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Chapter 4: Introduction to Representation Theory

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Abstract: BASED MOSTLY ON GTLECT4 FROM 2009. BUT LOTS OF MATERIAL HAS BEEN IMPORTED AND REARRANGED FROM MATHMETHODS 511, 2014 AND GMP 2010 ON ASSOCIATED BUNDLES. SOMETHING ABOUT SU(2) REPS AND INDUCED REPS NOW RESTORED. BECAUSE OF IMPORTS THERE IS MUCH REDUNDANCY. THIS CHAPTER NEEDS A LOT OF WORK. April 27, 2018

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1. Symmetries of physical systems

♣Much of this is redundant with discussion of Wigner's theorem which should be done in general discussion of quantum mechanics in Chapter 2. ♣

One of the most important applications of group theory in physics is in quantum mechanics. The basic principle is that if G is a symmetry group of a physical system(e.g., rotational symmetry, translational symmetry, ...) then each element $g \in G$ corresponds to a *unitary operator* acting on the Hilbert space $\mathcal{H}$ of physical states, i.e., we have an association:

$$\forall g \in G \mapsto U(g) : \mathcal{H} \to \mathcal{H} \tag{1.1}$$

where $U(g)$ is a unitary operator. Moreover, these unitary operators should "act in the same way as the physical symmetry." Mathematically, this means that we have the operator equation:

$$U(g_1)U(g_2) = U(g_1 g_2). \tag{1.2}$$

eq:repi

Moreover, if we have a symmetry of the physical system we should have the same kind of time-development of two systems related by the symmetry, so

$$U(g)HU(g)^{-1} = H \tag{1.3}$$

where H is the Hamiltonian.

The equation (1.2) is the essential equation defining what is called a *representation of a group*, and the above principle is one of the main motivations in physics for studying representation theory.

2. Basic Definitions

2.1 Representation of a group

subsec:ssFD

Let V be a vector space over a field κ and recall that $GL(V)$ denotes the group of all nonsingular linear transformations $V \to V$.

Definition 2.1.1 A *representation* of G with *representation space* V is a homomorphism

$$\begin{aligned} T : g \mapsto T(g) \\ G \to GL(V) \end{aligned} \tag{2.1}$$

eq:repii

The "dimension of the representation" is by definition the dimension $\dim V$ of the vector space V. This number can be finite or infinite. For convenience and mathematical rectitude we will often - but not always - focus on the case $\dim V < \infty$.

Terminology: We will often abbreviate "representation" to "rep." Moreover we sometimes refer to "the rep (T, V)" or to "the rep T," or to "the rep V," when the rest of the data is understood. Some authors call V the "carrier space."

Definition 2.1.2. A linear transformation $A : V \to V'$ between representations of G is called an *intertwiner* if for all $g \in G$ the diagram

$$\begin{array}{ccc} V & \stackrel{A}{\to} & V' \\ T(g) \downarrow & & \downarrow T'(g) \\ V & \stackrel{A}{\to} & V' \end{array} \tag{2.2}$$

commutes. Equivalently,

$$T'(g)A = AT(g) \tag{2.3}$$

eq:intertwin

for all $g \in G$.

Definition 2.1.3. Two representations are *equivalent* $(T, V) \cong (T', V')$ if there is an intertwiner which is an isomorphism. That is,

$$T'(g) = AT(g)A^{-1} \tag{2.4}$$

eq:wuivrep

for all $g \in G$.

Familiar notions of linear algebra generalize to representations:

1. The direct sum $\oplus$, tensor product $\otimes$ etc. of representations. Thus, the direct sum of (T_1, V_1) and (T_2, V_2) is the rep $(T_1 \oplus T_2, V_1 \oplus V_2)$ where the representation space is $V_1 \oplus V_2$ *and* the operators are:

$$(T_1 \oplus T_2)(g)\big(v_1, v_2\big) := \big(T_1(g)v_1, T_2(g)v_2\big) \tag{2.5}$$

2. Similarly, for the tensor product, the carrier space is $V_1 \otimes V_2$ and the group elements are represented by:

$$(T_1 \otimes T_2)(g)\big(v_1 \otimes v_2\big) := (T_1(g)v_1) \otimes (T_2(g)v_2) \tag{2.6}$$

3. Given a representation of V we get a dual representation on the dual space $V^\vee$ by demanding that under the pairing between V and $V^\vee$:

$$\langle T^\vee(g)\ell, T(g)v\rangle = \langle \ell, v\rangle, \tag{2.7}$$

   where $\ell \in V^\vee, v \in V$. Thus, if V, W are representation spaces then so is $Hom(V, W)$.

4. If V is a complex vector space then the complex conjugate representation sends $g \to \bar{T}(g) \in GL(\bar{V})$. A *real representation* is one where $(\bar{T}, \bar{V})$ is equivalent to (T, V).

2.2 Matrix Representations

Definition 2.1.3 A *matrix representation* is a homomorphism:

$$T : G \to GL(n, \kappa) \tag{2.8}$$

Given a representation, *and* an ordered basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ for V we can associate to a representation a matrix representation. The point is, with a basis we can identify $GL(V) \cong GL(n, \kappa)$. Specifically, we get the matrix from the linear transformation by:

$$T(g)\vec{v}_k = \sum_{j=1}^{n} T(g)_{jk}\vec{v}_j \tag{2.9}$$

so the $T(g)_{jk}$ are the matrix elements of an element of $GL(n, \kappa)$. We will sometimes denote this matrix as $T(g)$ if it is understood we are using an ordered basis for V.

Now, if we change basis

$$\vec{v}_i = \sum_{j=1}^{n} S_{ji}\vec{v}\,'_j \tag{2.10}$$

then the matrices change to

$$T'(g) = ST(g)S^{-1} \tag{2.11}$$

and this motivates

Definition Two n-dimensional matrix representations T and T' are *equivalent*, denoted $T \cong T'$ if $\exists S \in GL(n, \kappa)$ with

$$T'(g) = ST(g)S^{-1} \tag{2.12}$$

Exercise *Complex conjugate and transpose-inverse representations*

Given a matrix representation of a group $g \to T(g)$ show that

a.) $g \to (T(g))^{tr,-1}$ is also a representation.

If we choose a basis v_i for V then this is the matrix representation for the dual representation in the dual basis $\hat{v}_i$.

b.) If T is a complex matrix representation wrt basis v_i then the complex conjugate representation with respect to $\bar{v}_i$ is: $g \to T(g)^*$.

c.) If T is a real representation, then there exists an $S \in GL(n, \mathbb{C})$ such that for all $g \in G$:

$$T^*(g) = ST(g)S^{-1} \tag{2.13}$$

Warning: The matrix elements $T(g)_{ij}$ of a real representation can of course fail to be real numbers!

2.3 Examples

2.3.1 The fundamental representation of a matrix Lie group

$GL(n,\kappa), SL(n,\kappa), O(n,\kappa), U(n)$ are all matrix representations of themselves! In the first three examples $V = \kappa^n$. In the fourth $V = \mathbb{C}^n$. These are called "the fundamental representation." Note that when $\kappa = \mathbb{C}^n$ and $n > 2$ the fundamental representation is not equivalent to its complex conjugate [1] so the fundamental representation is not the same as the "minimal-dimensional nontrivial representation."

2.3.2 The determinant representation

The general linear group $GL(n,\kappa)$ always has a family of one-dimensional real representations $\det^\mu$, $\mu \in \mathbb{R}$, given by

$$T(A) := |\det A|^\mu \tag{2.14}$$ eq:dtrmrep

This is a representation because:

$$T(AB) = |\det AB|^\mu = |\det A|^\mu |\det B|^\mu = T(A)T(B) \tag{2.15}$$

2.3.3 A representation of the symmetric groups S_n

The symmetric group S_n has an n-dimensional representation with $V = \mathbb{R}^n$ (or $\mathbb{C}^n$) defined by choosing the standard basis $\{e_i\}$ and defining

$$T(\sigma) \cdot e_i := e_{\sigma(i)} \tag{2.16}$$

Note that acting on coordinates $x_i e_i$ this takes $x_i \to x_{\sigma^{-1}(i)}$, i.e.

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to \begin{pmatrix} x_{\sigma^{-1}(1)} \\ \vdots \\ x_{\sigma^{-1}(n)} \end{pmatrix} \tag{2.17}$$

and thus the matrix representation has matrix elements which are just 0's and 1's with a single nonzero entry in each row and column.

[1] This is easily proven using characters, see below.

Explicitly for $n = 2$ we have, for examples:

$$(12) \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.18}$$ eq:symmrep

while for $n = 3$ we have:

$$\begin{aligned} (12) &\to \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ (23) &\to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ (123) &\to \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned} \tag{2.19}$$ eq:symmrepii

2.3.4 $\mathbb{Z}$ and $\mathbb{Z}_2$

Sometimes relations come in families. If $\theta \in \mathbb{R}/\mathbb{Z}$ then we can define a representation T_θ of the group $\mathbb{Z}$ by $T_\theta(n) = e^{2\pi i n\theta}$.

At special values of parameters, special things can happen. For example note that when $\theta = 1/2$ the subgroup of even integers is represented trivially, and the representation "factors through" the sign representation of S_2. That is, $T_{1/2}$ is the composition of the projection $\mathbb{Z} \to \mathbb{Z}_2$ and the sign representation ε of $\mathbb{Z}_2$.

Similar, but more elaborate and interesting things happen for the representations of the braid group and its projection $B_n \to S_n$.

2.3.5 The Heisenberg group

An interesting group, the Heisenberg group, can be defined in terms of generators q, U, V where q is central and $UV = qVU$:

$$H = \langle q, U, V | qU = Uq, qV = Vq, UV = qVU \rangle \tag{2.20}$$ eq:heisn

If we we add the relations $q^N = U^N = V^N = 1$ then it is the finite Heisenberg group extension of $\mathbb{Z}_N \times \mathbb{Z}_N$ by $\mathbb{Z}_N$:

$$1 \to \mathbb{Z}_N \to \mathrm{Heis}_N \to \mathbb{Z}_N \times \mathbb{Z}_N \to 1 \tag{2.21}$$ eq:heisent

described in chapter 1, section 11.3.

Let ω be an N^{th} root of 1.

Let $U = Diag\{1, \omega, \omega^2, \dots, \omega^{N-1}\}$ be the "clock operator."

Let V be the "shift operator:"

$$V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad V = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad V = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix} \tag{2.22}$$

It is easy to check that:

$$UV = qVU \tag{2.23}$$

with $q = \omega$.

If we let ω range over the N^{th} roots of unity we obtain N inequivalent $N \times N$ (irreducible) representations.

If we consider the group without imposing $q^N = U^N = V^N = 1$ there are infinitely many inequivalent representations.

These assertions are obvious since different values of the central element q cannot be conjugated into each other.

3. Unitary Representations

sec:sUR

Of particular importance in physics are the unitary representations. By Wigner's theorem described in section **** above we know that symmetry transformations will act as unitary or anti-unitary transformations. The basic reason for this is that these preserve norms and hence probability amplitudes.

Definition 3.1. Let V be an inner product space. A *unitary representation* is a representation (T, V) such that $\forall g \in G$, $T(g)$ is a unitary operator on V, i.e.,

$$\langle T(g)v, T(g)v \rangle = \langle v, v \rangle \qquad \forall g \in G, v \in V \tag{3.1}$$

Definition 3.2. A unitary matrix representation is a homomorphism

$$T : G \to U(n) \tag{3.2}$$

Exercise

a.) Show that if $T(g)$ is a rep on an inner product space then $T(g^{-1})^\dagger$ is a rep also.

b.) Suppose $T : G \to GL(V)$ is a unitary rep on an inner product space V. Let $\{\vec{v}_i\}$ be an ON basis for V. Show that the corresponding matrix rep $T(g)_{ij}$ is a unitary matrix rep.

c.) Show that for a unitary matrix rep the transpose-inverse and complex conjugate representations are equivalent.

Definition 3.2. If a rep (T, V) is equivalent to a unitary rep then such a rep is said to be *unitarizable*.

