in the centralizer, $\phi SO(2)\phi^{-1}$ is a one parameter subgroup of $Diff(S^1)$ consisting of elements $\{\phi R_\theta \phi^{-1}|\theta\in\mathbb{R}\}$. However, $R_{2\pi/n}$ lies on all these one-parameter subgroups! Thus, the exponential map is not 1-1. Note that we can make n arbitrarily large, so this failure of injectivity holds arbitrarily close to the identity.

Regarding (2) consider the following diffeomorphism

$$f(\theta)=\theta+\pi/n+\epsilon\sin^2(n\theta) \tag{13.20}$$

where $0<\epsilon<1/n$. One can prove that this diffeomorphism has precisely one periodic orbit of length $2n$: $0\to\pi/n\to 2\pi/n\to\cdots\to 2\pi(n-1)/n\to 0$. Indeed, if $0<\theta_0<\pi/n$ then defining $\theta_{j+1}=f(\theta_j)$ one can check with a little calculus that

$$\theta_0<\theta_1-\pi/n<\theta_2-2\pi/n<\cdots<\pi/n \tag{13.21}$$

so no θ_j is ever equal to θ_0 modulo 2π. Thus, our diffeomorphism has precisely one periodic orbit, and it is of length $2n$. It is remarkable that it has only one orbit with an even number of elements in the orbit. Why?

Consider a diffeomorphism F of *any* manifold. It is useful to compare the periodic orbits of F with those of $F\circ F$. If F has an odd periodic orbit then, $F\circ F$ has the same odd periodic orbit, essentially because the order of the orbit and 2 are relatively prime. (Draw the case of $n=3,5$.) On the other hand, if F has an *even* periodic orbit of order $n=2m$, then the periodic orbits of $F\circ F$ split up into two distinct periodic orbits of length m. (Check the cases $n=4,6$.)

Now, we apply this remark to note that $f(\theta)$ above cannot be of the form $F\circ F$ for any diffeomorphism F, since the number of periodic orbits of even period must be even, but $f(\theta)$ has a single periodic orbit of even period. But this in turn means that $f(\theta)$ cannot be on any one-parameter subgroup $F_t(\theta)$ since if we had $F_{t_*}(\theta)=f(\theta)$ then we would have had $f=F\circ F$ with $F=F_{\frac{1}{2}t_*}$.

For a discussion of point (2) above see section 3.3.2 of Pressley-Segal.

♣Need to explain why. Helpful remarks in Graeme's email from November 19, 2009 on this. ♣