Example. A simple example of non-unitarizable reps are the $\det^{\mu}$ reps of $GL(n, k)$ described in section 2.3.4.

3.1 Invariant Integration

When proving facts about unitary representations a very important tool is the notion of *invariant integration*. We have already made use of it in our discussion of lattice gauge theory in Section *** of Chapter 1. Here is a reminder:

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Important Remark: The notion of averaging over the group can be extended to a much larger class of groups than finite groups. It is given by *invariant integration* over the group which is the analog of the operation

$$f \to \frac{1}{|G|} \sum_{g} f(g) \tag{3.3}$$

on functions.

In general we replace

$$\frac{1}{|G|} \sum_{g} f(g) \to \int_{G} f(g) dg \tag{3.4}$$

where $\int_G f(g)dg$ should be regarded as a rule such that:

1. $\int_G f(g)dg$ is a complex number depending linearly on $f \in R_G$.
2. It satisfies the *left invariance* property:

$$\int_{G} f(hg) dg = \int_{G} f(g) dg \tag{3.5}$$

for all $h \in G$. This is the generalization of the rearrangement lemma. (We can define right invariance in a similar way. Compact groups admit integration measures which are simultaneously left- and right- invariant.)

Examples:

$$\begin{aligned} \int_{G=U(1)} f(g) dg &\equiv \int_{0}^{+2\pi} \frac{d\theta}{2\pi} f(\theta) \\ \int_{G=\mathbb{Z}} f(g) dg &\equiv \sum_{n \in \mathbb{Z}} f(n) \\ \int_{G=\mathbb{R}} f(g) dg &\equiv \int_{-\infty}^{+\infty} dx f(x) \end{aligned} \tag{3.6}$$

eq:intgroup

The notion of invariant integration also extends to all compact groups. For the important case of $G = SU(2)$ we can write it as follows. First, every element of $SU(2)$ can be written as:

♣Definition of β here is backwards from what we use when we describe reps using homogeneous polynomials below. ♣

$$g = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \tag{3.7}$$

for 2 complex numbers α, β with

$$|\alpha|^2 + |\beta|^2 = 1. \tag{3.8}$$

In this way we identify the group as a manifold as S^3. That manifold has no globally well-defined coordinate chart. The best we can do is define coordinates that cover "most" of the group but will have singularities are some places. (It is always important to be careful about those singularities when using explicit coordinates!) One way to do this is to write

$$\begin{aligned} \alpha &= e^{i\frac{1}{2}(\phi+\psi)} \cos\theta/2 \\ \beta &= ie^{i\frac{1}{2}(\phi-\psi)} \sin\theta/2 \end{aligned} \tag{3.9}$$

With this definition we can write

$$\begin{aligned} g &= \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \begin{pmatrix} \cos\theta/2 & i\sin\theta/2 \\ i\sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix} \\ &= e^{i\frac{1}{2}\phi\sigma^3} e^{i\frac{1}{2}\theta\sigma^1} e^{i\frac{1}{2}\psi\sigma^3} \end{aligned} \tag{3.10}$$

If we take $0 \le \theta \le \pi$, and identify $\phi \sim \phi + 2\pi$ and $\psi \sim \psi + 4\pi$ we cover the manifold once, away from the singular points at $\theta = 0, \pi$.

The normalized Haar measure for $SU(2)$ in these coordinates is [2]

$$[dg] = \frac{1}{16\pi^2} d\psi \wedge d\phi \wedge \sin\theta d\theta \tag{3.11}$$

For much more about this, see Chapter 5 below.

We give formulae for the Haar measures on the classical compact matrix groups in Chapter 6 below.

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3.2 Unitarizable Representations

Proposition If T is a rep on an inner product space of a *compact group* G, then V is finite dimensional and T is unitarizable.

Proof for finite groups: If T is not already unitary with respect to the inner product $(\cdot,\cdot)_1$ then we can define a new inner product by:

$$\langle v, w\rangle_2 \equiv \frac{1}{\mid G \mid} \sum_{g\in G} \langle T(g)v, T(g)w\rangle_1 \tag{3.12}$$

This proof generalizes using a left- and right- invariant Haar measure, which always exists for a compact group.

[2] We have chosen an orientation so that, with a positive constant, this is $\mathrm{Tr}_{\mathbf{2}}(g^{-1}dg)^3)$.

Exercise

a.) Show that

$$\langle T(g)v, T(g)w\rangle_2 = \langle v, w\rangle_2 \tag{3.13}$$

and deduce that $T(g)$ is unitary with respect to $\langle\cdot,\cdot\rangle_2$.

b.) Show that if $T(g)$ is a finite-dimensional matrix rep of a finite group then it is equivalent to a unitary matrix rep.

3.3 Unitary representations and the Schrödinger equation

♣THIS SUBJECTION IS NOW OUT OF PLACE. ♣

Consider the Schrödinger equation for stationary states of energy E:

$$-\frac{\hbar^2}{2m}\Delta\psi(\vec{x}) + V(\vec{x})\psi(\vec{x}) = E\psi(\vec{x}) \tag{3.14}$$

eq:shro

and suppose V is rotationally invariant:

$$V(\vec{x}) = V(\mid \vec{x} \mid). \tag{3.15}$$

Now define

$$V_E = \{\psi(\vec{x}) : \psi solves (3.14) \ and \ is \ normalizable\} \tag{3.16}$$

Normalizable, or square integrable means:

$$\parallel \psi \parallel^2 = \int_{R^3} \mid \psi(\vec{x}) \mid^2 d^3x < \infty \tag{3.17}$$

so $V_E \subset L^2(\mathbb{R}^3)$.

Claim: V_E is a representation space of $O(3)$: $T : O(3) \to GL(V_E)$ given, as usual, by:

$$[T(g)\psi](\vec{x}) = \psi(g^{-1}\cdot\vec{x}) \tag{3.18}$$

Must check:

1. ψ solves (3.14) $\Rightarrow T(g)\psi$ solves (3.14).
2. $\parallel \psi \parallel^2 < \infty \Rightarrow \parallel T(g)\psi \parallel^2 < \infty$

Check of 1: If $x'_i = (g^{-1})_i^{\ j} x_j$ then

$$\Delta' = \sum_{i=1}^{3} \frac{\partial^2}{\partial {x'_i}^2} = \sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2} = \Delta \tag{3.19}$$

and therefore:

$$\Delta\big[T(g)\psi\big](x) = \Delta(\psi(x')) = \Delta'(\psi(x')) \tag{3.20}$$

Check of 2: $x' = g^{-1}x$ gives

$$d^3x' = \mid \det\frac{dx^{i'}}{dx^j} \mid d^3x = d^3x \tag{3.21}$$

so:

$$\parallel \psi \parallel^2 = \parallel T(g)\psi \parallel^2 \tag{3.22}$$

and hence this is an example of a unitary representation.

al states of a theory form unitary reps of the symmetry group. Since the group commutes with the Hamiltonian we can diag

Thus, for example, the wavefunctions in a spherically symmetric potential have quantum numbers, $|E, j, m, \cdots\rangle$.

Exercise *Helmholtz equation*
Consider the Helmholtz equation:

$$\Delta u(\vec{x}) + k^2 u(\vec{x}) = 0$$
$$\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} \tag{3.23}$$

eq:hlmhltz

Let

$$V = \{u(\vec{x}) : u \ \text{satisfies}(3.23)\} \quad \text{eq:hlmhltziii} \tag{3.24}$$

eq:hlmhltziii

Show that V in (3.24) is a representation of the isometry group $E(3)$.

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4. Projective Representations and Central Extensions

♣GIVEN THE EXTENSIVE MATERIAL ON CENTRAL EXTENSIONS IN CHAPTER ONE THIS NEEDS TO BE REWRITTEN ♣

A *projective representation* is a pair (T, V) where $T : G \to GL(V)$ "almost" is a homomorphism, but is allowed to deviate from a homomorphism by a phase:

$$T(g_1)T(g_2) = c(g_1, g_2)T(g_1 g_2) \tag{4.1}$$

eq:projrep

where $c(g_1, g_2) \in \mathbb{C}^*$. For unitary representations $c(g_1, g_2) \in U(1)$.

From the associativity of a product of three transformations $T(g_1)T(g_2)T(g_3)$ we find that $c : G \times G \to \mathbb{C}^*$ is a 2-cocycle.

Recalling our discussion of central extensions we see that T should really be regarded as a representation of the central extension of G by $\mathbb{C}^*$ (or $U(1)$) constructed from the cocycle c. Put more formally

$$\hat{T}(z, g) := zT(g) \tag{4.2}$$

eq:cntpfjr

defines a true representation of the central extension $\hat{G}$ defined by c.

Moreover, if we allow ourselves the freedom to redefine $T(g) \to \tilde{T}(g) := f(g)T(g)$ where $f : G \to \mathbb{C}^*$ is a function then c changes by a coboundary.

Example Returning to our clock and shift operators, we can define a *projective* representation of $\mathbb{Z}_n \times \mathbb{Z}_n$ by

$$(\bar{s}, \bar{s}') \to U^{\bar{s}} V^{\bar{s}'} \tag{4.3}$$

but these only satisfy the group law up to a power of ω. In fact, what we have is a representation of $\mathrm{Heis}(\mathbb{Z}_n \times \mathbb{Z}_n)$.

5. Induced Group Actions On Function Spaces

:InducedAction

♣NOTE THERE IS OVERLAP OF THIS MATERIAL WITH EXAMPLES 5,6 OF FUNCTORS IN CATEGORY SECTION OF CHAPTER 1 ♣

Let X be a G-set and let Y be any set. There are natural left- and right- actions on the function space $\mathrm{Map}(X,Y)$. Given $\Psi \in \mathrm{Map}(X,Y)$ and $g \in G$ we need to produce a new function $\phi(g,\Psi) \in \mathrm{Map}(X,Y)$. The rules are as follows:

1. If G is a left-action on X then

$$\phi(g,\Psi)(x) := \Psi(g\cdot x) \qquad \text{right action on } \mathrm{Map}(X,Y) \tag{5.1}$$

2. If G is a left-action on X then

$$\phi(g,\Psi)(x) := \Psi(g^{-1}\cdot x) \qquad \text{left action on } \mathrm{Map}(X,Y) \tag{5.2}$$

3. If G is a right-action on X then

$$\phi(g,\Psi)(x) := \Psi(x\cdot g) \qquad \text{left action on } \mathrm{Map}(X,Y) \tag{5.3}$$

4. If G is a right-action on X then

$$\phi(g,\Psi)(x) := \Psi(x\cdot g^{-1}) \qquad \text{right action on } \mathrm{Map}(X,Y) \tag{5.4}$$

Example: Consider a spacetime $\mathcal{S}$. With suitable analytic restrictions the space of scalar fields on $\mathcal{S}$ is $\mathrm{Map}(\mathcal{S},\kappa)$, where $\kappa = \mathbb{R}$ or $\mathbb{C}$ for real or complex scalar fields. If a group G acts on the spacetime, there is automatically an induced action on the space of scalar fields. To be even specific, suppose $X = \mathbb{M}^{1,d-1}$ is d-dimensional Minkowski space time, G is the Poincaré group, and $Y = \mathbb{R}$. Given one scalar field Ψ and a Poincaré transformation $g^{-1}\cdot x = \Lambda x + v$ we have $(g\cdot\Psi)(x) = \Psi(\Lambda x + v)$.

Similarly, suppose that X is any set, but now Y is a G-set. Then again there is a G-action on $\mathrm{Map}(X,Y)$:

$$(g\cdot\Psi)(x) := g\cdot\Psi(x) \qquad \text{or} \qquad \Psi(x)\cdot g \tag{5.5}$$

according to whether the G action on Y is a left- or a right-action, respectively. These are left- or right-actions, respectively.

We can now combine these two observations and get the general statement: We assume that both X is a G_1-set and Y is a G_2-set. We can assume, without loss of generality, that we have left-actions on both X and Y. Then there is a natural $G_1 \times G_2$-action on $\mathrm{Map}(X, Y)$ defined by:

$$\phi((g_1, g_2), \Psi)(x) := g_2 \cdot (\Psi(g_1^{-1} \cdot x)) \tag{5.6}$$

eq:GenAction

note that if one writes instead $g_2 \cdot (\Psi(g_1 \cdot x))$ on the RHS then we do not have a well-defined $G_1 \times G_2$-action (if G_1 and G_2 are both nonabelian). In most applications X and Y both have a G action for a single group and we write

$$\phi(g, \Psi)(x) := g \cdot (\Psi(g^{-1} \cdot x)) \tag{5.7}$$

This is a special case of the general action (5.6), with $G_1 = G_2 = G$ and specialized to the diagonal $\Delta \subset G \times G$.

Example: Again let $X = \mathbb{M}^{1,d-1}$ be a Minkowski space time. Take $G_1 = G_2$ and let $G = \Delta \subset G \times G$ be the diagonal subgroup, and take G to be the Poincaré group. Now let $Y = V$ be a finite-dimensional representation of the Poincaré group. Let us denote the action of $g \in G$ on V by $\rho(g)$. Then a field $\Psi \in \mathrm{Map}(X, Y)$ has an action of the Poincaré group defined by

$$g \cdot \Psi(x) := \rho(g)\Psi(g^{-1}x) \tag{5.8}$$

This is the standard way that fields with nonzero "spin" transform under the Poincaré group in field theory. As a very concrete related example, consider the transformation of electron wavefunctions in nonrelativistic quantum mechanics. The electron wavefunction is governed by a two-component function on $\mathbb{R}^3$:

$$\Psi(\vec{x}) = \begin{pmatrix} \psi_+(\vec{x}) \\ \psi_-(\vec{x}) \end{pmatrix} \tag{5.9}$$

Then, suppose $G = SU(2)$. Recall there is a surjective homomorphism $\pi : G \to SO(3)$ defined by $\pi(u) = R$ where

$$u\vec{x} \cdot \vec{\sigma} u^{-1} = (R\vec{x}) \cdot \vec{\sigma} \tag{5.10}$$

Then the (double-cover) of the rotation group acts to define the transformed electron wavefunction $u \cdot \Psi$ by

$$(u \cdot \Psi)(\vec{x}) := u \begin{pmatrix} \psi_+(R^{-1}\vec{x}) \\ \psi_-(R^{-1}\vec{x}) \end{pmatrix} \tag{5.11}$$

In particular, $u = -1$ acts trivially on $\vec{x}$ but nontrivially on the wavefunction.

6. The regular representation

Now that we have seen several examples of representations we would like to introduce a "universal example," of a representation of G, called the *regular representation*. We will see that from this representation we can learn about *all* of the representations of G, at least when G is compact.

Recall once again the principle of section 5: If X has G-action then the space $\mathcal{F}_{X,Y}$ of all functions $X \to Y$ also has a G-action. In particular, if Y is a vector space then $\mathcal{F}_{X,Y}$ is a vector space and we have a natural source of representations of G. In particular, we can make the simplest choice $Y = \mathbb{C}$, so the space of complex-valued functions on X, $\mathcal{F}_{X,\mathbb{C}}$ is a very natural source of representations of G.

♣THIS IS OLD. SHOULD MAKE IT A REPRESENTATION OF $G \times G$ FROM THE START. ♣

If G acts on nothing else, it certainly acts on itself as a group of transformations of $X = G$ with left or right action. Applying the above general principle we see that the space of complex-valued functions on G:

$$\mathcal{F}_{G,\mathbb{C}} := \{Maps \qquad \phi : G \to \mathbb{C}\} \tag{6.1}$$

forms a natural - "God given" - representation space of G. $\mathcal{F}_{G,\mathbb{C}}$ is also denoted by R_G. When G is infinite we usually want to put some conditions on the functions in question. If G admits a left-invariant integration measure then by R_G we understand $L^2(G)$ in this measure.

In fact, R_G is a representation space in two ways:

1. Left regular representation (LRR):

$$\phi \mapsto L(g) \cdot \phi \tag{6.2}$$

where $L(g) \cdot \phi$ has values defined by

$$(L(g) \cdot \phi)(h) \equiv \phi(g^{-1}h) \tag{6.3}$$

2. Right regular representation (RRR):

$$\phi \mapsto R(g) \cdot \phi \tag{6.4}$$

where $R(g) \cdot \phi$ has values defined by

$$(R(g) \cdot \phi)(h) \equiv \phi(hg) \tag{6.5}$$

Exercise

Check that

$$L(g_1)L(g_2) = L(g_1 g_2) \tag{6.6}$$

and that

$$R(g_1)R(g_2) = R(g_1 g_2) \tag{6.7}$$

Note too that for all $g, h \in G$

$$R(g)L(h) = L(h)R(g) \tag{6.8}$$

so, in fact, R_G is a representation of $G \times G = G_{\text{left}} \times G_{\text{right}}$.

Now let us assume $\mid G \mid < \infty$. Then R_G is a finite dimensional vector space. we can label a function f by its values $f(g)$.

Example. $G = \mathbb{Z}_2 = \{1, \sigma\}$. Then $R_{\mathbb{Z}_2} \cong \mathbb{C}^2$ as a vector space, since a function on $\mathbb{Z}_2$ is labelled by (a, b) where $a = f(1), b = f(\sigma)$.

Quite generally, we can introduce a basis of "delta functions" concentrated at g:

$$\delta_g(g') = \begin{cases} 1 & for g' = g \\ 0 & else \end{cases} \tag{6.9}$$

The functions $\delta_g \in R_G$ span R_G:

$$f = \sum_g f(g)\delta_g \tag{6.10}$$

Exercise

If G is a finite group, then show that

$$\dim R_G = |G| \tag{6.11}$$

Exercise

a.) Let $\delta_0, \delta_1, \delta_2$ be a basis of functions in the regular representation of $\mathbb{Z}_3$ which are 1 on $1, \omega, \omega^2$, respectively, and zero elsewhere. Show that ω is represented as

$$L(\omega) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{6.12}$$

b.) Show that

$$\begin{aligned} L(h) \cdot \delta_g &= \delta_{h \cdot g} \\ R(h) \cdot \delta_g &= \delta_{g \cdot h^{-1}} \end{aligned} \tag{6.13}$$

and conclude that for the left, or right, regular representation of a finite group the representation matrices in the δ-function basis have entries which are simply zeroes or ones in the natural basis.

6.1 Matrix elements as functions on G

Now we make an important conceptual step forward:

Let $g \to T_{ij}(g)$ be any matrix representation of G.

♣Actually, for current level of earlier chapters this is pretty obvious to the reader. ♣

We may regard the matrix elements $T_{m\ell}$ as complex-valued *functions* on G. Therefore, we may regard $T_{m\ell}$ as *vectors* in the regular representation R_G.

6.2 R_G as a unitary rep: Invariant integration on the group

When G is a finite group we can turn the vector space R_G into an inner product space by the defining an inner product using averaging over the group:

$$\langle \phi_1, \phi_2 \rangle := \frac{1}{|G|} \sum_{g \in G} \phi_1^*(g) \phi_2(g) \tag{6.14}$$

eq:innprod

Theorem. With the inner product (6.14) the representation space R_G is *unitary* for both the RRR and the LRR.

Proof: Straightforward application of the rearrangement lemma.

6.3 A More Conceptual Description

♣MUCH OF THIS IS REDUNDANT WITH ABOVE BUT SOME THINGS ARE SAID MORE NICELY, esp. map of $End(V)$ into $L^2(G)$. ♣

As a very nice example of the idea of how group actions on a space induce group actions on the functions on that space we touch briefly on the Peter-Weyl theorem and the idea of induced representations.

We first describe the Peter-Weyl theorem:

♣Notation in this subsection is at variance with rest of chapter: Here ρ is the rep homomorphism and T is a general operator... ♣

Let G be a group. Then there is a left action of $G \times G$ on G: $(g_1, g_2) \mapsto L(g_1)R(g_2^{-1})$. Now let $Y = \mathbb{C}$. Then $\mathrm{Map}(G, \mathbb{C})$ is known as the *regular representation* of G because the induced left-action:

$$((g_1, g_2) \cdot \Psi)(h) := \Psi(g_1^{-1} h g_2) \tag{6.15}$$

converts the vector space $\mathrm{Map}(G \to \mathbb{C})$ of functions $\Psi : G \to \mathbb{C}$ into a representation space for $G \times G$.

Suppose, on the other hand that V is a linear representation of G. As mentioned above this means we have a group homomorphism $\rho : G \to GL(V)$. Then consider the vector space of linear transformations $\mathrm{End}(V)$ of V to itself. This is also a representation of $G \times G$ because if $T \in \mathrm{End}(V)$ then we can define a linear left-action of $G \times G$ on $\mathrm{End}(V)$ by:

$$(g_1, g_2) \cdot T := \rho(g_1) \circ T \circ \rho(g_2)^{-1} \tag{6.16}$$

Now, we have two representations of $G \times G$. How are they related? If V is finite-dimensional we have a map

$$\iota : \mathrm{End}(V) \to \mathrm{Map}(G, \mathbb{C}) \tag{6.17}$$

The map ι takes a linear transformation $T : V \to V$ to the complex-valued function $\Psi_T : G \to \mathbb{C}$ defined by

$$\Psi_T(g) := \mathrm{Tr}_V(T \rho(g^{-1})) \tag{6.18}$$

eq:PsiTee

If we choose a basis w_μ for V then the operators $\rho(g)$ are represented by matrices:

$$\rho(g) \cdot w_\nu = \sum_\mu D(g)_{\mu\nu} w_\mu \tag{6.19}$$

If we take $T = e_{\nu\mu}$ to be the matrix unit in this basis then Ψ_T is the function on G given by the matrix element $D(g^{-1})_{\mu\nu}$. So the Ψ_T's are linear combinations of matrix elements of the representation matrices of G. The advantage of (6.18) is that it is completely canonical and basis-independent.

Note that $\iota : T \mapsto \Psi_T$ "commutes with the $G \times G$ action." What this means is that

$$(g_1, g_2) \cdot \Psi_T = \Psi_{(g_1,g_2)\cdot T} \qquad (6.20)$$

eq:GxG-commute

(The reader should check this carefully.) Such a map is said to be *equivariant*. Put differently, denoting by $\rho_{\mathrm{End}(V)}$ the representation of $G \times G$ on $\mathrm{End}(V)$ and $\rho_{Reg.Rep.}$ the representation of $G \times G$ on $\mathrm{Map}(G, \mathbb{C})$ we get a commutative diagram:

$$\begin{array}{ccc} \mathrm{End}(V) & \xrightarrow{\iota} & \mathrm{Map}(G,\mathbb{C}) \\ \downarrow{\scriptstyle \rho_{\mathrm{End}(V)}} & & \downarrow{\scriptstyle \rho_{Reg.Rep.}} \\ \mathrm{End}(V) & \xrightarrow{\iota} & \mathrm{Map}(G,\mathbb{C}) \end{array} \qquad (6.21)$$

In particular if we have a collection of finite-dimensional representations $\{V_\lambda\}$ of G then we have

$$\oplus_\lambda \mathrm{End}(V_\lambda) \hookrightarrow \mathrm{Map}(G, \mathbb{C}) \qquad (6.22)$$

eq:PW-1

Thanks to the equivariance, the image of (6.22) is a $G \times G$-invariant subspace, i.e. a subrepresentation of $\mathrm{Map}(G, \mathbb{C})$. The very beautiful *Peter-Weyl theorem* states that, if G is a compact group, then, as representations of $G \times G$, (6.22) is an isomorphism if the sum is over the distinct isomorphism classes λ of irreducible representations V_λ of G and we restrict to the the subspace of $\mathrm{Map}(G, \mathbb{C})$ of L^2-normalizable functions with respect to a left-right-invariant measure on G. [3]

♣IRREPS NOT YET EXPLAINED ♣

7. Reducible and Irreducible representations

7.1 Definitions

Sometimes reps are "too big" and one wants to reduce them to their "essential parts." Reducing a representation to smaller representations is closely analogous to diagonalization of matrices.

For example – we will see that the natural representation of a group - its regular representation - is in fact highly reducible.

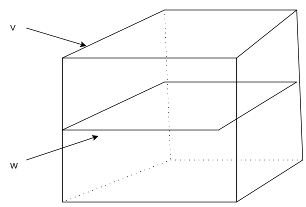

Figure 1: $T(g)$ preserves the subspace W.

fig:redrep

[3] In order for this to be completely correct we need to assume that G is a compact group. Then we introduce a left-right-invariant measure on G and replace the LHS by $L^2(G)$.

Definition. Let $W \subset V$ be a subspace of a group representation $T : G \to GL(V)$. Then W is *invariant* under T if $\forall g \in G, w \in W$

$$T(g)w \in W \tag{7.1}$$

This may be pictured as in 1. Example: Consider the three-dimensional representation of $SO(2)$ as rotations around the z-axis. Then the vector subspace of the xy plane is an invariant subspace.

Since T preserves the smaller vector space W, we can define a smaller group representation (T, W). This is called the *restriction of T to W.*

Remarks

•

If T is unitary on V then it is unitary on W.

• Both $\{\vec{0}\}$ and V are always invariant subspaces.

Definition. A representation T is called *reducible* if there is an invariant subspace $W \subset V$, under T, which is nontrivial, i.e., such that $W \neq 0, V$, If V is not reducible we say V is *irreducible*. That is, in an irreducible rep, the only invariant subspaces are $\{\vec{0}\}$ and V. We often shorten the term "irreducible representation" to "irrep."

Note:

• Given any nonzero vector $v \in V$, the linear span of $\{T(g)v\}_{g \in G}$ is an invariant subspace. In an irrep this will span all of V.

• If W is an subrepresentation of V then T descends to a representation on V/W.

Let us see how this works in terms of matrix representations. Suppose T is reducible. Then we can choose a basis

$\{v_1, \dots v_k\}$ for W

and

$\{v_1, \dots v_k, v_{k+1}, \dots v_n\}$ for V

In this basis the matrix representation of T looks like:

$$T(g) = \begin{pmatrix} T_1(g) & T_{12}(g) \\ 0 & T_2(g) \end{pmatrix} \tag{7.2}$$

eq:redrep

where

$T_1(g) \in Mat_{k \times k}$

$T_{12}(g) \in Mat_{k \times n-k}$

$T_2(g) \in Mat_{(n-k)\times(n-k)}$

Writing out $T(g_1)T(g_2) = T(g_1 g_2)$ we see that T_1 is the representation on W and T_2 is the representation on V/W. T_{12} transforms in a more complicated way.

Definition. A representation T is called *completely reducible* if it is isomorphic to

$$W_1 \oplus \cdots \oplus W_n \tag{7.3}$$

where the W_i are irreducible reps. Thus, there is a basis in which the matrices look like:

$$T(g) = \begin{pmatrix} T_1(g) & 0 & 0 & \cdots \\ 0 & T_2(g) & 0 & \cdots \\ 0 & 0 & T_3(g) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{7.4}$$

eq:comredrep

Examples

• $G = \mathbb{Z}_2$

$$\begin{aligned} 1 &\rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ (12) &\rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{7.5}$$

is a 2-dimensional *reducible* rep on $\mathbb{R}^2$ because $W = \{(x, x)\} \subset \mathbb{R}^2$ is a nontrivial invariant subspace. Indeed, σ^1 is diagonalizable, so this rep is equivalent to

$$\begin{aligned} 1 &\rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ (12) &\rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{7.6}$$

which is a direct sum of two 1×1 reps.

• Consider the representation of S_n on $\mathbb{R}^n$. Then the one-dimensional subspace $L = \{(x, \cdots, x)\}$ is a subrepresentation. Moreover we can take $L^{\perp} = \{x_1, \cdots x_n | \sum x_i = 0\}$. Then $L^{\perp}$ is an $(n-1)$-dimensional representation of S_n and the representation is block diagonal on $L \oplus L^{\perp}$.

♣THIS IS OLD. MAYBE START WITH SPAN OF T_{ij} WHERE BOTH i, j VARY AS REP OF $G \times G$. WE CAN ALWAYS THEN RESTRICT TO $G \times 1$ AND $1 \times G$. ♣

• Invariant subspaces of the regular representation. Let T_{ij} be *any* matrix n-dimensional representation of G.

Claim: The linear span of functions

$$\mathcal{R} := Span\{T_{ij}\}_{j=1,\ldots,n_\mu} \tag{7.7}$$

is an invariant subspace of R_G, considered as the RRR.

Proof:

$$\begin{aligned} (R(g) \cdot T_{ij})(h) &= T_{ij}(hg) \\ &= \sum_{s=1}^{n_\mu} T_{is}(h) T_{sj}(g) \end{aligned} \tag{7.8}$$

which is equivalent to the equation on functions:

$$\overbrace{R(g) \cdot T_{ij}}^{\text{function on G}} = \sum_{s=1}^{n_\mu} \underbrace{T_{sj}(g)}_{\text{matrix element for } R_G} \overbrace{T_{is}}^{\text{function on G}} \tag{7.9}$$

eq:mtxeltinvt

7.2 Reducible vs. Completely reducible representations

subsec:ssRedC

Irreps are the "atoms" out of which all reps are made. Thus we are naturally led to study the *irreducible* reps of G. In real life it can and does actually happen that a group G has representations which are reducible but not completely reducible. Reducible, but not completely reducible reps are sometimes called *indecomposable*.

Example . An example of an indecomposible rep which is not completely reducible is the rep

$$A \to \begin{pmatrix} 1 & \log|\det A| \\ 0 & 1 \end{pmatrix} \tag{7.10}$$

of $GL(n, \mathbb{R})$. As we will see, the Poincare group has indecomposable reps.

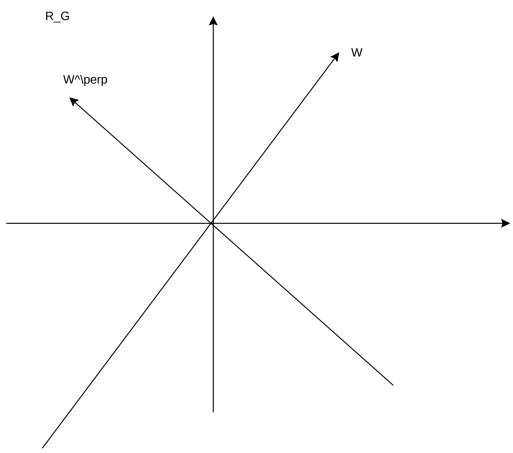

Figure 2: The orthogonal complement of an invariant subspace is an invariant subspace. fig:orthcomp

It is useful to have criteria for when this complication cannot occur:

Proposition 7.2.1 Suppose that the representation (T, V) is a unitary representation on an inner product space V and $W \subset V$ is an invariant subspace then $W^\perp$ is an invariant subspace:

Proof: Recall that

$$y \in W^\perp \Leftrightarrow \forall x \in W, \quad \langle y, x \rangle = 0 \tag{7.11}$$

Let $g \in G$, $y \in W^\perp$. Compute

$$\begin{aligned} \big(T(g)y, x\big) &= \big(y, T(g)^\dagger x\big) \\ &= \big(y, T(g^{-1})x\big) \end{aligned} \tag{7.12}$$

But, $T(g^{-1})x \in W$, since $W \subset V$ is an invariant subspace.

therefore: $\forall x \in W$, $\langle T(g)y, x \rangle = 0$.

therefore: $T(g)y \in W^\perp$
therefore: $W^\perp$ is an invariant subspace. ♠

Corollary: *Finite dimensional unitary representations are always completely reducible.*

In particular, by section 4:
1. Finite dimensional reps of finite groups are completely reducible.
2. More generally, finite dimensional reps of compact Lie groups are completely reducible.

It follows that for a finite group the regular representation R_G is completely reducible. In the next several sections we will show how to decompose R_G in terms of the irreducible representations of G.

If V is a completely decomposable representation (always true for compact groups) then we can write

$$V \cong \oplus_{\mu=1}^{r} \oplus_{i=1}^{a_\mu} T^{(\mu)} := \oplus_\mu a_\mu T^{(\mu)} \tag{7.13}$$

For a fixed μ let let

$$V^{(\mu)} := \oplus_{i=1}^{a_\mu} T^{(\mu)} \tag{7.14}$$

It contains the representation μ with the correct degeneracy in V. It is called the *isotypical component* belonging to μ. Note that it can be written as

$$V^{(\mu)} = \mathbb{C}^{a_\mu} \otimes T^{(\mu)} \tag{7.15}$$ eq:isotpy

where $\mathbb{C}^{a_\mu}$ is understood to be the trivial representation of G.

We abbreviate $\mathbb{C}^{a_\mu} \otimes T^{(\mu)}$ to $a_\mu T^\mu$ and with this understood we write the decomposition into isotypical components as:

$$V = \oplus a_\mu T^\mu \tag{7.16}$$

8. Schur's Lemmas

When thinking about irreducible representations it is important to understand how they are related to each other. A key technical tool is Schur's lemma. It is usually stated as two separate statements, each of which is practically a triviality.

♣This section needs improvement: Schur's lemma says that the algebra of intertwiners of an irrep is a division algebra. So $\mathbb{R}, \mathbb{C}, \mathbb{H}$. What is here is correct, but not the best viewpoint. ♣

Theorem 1: If $A : V \to V'$ is an intertwiner between two irreps then it is either an isomorphism or zero.

Proof: ker$A$ and Im A are both invariant subspaces. Of course, if V and V' are inequivalent then $A = 0$. ♠.

Theorem 2: If A is a complex matrix commuting with an irreducible matrix representation then A is proportional to the identity matrix.

Proof: Since we are working over the complex field A has a nonzero eigenvector $Av = \lambda v$. The eigenspace $C = \{w : Aw = \lambda w\}$ is therefore not the zero vector space. But it is also an invariant subspace. Therefore, it must be the entire carrier space. ♠.

Remarks:

• Consider the isotypical decomposition of a completely reducible representation. By Schur's lemma, the intertwiners of $V^{(\mu)}$ with itself are of the form $K \otimes 1$ where $K \in \mathrm{End}(\mathbb{C}^{a_\mu})$ is arbitrary.

• Schur's lemma is used in quantum mechanics very often. Note that a Hamiltonian invariant under some symmetry group $U(g)HU(g)^{-1} = H$ is an intertwiner. Therefore, if we decompose the Hilbert space of states into irreps of G then H must be scalar on each irrep. In particular this leads to important *selection rules* in physics. For example, if ΔH is an invariant operator which commutes with some symmetry G of a physical system then the matrix elements of ΔH between states in different irreducible representations of G must vanish.

9. Orthogonality relations for matrix elements

Let G be a finite group. We are interested in finding the reduction of R_G to its irreducible representations. Our next technical tool - very useful in a variety of contexts - are the orthogonality relations of matrix elements of irreps.

Label the distinct, i.e. inequivalent, *irreducible representations* by

$$T^{(\mu)} \qquad \mu = 1, 2, \ldots, r \tag{9.1}$$

and let $n_\mu = \dim T^{(\mu)}$. (We will prove later that $r < \infty$ and $n_\mu < \infty$.) In order not to overburden the notation we will denote the carrier space and the homomorphism by the same name $T^{(\mu)}$, relying on context to tell which is intended. Choose bases for $T^{(\mu)}$ and consider these to be matrix irreps.

Let $B \in Mat_{n_\mu \times n_\nu}$, and define a matrix $A \in Mat_{n_\mu \times n_\nu}$ by:

♣SHOULD GIVE A MORE CONCEPTUAL DESCRIPTION OF WHAT IS GOING ON HERE USING $\sum_g T^\vee(g) \otimes T(g)$. ♣

$$A \equiv \frac{1}{|G|} \sum_{g \in G} T^{(\mu)}(g) B T^{(\nu)}(g^{-1}) \tag{9.2}$$

eq:orthi

We claim that A is an intertwiner, that is, $\forall h \in G$:

$$\boxed{T^{(\mu)}(h) A = A T^{(\nu)}(h)} \tag{9.3}$$

Proof:

$$\begin{aligned}
T^{(\mu)}(h)A &= \frac{1}{|G|}\sum_{g\in G} T^{(\mu)}(h)T^{(\mu)}(g)BT^{(\nu)}(g^{-1}) \\
&= \frac{1}{|G|}\sum_{g\in G} T^{(\mu)}(hg)BT^{(\nu)}(g^{-1}) \\
&= \frac{1}{|G|}\sum_{g'\in G} T^{(\mu)}(g')BT^{(\nu)}((h^{-1}g')^{-1}) \\
&= \frac{1}{|G|}\sum_{g\in G} T^{(\mu)}(g)BT^{(\nu)}((g)^{-1})T^{(\nu)}(h) \\
&= AT^{(\nu)}(h)
\end{aligned} \tag{9.4}$$

In the third line let $g' = hg$, and use the rearrangement lemma.

Therefore, by Schur's lemma:

$$A = \lambda\delta_{\mu,\nu}\mathbf{1}_{n_\nu\times n_\nu} \tag{9.5}$$

eq:orthii

where λ can depend on $\mu = \nu, B$.

Now, to extract useful information from this let $B = e_{\ell m}$ be a matrix unit, $1 \le \ell \le n_\mu$, $1 \le m \le n_\nu$. Then (9.2)(9.5) imply:

$$\sum_{g\in G} T^{(\mu)}_{i\ell}(g)T^{(\nu)}_{ms}(g^{-1}) = |G|\lambda\delta_{\mu\nu}\delta_{is} \tag{9.6}$$

eq:orthiii

where now λ depends on m, ℓ, μ, but does *not* depend on i, s.

We can determine λ by setting $\mu = \nu$, $i = s$ and summing on i:

Then (9.6) becomes:

$$\begin{aligned}
\sum_{g\in G} T^{(\mu)}_{m\ell}(g^{-1}g) &= |G|\lambda n_\mu \\
|G|\delta_{\ell m} &= |G|\lambda n_\mu \\
\Rightarrow \lambda &= \frac{\delta_{\ell m}}{n_\mu}
\end{aligned} \tag{9.7}$$

Putting it all together we get the main result of this section:

Theorem The orthogonality relations for the group matrix elements are:

$$\boxed{\frac{1}{|G|}\sum_{g\in G} T^{(\mu)}_{i\ell}(g)\hat{T}^{(\nu)}_{sm}(g) = \frac{1}{n_\mu}\delta_{\mu\nu}\delta_{is}\delta_{\ell m}} \tag{9.8}$$

eq:orthogmatrx

or, for a general compact group:

$$\boxed{\int_G T^{(\mu)}_{i\ell}(g)\hat{T}^{(\nu)}_{sm}(g)[dg] = \frac{1}{n_\mu}\delta_{\mu\nu}\delta_{is}\delta_{\ell m}} \tag{9.9}$$

eq:orthogmatrx

where the Haar measure $[dg]$ is normalized to unit volume. Recall that

$$\hat{T}^{(\nu)}_{sm}(g) = (T^{(\nu)}(g))^{-1,tr}_{sm} \tag{9.10}$$

is the matrix representation on the dual space.

Note: If the matrix reps $T^{(\mu)}_{i\ell}$ are *unitary* then $T(g)^{tr,-1} = T(g)^*$ so we can rewrite this in a nicer form: Define

$$\phi^{(\mu)}_{i\ell} \equiv n_\mu^{1/2} T^{(\mu)}_{i\ell} \tag{9.11}$$

then we have:

$$\boxed{\frac{1}{|G|}\sum_{g\in G} \phi^{(\mu)}_{i\ell}(g)(\phi^{(\nu)}_{sm}(g))^* = \delta_{\mu\nu}\delta_{is}\delta_{\ell m}} \tag{9.12}$$

eq:onrels

Or, for a general compact group

$$\boxed{\int_G (\phi^{(\nu)}_{sm}(g))^* \phi^{(\mu)}_{i\ell}(g)[dg] = \delta_{\mu\nu}\delta_{is}\delta_{\ell m}} \tag{9.13}$$

eq:onrels

Interpretation: The vectors $\phi^{(\mu)}_{ij}$ form an *orthonormal set* of vectors in R_G.

Below we will apply this to the decomposition of the regular representation into irreps.

10. Decomposition of the Regular Representation

sec:sDRR

As before we let $T^{(\mu)}_{m\ell}(g)$ be matrix elements of the distinct irreps of G.

Now recall our important conceptual step forward: We may regard these matrix elements as complex-valued *functions* on G. That is $T^{(\mu)}_{m\ell}$ are *vectors* in R_G.

Moreover, as we showed in the example surrounding (7.9) the linear span of functions

$$\mathcal{R}^{(\mu)}_i \equiv Span\{T^{(\mu)}_{ij}\}_{j=1,\dots,n_\mu} \tag{10.1}$$

is an invariant subspace of R_G, considered as the RRR. Here we are holding μ, i fixed. Indeed, we have:

$$\overbrace{R(g)\cdot T^{(\mu)}_{ij}}^{\text{function on G}} = \sum_{s=1}^{n_\mu} \underbrace{T^{(\mu)}_{sj}(g)}_{\text{matrix element for } R_G} \overbrace{T^{(\mu)}_{is}}^{\text{function on G}} \tag{10.2}$$

We learn two things:

1. $\mathcal{R}^{(\mu)}_i$ is an invariant subspace, as claimed.
2. Moreover, the matrix elements are those of the rep $T^{(\mu)}$! Thus:

$$\mathcal{R}^{(\mu)}_i \cong T^{(\mu)} \tag{10.3}$$

independent of the value of i!

It follows that R_G is *highly reducible*. Indeed it contains invariant subspaces corresponding to each and every irrep of G. In fact, since the above holds for each $i = 1, \dots, n_\mu$ the same irrep occurs n_μ times, where $n_\mu = \text{dim}T^{(\mu)}$.

Let us define

$$
\begin{aligned}
W &:= \oplus_\mu \oplus_{i=1}^{n_\mu} \mathcal{R}_i^{(\mu)} \subset R_G \\
W &\cong \oplus_\mu \left(\overbrace{T^\mu \oplus \cdots \oplus T^\mu}^{n_\mu \ times} \right)
\end{aligned}
\tag{10.4}
$$

Note:

• $W = Span\{T_{ij}^{(\mu)}\}$.

• By the orthogonality relations the $T_{ij}^{(\mu)}$ are linearly independent. Thus we really do have a direct sum .

Exercise

Show that the set of functions $T_{ij}^{(\mu)}$ with μ, j held fixed forms a copy of the irrep $\overline{T^{(\mu)}}$ under the left regular rep. More on this below.

At this point the possibility remains that W is a proper subspace of R_G. We will now show that in fact $W = R_G$ exactly, by showing that $T_{ij}^{(\mu)}$ span R_G.

To do this recall the vector space R_G is an inner product space by the rule

$$
(\phi_1, \phi_2) \equiv \frac{1}{|G|} \sum_{g \in G} \phi_1^*(g) \phi_2(g) \tag{10.5}
$$

eq:innproda

Recall:

1. R_G is a *unitary* representation, wrt this product.

2. The normalized matrix elements ϕ_{ij}^μ are orthonormal wrt this inner product. This follows from (9.13).

By the proposition 7.2.1 we can use the inner product to decompose

$$
R_G \cong W \oplus W^\perp \tag{10.6}
$$

so $W^\perp$ is also a unitary representation. Moreover, again by 7.2.1 it can be reduced to a sum of irreducible representations, $T^{(\mu)}$.

Suppose

$$
Span\{f_i(g)\}_{i=1,\ldots,n_\mu} \subset W^\perp \tag{10.7}
$$

is a subspace transforming in a rep equivalent to T^μ. Then, by definition:

$$
\begin{aligned}
(R(h)f_i)(g) &= \sum_s T^{\mu}_{si}(h) f_s(g) \\
&\Rightarrow \\
f_i(gh) &= \sum_s T^{\mu}_{si}(h) f_s(g) \\
&\Rightarrow \\
f_i(g') &= \sum_s T^{\mu}_{si}(g^{-1}g') f_s(g) \\
&= \sum_{s,k} T^{\mu}_{sk}(g^{-1}) T^{\mu}_{ki}(g') f_s(g)
\end{aligned}
\tag{10.8}
$$

where we have set $g' = gh$.

Therefore, we have the equality of functions on G:

$$
\begin{aligned}
f_i(\cdot) &= \sum_k \alpha_k T^{\mu}_{ki}(\cdot) \\
\alpha_k &= \sum_s \Big(T^{\mu}_{sk}(g^{-1}) f_s(g) \Big) \qquad \text{any} \qquad g \in G
\end{aligned}
\tag{10.9}
$$

That is, f_i is a linear combination of the functions ϕ^{μ}_{ij}. But this contradicts the assumption that

$$
\{f_i\} \subset W^{\perp} \tag{10.10}
$$

so $W^{\perp} = 0$. ♠.

Thus, we have arrived at the decomposition of the RRR into its irreducible pieces:

$$
R_G \cong \oplus_{\mu} \Big(\overbrace{T^{\mu} \oplus \cdots \oplus T^{\mu}}^{n_\mu \ times} \Big) \tag{10.11}
$$

eq:rrrep

But we are not quite done. To arrive at the most beautiful statement we should study R_G as a representation under the the left regular rep. So we consider the linear subspaces where we hold μ, j fixed and consider the span:

$$
\mathcal{L}^{(\mu)}_j \equiv Span\{\phi^{(\mu)}_{ij}\}_{i=1,\dots,n_\mu} \tag{10.12}
$$

This forms a copy of the dual rep:

$$
\begin{aligned}
(L(g)\cdot \phi^{(\mu)}_{ij})(h) &= \sum_s T^{(\mu)}_{is}(g^{-1}) \phi^{(\mu)}_{sj}(h) \\
&= \sum_s \hat{T}^{(\mu)}_{si}(g) \phi^{(\mu)}_{sj}(h)
\end{aligned}
\tag{10.13}
$$

so the span of the matrix elements of the rep μ

$$
Span\{\phi^{(\mu)}_{ij}\}_{i,j=1,\dots,n_\mu} \tag{10.14}
$$

forms the rep $\hat{T}^{(\mu)} \otimes T^{(\mu)}$ of $G_L \times G_R$. Finally, we can identify

$$\hat{T}^{(\mu)} \otimes T^{(\mu)} \cong \text{End}(T^{(\mu)}) \tag{10.15}$$

This is a representation of $G_L \times G_R$ of dimension n_μ^2 with action

$$T(g_L, g_R) \cdot \Phi = T^{(\mu)}(g_L)\Phi T^{(\mu)}(g_R^{-1}) \tag{10.16}$$

We have finally arrived at the beautiful :

Theorem 10.1 Let G be a finite group. The matrix elements ϕ_{ij}^{μ} of the distinct irreps of G decompose R_G into irreps of $G_{\text{Left}} \times G_{\text{Right}}$:

$$R_G \cong \oplus_\mu \hat{T}^{(\mu)} \otimes T^{(\mu)} \cong \oplus_\mu \text{End}(T^{(\mu)}) \tag{10.17}$$

Moreover, if we use matrix elements of unitary irreps in an ON basis then the ϕ_{ij}^{μ} form an ON basis for R_G and

$$R_G \cong \oplus_\mu \overline{T^{(\mu)}} \otimes T^{(\mu)} \tag{10.18}$$

Corollary. Comparing dimensions we have the beautiful relation:

$$|G| = \sum_\mu n_\mu^2 \tag{10.19}$$ eq:grouporder

Remarks:

• This theorem generalizes beautifully to all compact groups and is known as the Peter-Weyl theorem.

• This leaves the separate question of actually *constructing* the representations of the finite group G. Until recently there has been no general algorithm for doing this.

Recently there has been a claim that it can be done. See

Vahid Dabbaghian-Abdoly, "An Algorithm for Constructing Representations of Finite Groups." Journal of Symbolic Computation Volume 39, Issue 6, June 2005, Pages 671-688

Exercise

Show that the number of distinct one-dimensional representations of a finite group G is the same as the index of the commutator subgroup $[G, G]$ in G. [4]

*********************************** *************************************

MATERIAL FROM MATHMETHODS 511 2015:

For now, we just content ourselves with the statement of the theorem for G a finite group:

[4] Hint: A one-dimensional representation is trivial on $[G, G]$ and hence descends to a representation of the abelianization of G, namely the quotient group $G/[G, G]$.

Theorem: Let G be a finite group, and define an Hermitian inner product on $L^2(G) = \mathrm{Map}(G, \mathbb{C})$ by

$$(\Psi_1, \Psi_2) := \frac{1}{|G|} \sum_g \Psi_1^*(g) \Psi_2(g) \tag{10.20}$$

Then let $\{V_\lambda\}$ be a set of representatives of the distinct isomorphism classes of irreducible unitary representations for G. For each representation V_λ choose an ON basis $w_\mu^{(\lambda)}$, $\mu = 1, \dots, n_\lambda := \dim_{\mathbb{C}} V_\lambda$. Then the matrix elements $D_{\mu\nu}^\lambda(g)$ defined by

$$\rho(g) w_\nu^{(\lambda)} = \sum_{\mu=1}^{n_\lambda} D_{\mu\nu}^\lambda(g) w_\mu^{(\lambda)} \tag{10.21}$$

form a complete orthogonal set of functions on $L^2(G)$ so that

$$(D_{\mu_1\nu_1}^{\lambda_1}, D_{\mu_2\nu_2}^{\lambda_2}) = \frac{1}{n_\lambda} \delta^{\lambda_1,\lambda_2} \delta_{\mu_1,\mu_2} \delta_{\nu_1,\nu_2} \tag{10.22}$$ eq:PW-2

Idea of proof: The proof is based on linear algebra and Schur's lemma. The normalization constant on the RHS of (10.22) is easily determined by setting $\lambda_1 = \lambda_2$ and $\nu_1 = \nu_2 = \nu$ and summing on ν, and using the hypothesis that these are matrix elements in a *unitary* representation. The relation to (6.22) is obtained by noting that the linear transformations $T = e_{\nu\mu}$, given by matrix units relative to the basis $w_\mu^{(\lambda)}$ form a basis for $\mathrm{End}(V_\lambda)$. ♠

Example 1: Let $G = \mathbb{Z}_2 = \{1, \sigma\}$ with $\sigma^2 = 1$. Then the general complex valued-function on G is specified by two complex numbers $(\psi_+, \psi_-) \in \mathbb{C}^2$:

$$\Psi(1) = \psi_+ \qquad \Psi(\sigma) = \psi_- \tag{10.23}$$

This identifies $\mathrm{Map}(G, \mathbb{C}) \cong \mathbb{C}^2$ as a vector space. There are just two irreducible representations $V_\pm \cong \mathbb{C}$ with $\rho_\pm(\sigma) = \pm 1$, because for any representation $\rho(\sigma)$ on a vector space V we can form orthogonal projection operators $P_\pm = \frac{1}{2}(1 \pm \rho(\sigma))$ onto direct sums of the irreps. They are obviously unitary with the standard Euclidean norm on $\mathbb{C}$. The matrix elements give two functions on the group $D^\pm$:

$$D^+(1) = 1 \qquad D^+(\sigma) = 1 \tag{10.24}$$

$$D^-(1) = 1 \qquad D^-(\sigma) = -1 \tag{10.25}$$

(Here and in the next examples when working with 1×1 matrices we drop the $\mu\nu$ subscript!) The reader can check they are orthonormal, and they are complete because any function Ψ can be expressed as:

$$\Psi = \frac{\psi_+ + \psi_-}{2} D^+ + \frac{\psi_+ - \psi_-}{2} D^- \tag{10.26}$$

Example 2: We can generalize the previous example slightly by taking $G = \mathbb{Z}/n\mathbb{Z} = \langle \omega | \omega^n = 1 \rangle$. Let us identify this group with the group of n^{th} roots of unity and choose a generator $\omega = \exp[2\pi \mathrm{i}/n]$. Since G is abelian all the representation matrices can be simultaneously diagonalized so all the irreps are one-dimensional. They are:

$V = \mathbb{C}$ and $\rho_m(\omega) = \omega^m$ where m is an integer. Note that $m \sim m + n$ so the set of irreps is again labeled by $\mathbb{Z}/n\mathbb{Z}$ and in fact, under tensor product the set of irreps itself forms a group isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

The matrix elements in the irrep (ρ_m, V) are

$$D^{(m)}(\omega^j) = \omega^{mj} = e^{2\pi \mathrm{i} \frac{mj}{n}} \tag{10.27}$$

Now we can check that indeed

$$\frac{1}{|G|} \sum_{g \in G} (D^{(m_1)}(g))^* D^{(m_2)}(g) = \delta_{m_1 - m_2 = 0 \mathrm{mod} n} \tag{10.28}$$

The decomposition of a function Ψ on the group G is known as the discrete Fourier transform.

Remark: The theorem applies to all compact Lie groups. For example, when $G = U(1) = \{z||z| = 1\}$ then the invariant measure on the group is just $-\mathrm{i}\frac{dz}{z} = \frac{d\theta}{2\pi}$ where $z = e^{\mathrm{i}\theta}$:

$$(\Psi_1, \Psi_2) = \int_0^{2\pi} \frac{d\theta}{2\pi} (\Psi_1(\theta))^* \Psi_2(\theta) \tag{10.29}$$

Now, again since G is abelian the irreducible representations are 1-dimensional and the unitary representations are (ρ_n, V_n) where $n \in \mathbb{Z}$, $V_n \cong \mathbb{C}$ and

$$\rho_n(z) := z^n \tag{10.30}$$

Now, the orthonormality of the matrix elements is the standard orthonormality of $e^{\mathrm{i}n\theta}$ and the Peter-Weyl theorem specializes to Fourier analysis: An L^2-function $\Psi(\theta)$ on the circle can be expanded in terms of the matrix elements of the irreps:

$$\Psi = \sum_{\text{Irreps } \rho_n} \hat{\Psi}_n D^{(n)} \tag{10.31}$$

When applied to $G = SU(2)$ the matrix elements are known as *Wigner functions* or *monopole harmonics*. They are the matrix elements

$$D^j_{m_L, m_R}(g) := \langle m_L | \rho^j(g) | m_R \rangle \tag{10.32}$$

in the standard ON basis of the unitary spin j representation diagonalizing the diagonal subgroup of $SU(2)$. So

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \qquad m_L, m_R \in \{-j, -j+1, \ldots, j-1, j\} \tag{10.33}$$

Recall that $SU(2) \cong S^3$ as a manifold. Using the standard volume form, with unit volume we can define $L^2(SU(2))$. The entire theory of spherical harmonics and Legendre polynomials is easily derived from basic group theory.

♣MAKE GOOD ON THIS CLAIM BELOW. ♣

******************************** ************************************

11. Fourier Analysis as a branch of Representation Theory

11.1 The irreducible representations of abelian groups

Suppose G is abelian. Let T be an irrep of G. Fix $g_0 \in G$. Then

$$T(g_0)T(g) = T(g_0 g) = T(g g_0) = T(g)T(g_0) \tag{11.1}$$

Therefore, by Schur's lemma:

$$T(g_0) = \lambda(g_0)1 \tag{11.2}$$

Therefore, all representation matrices of irreducible representations are proportional to 1 $\Rightarrow$

All finite-dimensional irreducible complex representations of an abelian group are 1-dimensional.

This is just simultaneous diagonalization of commuting operators. Note that the adjective "complex" is essential here. Consider $SO(2)$ acting on $\mathbb{R}^2$.

As we mentioned before noncompact groups can have indecomposable representations, e.g. $\mathbb{R}$ has the representation

$$x \to \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix} \tag{11.3}$$

Note that this is not unitary.

11.2 The character group

Notice that the space of 1 dimensional irreps of any group G is *itself* a group under $\otimes$ product:

$$T_1 \otimes T_2(g) = T_1(g)T_2(g) \tag{11.4}$$

The identity is the trivial representation $T_0(g) = 1$, and inverses exist

$$(T)^{-1}(g) = (T(g))^{-1} \tag{11.5}$$

By the same reasoning the space of 1-dimensional unitary irreps of a group is a group. Note that in this case $(T)^{-1}(g) = T^*(g)$ and $T(g)$ is always represented by a phase.

Definition The space of unitary irreps of a group G is called the *unitary dual* and is denoted by $\hat{G}$.

In the case when G is abelian, the unitary dual $\hat{G}$ is itself a group. It is sometimes called the *Pontryagin dual group.*

Examples

• Finally, let us consider $\mathbb{Z}_n$ thought of as multiplicative n^{th} roots of unity. For any integer s we can define a representation by taking

$$T^{(s)}(\omega) = \omega^s \tag{11.6}$$ eq:srep

where ω is any n^{th} root of unity. These are clearly unitary irreps. Note that

$$T^{(s+n)} = T^{(s)} \tag{11.7}$$

and

$$T^{(s_1)} \otimes T^{(s_2)} \cong T^{(s_1+s_2)} \tag{11.8}$$

so the dual group is another copy of $\mathbb{Z}_n$:

$$\widehat{\mathbb{Z}_n} \cong \mathbb{Z}_n \tag{11.9}$$

Note that the relation (10.19) is easily verified:

$$n = 1^2 + 1^2 + \cdots + 1^2 \tag{11.10}$$

• The theory we are discussing in this section extends to locally compact abelian groups such as $G = \mathbb{R}$. The unitary irreps are specified by a "momentum" $k \in \mathbb{R}$.

$$T^{(k)} : x \to e^{ikx} \tag{11.11}$$

This is a rep because $T^{(k)}(x+y) = T^{(k)}(x)T^{(k)}(y)$. It is unitary for k real.

Now notice that

$$T^{(k)} \otimes T^{(k')} = T^{(k+k')} \tag{11.12}$$

so we conclude:

$$\widehat{\mathbb{R}} \cong \mathbb{R} \tag{11.13}$$

• $G = \mathbb{Z}$. Unitary irreps of $\mathbb{Z}$ are labelled by a real number θ:

$$T^{(\theta)} : n \to e^{in\theta} \tag{11.14}$$

Note that

$$\begin{aligned} T^{(\theta)} &= T^{(\theta+2\pi)} \\ T^{(\theta)} \otimes T^{(\theta')} &= T^{(\theta+\theta')} \end{aligned} \tag{11.15}$$

eq:unirri

So the *space of representations* is parametrized by θ mod 2π, or, equivalently by $e^{i\theta} \in U(1)$. Moreover (11.15) shows that

$$\widehat{\mathbb{Z}} \cong U(1) \tag{11.16}$$

eq:dlgrp

• $G = \mathbb{R}/\mathbb{Z} \cong U(1)$. Now the unitary irreps are labelled by $n \in \mathbb{Z}$:

$$T^{(n)} : e^{i\theta} \to e^{2\pi in\theta} \tag{11.17}$$

eq:unirrii

and we check:

$$T^{(n)} \otimes T^{(n')} = T^{(n+n')} \tag{11.18}$$

eq:unirriii

so

$$\widehat{U(1)} \cong \mathbb{Z} \tag{11.19}$$

eq:dlgrpp

• $G = \Lambda \subset \mathbb{R}^d$, a lattice. Given a vector $\vec{k} \in \mathbb{R}^d$ we can define a unitary irrep by:

$$T^{(\vec{k})} : \vec{n} \to e^{2\pi i \vec{k}\cdot\vec{n}} \tag{11.20}$$

where we use the standard inner product on $\mathbb{R}^d$.

As before $T^{(\vec{k})} \otimes T^{(\vec{k}')} = T^{(\vec{k}+\vec{k}')}$. Note however that

$$T^{(\vec{k}+\vec{b})} = T^{(\vec{k})} \tag{11.21}$$

if $\vec{b}$ is in the dual lattice:

$$\Lambda^* = \{\vec{b} \in \mathbb{R}^d | \forall \vec{n} \in \Lambda, \vec{b}\cdot\vec{n} \in \mathbb{Z}\} \tag{11.22}$$

Thus, the space of unitary irreps of Λ is a compact d-dimensional torus:

$$\widehat{\Lambda} \cong \mathbb{R}^d/\Lambda^* \tag{11.23}$$ eq:torirep

thus generalizing $\widehat{\mathbb{Z}} = U(1)$.

• . $G = \mathbb{R}^d/\Lambda$ is a d-dimensional torus.

Remarks Note that in all the above examples,

$$\widehat{\widehat{G}} = G \tag{11.24}$$

This is a general fact for abelian groups.

11.3 Fourier duality

Let us now return to Theorem 10.1 decomposing R_G in terms of the irreducible representations through the matrix elements of unitary irreps and apply it to finite abelian groups:

• $G = \mathbb{Z}_2 = \{1, \sigma\}$. A function $f : G \to \mathbb{C}$ is specified by its two values

$$f(1) = f_0 \qquad f(\sigma) = f_1 \tag{11.25}$$

The unitary irreps are just:

$$\begin{aligned} T^{(0)}(1) &= 1 \qquad & T^{(0)}(\sigma) &= 1 \\ T^{(1)}(1) &= 1 \qquad & T^{(1)}(\sigma) &= -1 \end{aligned} \tag{11.26}$$

Clearly:

$$f = \frac{f_0 + f_1}{2} T^{(0)} + \frac{f_0 - f_1}{2} T^{(1)} \tag{11.27}$$

Let us generalize this:

•

$G = \mathbb{Z}_n \cong \{1, \omega, \omega^2, \dots, \omega^{n-1}\}$ where $\omega = e^{2\pi i/n}$. We worked out the unitary irreps above.

The orthogonality relations on the irreducible matrix elements (11.6) flow from:

$$\begin{aligned}\frac{1}{n}\sum_{\ell} T^{(j)}(\omega^{\ell})(T^{(k)}(\omega^{\ell}))^* &= \frac{1}{n}\sum_{\ell} e^{2\pi i(j\ell-k\ell)/n} \\ &= \delta_{j,k}\end{aligned} \tag{11.28}$$

where j, k are understood modulo n.

Now, according to the theorem, we can decompose any function on $G = \mathbb{Z}_n$ as:

$$\boxed{f = \sum_{j=0}^{n-1} \hat{f}_j T^{(j)}} \tag{11.29}$$

and we see that

$$\begin{aligned}\hat{f}_j &= \frac{1}{n}\sum_{s} f(\omega^s)(T^{(j)}(\omega^s))^* \\ &= \frac{1}{n}\sum_{s} f(\omega^s)e^{-2\pi isj/n}\end{aligned} \tag{11.30}$$

is the *discrete Fourier transform.*

In fact, our discussion generalizes to all (locally compact[5] abelian groups including infinite groups like $\mathbb{R}, \mathbb{Z}, U(1)$. The main new ingredient we need is *integration over the group*, which is the analog of $\frac{1}{|G|}\sum_g$.

For these groups the integration of a function $f \in Fun(G)$ is the obvious one:

$$\begin{aligned}\int_{G=U(1)} f(g)dg &\equiv \int_0^{+2\pi} \frac{d\theta}{2\pi} f(\theta) \\ \int_{G=\mathbb{Z}} f(g)dg &\equiv \sum_{n\in\mathbb{Z}} f(n) \\ \int_{G=\mathbb{R}} f(g)dg &\equiv \int_{-\infty}^{+\infty} dx f(x)\end{aligned} \tag{11.31}$$

eq:intgroupp

With this in mind the above Theorem 10.1 becomes the following statements:

• Let us begin with $G = (\mathbb{R}, +)$ then $\hat{G} = (\mathbb{R}, +)$. Unitary irreps are labelled by $k \in \mathbb{R}$:

$$T^{(k)}(x) = \frac{1}{\sqrt{2\pi}} e^{2\pi ikx} \tag{11.32}$$

The orthogonality relations are:

$$\int_{\mathbb{R}} dx T^{(k)}(x)(T^{(\ell)}(x))^* = \delta(k-\ell) \tag{11.33}$$

[5] A Hausdorff space is called *locally compact* if every point has a compact neighborhood. All the abelian groups we will meet in this course are locally compact. **** GIVE AN EXAMPLE OF A non-locally-compact abelian group ***

The statement that any L^2 function on the group $\mathbb{R}$ can be expanded in the matrix elements of its unitary irreps is that statement that any L^2 function $f(x)$ can be expanded:

$$\boxed{f(x) = \int_{\widehat{\mathbb{R}}} dk \hat{f}(k) T^{(k)}(x)} \tag{11.34}$$

So we recognize the usual Fourier expansion.

Of course, since $\hat{\widehat{\mathbb{R}}} \cong \mathbb{R}$ we have the inverse relation:

$$\hat{f}(k) = \int_{\mathbb{R}} dx f(x) \widehat{T}^{(x)}(k) \tag{11.35}$$

Of course, there remarks generalize to $\mathbb{R}^d$.

•

We showed that $\mathbb{Z}$ and $U(1)$ are dual groups. Thus, an arbitrary (L^2) function $f(\theta)$ on $G = U(1)$, can be decomposed into a sum of irreps of $U(1)$:

$$f(\theta) = \sum_{n \in \mathbb{Z}} \hat{f}(n) T^{(n)}(\theta) \tag{11.36}$$ eq:foursum

This is the usual Fourier series. and we recognize the Fourier decomposition. The orthogonality relations on matrix elements read:

$$\int_0^{2\pi} \frac{d\theta}{2\pi} (T^{(n)}(\theta))^* T^{(m)}(\theta) = \delta_{n,m} \tag{11.37}$$ eq:orthogrels

Conversely, an arbitrary L^2 function $\hat{f}(n)$ on the abelian group $G = \mathbb{Z}$ can be decomposed into a sum over the irreps $T^{(\theta)}$ of $\mathbb{Z}$, so we have the expansion:

$$\hat{f}(n) = \int_0^{2\pi} \frac{d\theta}{2\pi} f(\theta) T^{(\theta)}(n) \tag{11.38}$$ eq:invfour

The orthogonality relations read:

$$\begin{aligned} \sum_{n \in \mathbb{Z}} (T^{(\theta_1)}(n))^* T^{(\theta_2)}(n) &= \delta_{\text{periodic}}(\theta_1 - \theta_2) \\ &\equiv \sum_{m \in \mathbb{Z}} \delta(\theta_1 - \theta_2 - 2\pi m) \end{aligned} \tag{11.39}$$ eq:orthogrelsa

In summary, we have the

Theorem Let G be a locally compact group. Then there is a natural isomorphism $L^2(G) \cong L^2(\hat{G})$ which expands any function on G in terms of the unitary irreps of G:

$$f(g) := \int_{\hat{G}} d\chi \chi(g)^* \hat{f}(\chi) \tag{11.40}$$ eq:svnthm

Moreover, this isomorphism is an isometry:

$$\int_G dg f_1^* f_2 = \int_{\hat{G}} d\chi \hat{f}_1^* \hat{f}_2 \tag{11.41}$$

Remarks

• The functional analysis of $L^2(G)$ is often called *"nonabelian harmonic analysis."* For more about this point of view see,

A.A. Kirillov, *Elements of the Theory of Representations*, Springer-Verlag, 1976.

11.4 The Poisson summation formula

Let us return to the orthogonality relations for the irreps of Λ and $\mathbb{R}^d/\Lambda^*$.

These imply a beautiful formula known as the Poisson summation formula.

Suppose $f \in \mathcal{S}(\mathbb{R})$ (the Schwarz space of functions of rapid decrese for $\mid x \mid \rightarrow \infty$). Define the Fourier transform:

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(y) e^{-iky} dy \ . \tag{11.42}$$

Then:

Theorem [Poisson summation formula].

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(2\pi n) \tag{11.43}$$

Proof: The orthogonality relations tell us that

$$\boxed{\sum_{n\epsilon\mathbb{Z}} e^{2\pi i n t} = \sum_{k\epsilon\mathbb{Z}} \delta(k-t)} \tag{11.44}$$

eq:basicpsf

Now integrate both sides against $f(t)$. ♠

Remarks

• If it possible to give other proofs of this formula. We will content ourselves with its being a consequence of the orthogonality relations.

• The rapid decrease condition can be relaxed to

$$\mid f(x) \mid \ + \ \mid f'(x) \mid \ + \ \mid f''(x) \mid \ \leq \frac{const}{1+x^2} \tag{11.45}$$

Remark The above result has a beautiful generalization to arbitrary lattices in Euclidean space $\mathbb{R}^d$. The generalized Poisson summation formula states that:

$$\boxed{\sum_{\vec{v}\in\Lambda} e^{2\pi i \vec{v}\cdot\vec{x}} = \sum_{\vec{l}\in\Lambda^*} \delta^{(d)}(\vec{x}-\vec{l})} \tag{11.46}$$

or - in terms or functions

$$\sum_{\vec{n}\in Z^d} f(\vec{n}) = \sum_{\vec{m}\in Z^d}\int_{R^d} e^{-2\pi i\vec{m}\cdot\vec{t}} f(t)dt = \sum_{\vec{m}\in Z^d}\int_{R^d} e^{+2\pi i\vec{m}\cdot\vec{t}} f(t)dt \qquad (11.47)$$ eq:poissonsumm

That is:

$$\sum_{\vec{v}\in\Lambda} f(\vec{v}) = \sum_{\vec{l}\in\Lambda^*} \hat{f}(\vec{l}) \ . \qquad (11.48)$$

Remarks

• Since people have different conventions for the factors of 2π in Fourier transforms it is hard to remember the factors of 2π in the PSF. The equation (11.44) has no factors of 2π. One easy way to see this is to integrate both sides from $t = -1/2$ to $t = +1/2$.

• One application of this is the x-ray crystallography: The LHS is the sum of scattered waves. The RHS constitutes the bright peaks measured on a photographic plate.

• Another application is in the theory of elliptic functions and θ-functions described later.

Exercise
a.) Show that

$$\sum_{n\in Z} e^{-\pi a n^2 + 2\pi i b n} = \sqrt{\frac{1}{a}} \sum_{m\in Z} e^{-\frac{\pi(m-b)^2}{a}} \qquad (11.49)$$ eq:formtwo

b.) If τ is in the upper half complex plane, and θ, ϕ, z are complex numbers define the *theta function*

$$\vartheta_{\theta,\phi} z = \sum_{n\in\mathbb{Z}} e^{i\pi\tau(n+\theta)^2 + 2\pi i(n+\theta)(z+\phi)} \qquad (11.50)$$ eq:thefun

(usually, θ, ϕ are taken to be real numbers, but z is complex).
Using the Poisson summation formula show that

$$\vartheta[{\theta \atop \phi}](\frac{-z}{\tau}|\frac{-1}{\tau}) = (-i\tau)^{1/2} e^{2\pi i\theta\phi} e^{i\pi z^2/\tau} \vartheta[{-\phi \atop \theta}](z|\tau) \qquad (11.51)$$ eq:esstmn

---

11.5 Application: Bloch's Theorem in Solid State Physics

Consider an electron moving in a crystalline material. Atoms or ions live on a lattice

$$\Lambda \subset \mathbb{R}^3 \qquad (11.52)$$

and an electron (in the single-electron approximation) satisfies the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(\vec{x})\psi = E\psi. \qquad (11.53)$$

Now, $V(\vec{x})$ will be complicated in general, but we do know that

$$V(\vec{x}+\vec{a}) = V(\vec{x}) \tag{11.54}$$

$\forall \vec{a} \in \Lambda$. Therefore the energy eigenspace $\mathcal{V}_E$ is some representation of the lattice translation group Λ.

As we have seen, the *one dimensional* reps of Λ are

$$\vec{a} \to e^{2\pi i \vec{k}\cdot\vec{a}} \tag{11.55}$$

and are labeled by $\vec{k} \in \hat{\Lambda} = \mathbb{R}^n/\Lambda^*$. Therefore, $\mathcal{V}_E$ can be written as a sum of reps labeled by $\vec{k}$:

$$\mathcal{V}_E = \oplus_{\vec{k}\in\hat{\Lambda}} \mathcal{V}_{E,\vec{k}} \tag{11.56}$$

eq:decompvx

where eigenfunctions in $\mathcal{V}_{E,\vec{k}}$ satisfy:

$$\psi(\vec{x}+\vec{a}) = e^{2\pi i \vec{k}\cdot\vec{a}}\psi(\vec{x}) \tag{11.57}$$

that is, $\psi(\vec{x}) = e^{2\pi i \vec{k}\cdot\vec{x}} u(\vec{x})$ where u is periodic. $\vec{k}$, the "crystal momentum," is only defined modulo Λ^*. It is properly an element of a torus.

Since we are working with infinite groups the sum on $\vec{k}$ in (11.56) in general must be interpreted as an integral.

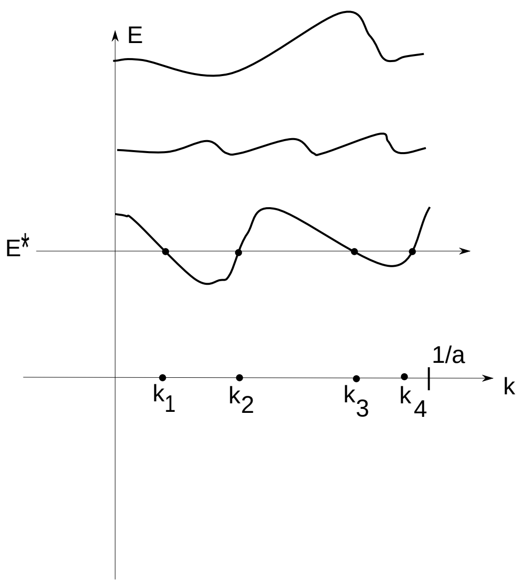

Figure 3: Four irreps occur at energy E_* in this one-dimensional band structure. For $d = 2, 3$ dimensional bandstructures there are infinity many k's forming a surface. The surface at the Fermi energy is the "Fermi surface."

fig:bandstruct

Remarks

• If we substitute $\psi(\vec{x}) = e^{2\pi i \vec{k}\cdot\vec{x}} u(\vec{x})$ into the Schrödinger equation we obtain an elliptic eigenvalue problem on a compact space. It will therefore have a tower of discrete eigenvalues. This leads to the band structure illustrated schematically in 3.

• In solid state physics the dual lattice is referred to as the *reciprocal lattice*. Physicists generally choose a fundamental domain for the torus $\mathbb{R}^d/\Lambda^*$ and refer to it as a *Brillouin zone.*

• If the lattice Λ has a symmetry group then it will act on the torus. At fixed points there is enhanced symmetry and bands will cross.

11.6 The Heisenberg group extension of $\hat{S} \times S$ for an abelian group S

******

Change notation $S \to G$ below. Also need to explain that $G \times \hat{G} \to U(1)$ is a *perfect pairing.*

******

Let S be a locally compact abelian group with a measure. Let $\hat{S}$ be the dual group. There is a very natural Heisenberg extension of $S \times \hat{S}$ defined by the cocycle to choose the cocycle

$$c\big((s_1,\chi_1),(s_2,\chi_2)\big) = \frac{1}{\chi_1(s_2)} \qquad (11.58)$$

eq:cocyclechoi

whose antisymmetrization is

$$s\big((s_1,\chi_1),(s_2,\chi_2)\big) = \frac{\chi_2(s_1)}{\chi_1(s_2)}. \qquad (11.59)$$

eq:cocyclechoi

This gives

$$1 \to U(1) \to \mathrm{Heis}(S \times \hat{S}) \to S \times \hat{S} \to 1 \qquad (11.60)$$

eq:heisext

There are two very natural representations of this group. First we consider $\mathcal{H} = L^2(S)$.

First of all $L^2(S)$ is a representation of S. After all for $s_0 \in S$ we can define the translation operator:

$$(T_{s_0}\psi)(s) := \psi(s+s_0). \qquad (11.61)$$

eq:transop

On the other hand, $\mathcal{H}$ is also a representation of the Pontrjagin dual group of characters, denoted $\hat{S}$. If $\chi \in \hat{S}$ is a character on S then we define the multiplication operator M_χ on $\mathcal{H}$ via

$$(M_\chi\psi)(s) := \chi(s)\psi(s). \qquad (11.62)$$

eq:multop

Note that $\mathcal{H}$ is *not* a representation of $S\times\hat{S}$. This simply follows from the easily-verified relation

$$T_{s_0} M_\chi = \chi(s_0) M_\chi T_{s_0}. \qquad (11.63)$$

eq:noncomm

However, $\mathcal{H}$ *is* a representation of $\mathrm{Heis}(S \times \hat{S})$. It is defined so that

$$(z, (s, \chi)) \to z T_s M_\chi \tag{11.64}$$ eq:heishom

where $z \in U(1)$. Equation (11.64) defines a homomorphism into the group of invertible operators on $L^2(S)$.

Notice the complete symmetry in the construction between S and $\hat{S}$ (since the double-dual gives S again). Thus, we could also provide a representation from $\hat{\mathcal{H}} = L^2(\hat{S})$. The two representations are in fact equivalent under Fourier transform:

$$\hat{\psi}(\chi) := \int ds \chi(s)^* \psi(s) \tag{11.65}$$ eq:svnthmi

In fact, a theorem, the Stone-von Neumann theorem guarantees the essential uniqueness of the unitary representations of Heisenberg groups:

Theorem If $\tilde{A}$ is a central extension of a locally compact topological abelian group A by $U(1)$, then the unitary irreps of $\tilde{A}$ where $U(1)$ acts by scalar multiplication are in $1-1$ correspondence with the unitary irreps of the center of $\tilde{A}$, where $U(1)$ acts by scalar multiplication.

Example 1 Consider $L^2(\mathbb{Z}_n)$. Regarding $\mathbb{Z}_n$ as the integers modulo n we have

$$(T(\bar{s}_0) \cdot f)(\bar{s}) = f(\bar{s} + \bar{s}_0) \tag{11.66}$$

while the dual group $\chi_{\bar{t}}$ has character $\chi_{\bar{t}}(\bar{s}) = e^{2\pi i st/n}$ acts by multiplication.

Example 2 $S = \mathbb{R}$ and $\hat{S} = \mathbb{R}$. Denote elements $q \in \mathbb{R}$ and $p \in \widehat{\mathbb{R}} \cong \mathbb{R}$.

$$(T(p) \cdot \psi)(q) = e^{2\pi i p q \theta} \psi(q) \tag{11.67}$$

$$(T(q_0) \cdot \psi)(q) = \psi(q + q_0) \tag{11.68}$$

Then

$$(T(q_0)(T(p) \cdot \psi))(q) = e^{2\pi i \theta p (q+q_0)} \psi(q + q_0) \tag{11.69}$$

$$(T(p)(T(q_0) \cdot \psi))(q) = e^{2\pi i \theta p q} \psi(q + q_0) \tag{11.70}$$

So $T(q_0)T(p) = e^{2\pi i p q_0} T(p) T(q_0)$. We recognize standard relations from quantum mechanics with θ playing the role of Planck's constant.

Example 3 $S = \mathbb{Z}$ and $\hat{S} = U(1)$ ... This generalizes to the duality between lattices and tori.

Remarks

• Most free field theories can be interpreted in this framework, with proper regard for the infinite-dimensional aspects of the problem. There is the interesting exception of self-dual field theories which are more subtle.

12. Induced Representations

♣MATERIAL HERE IMPORTED FROM GMP 2010 ASSUMES KNOWLEDGE OF BUNDLES. NEED TO EDIT THOSE PARTS AWAY. ♣

Induced representations are certain representations defined by taking sections of associated bundles. They form a unifying theme for a broad class of representations in physics:

1. Finite groups
2. Compact groups: The Borel-Weil-Bott Theorem
3. Lorentz groups: The Wigner construction.
4. Representations of Heisenberg groups and coherent states
5. Representations of Loop Groups

Suppose $H \subset G$ is a subgroup.

Suppose $\rho : H \to GL(V)$ is a representation of H with representation space V.

Note that

$$\begin{array}{ccc} H \to & G & \\ & \downarrow \pi & \\ & G/H & \end{array} \tag{12.1}$$

is a principal H bundle with total space $P = G$.

Therefore, according to what we have said above, using (ρ, V) we can form the associated vector bundle

$$G \times_H V \tag{12.2}$$

Now, the key point is that $\Gamma(G \times_H V)$ *is also a representation of* G. This representation is called the *induced representation* and sometimes denoted:

$$\Gamma(G \times_H V) = \mathrm{Ind}_H^G(V) \tag{12.3}$$

Using the grand tautology (**??**) we can equivalently define the induced representation as:

Definition Let (V, ρ) be a representation of H. As a vector space, $\mathrm{Ind}_H^G(V)$ is the set of all smooth functions $\psi : G \to V$ satisfying the equivariance condition:

$$\psi(gh) = \rho(h^{-1}) \cdot \psi(g) \tag{12.4}$$

for all $h \in H, g \in G$.

Note that, because G is a group we can multiply both on the left, and on the right. This allows us to define a representation of G in a way that does not spoil the equivariance condition:

Let us check that this is indeed a representation. If $\psi \in \mathrm{Ind}_H^G(\rho)$, and $g \in G$ we define $T(g) \cdot \psi$ as the new function whose values are given in terms of the old one by

$$(T(g) \cdot \psi)(g') := \psi(g^{-1} g') \tag{12.5}$$

eq:indrepi

The inverse is needed so that we actually have a representation:

$$\begin{aligned} (T(g_1)T(g_2) \cdot \psi)(g') &= (T(g_2) \cdot \psi)(g_1^{-1} g') \\ &= \psi(g_2^{-1} g_1^{-1} g') \\ &= (T(g_1 g_2) \cdot \psi)(g') \end{aligned} \tag{12.6}$$

However, the action (12.5) manifestly does not interfere with the equivariance property of ψ! In detail: if $h \in H$ then

$$(T(g)\cdot\psi)(g'h) = \psi(g^{-1}g'h) = \rho(h^{-1})\psi(g^{-1}g') = \rho(h^{-1})((T(g)\cdot\psi)(g')) \tag{12.7}$$ eq:explicit

So, $T(g)\cdot\psi$ is also equivariant.

Remarks: The associated vector bundle $(G\times_H V)\to G/H$ is an example of an *equivariant vector bundle*. In general if $E\to M$ is a vector bundle over a space M with a left G-action then E is an *equivariant vector bundle*, or *G-vector bundle* if there is an action of G on the total space of E, linear on the fibers, which covers the G action on M. Very often in physics (and mathematics) we are in a situation where there is a given G action on a space M and we must lift the action to make a vector bundle equivariant. This comes up, for example, in formulating Gauss laws.

12.1 Induced Representations of Finite Groups: Frobenius Reciprocity

The theory of induced representations is already interesting and nontrivial for G, H finite groups. In this case $G\to G/H$ is a finite cover (by H) of a discrete set of points. Nevertheless, the general geometrical ideas apply.

Let $\mathrm{Rep}(G)$ denote the category of finite-dimensional representations of G. Morphisms between $W_1, W_2\in\mathrm{Rep}(G)$ are linear transformations commuting with G, i.e. G-intertwiners, and the vector space of all morphisms is denoted $\mathrm{Hom}_G(W_1,W_2)$. The induced representation construction defines a functor

$$\mathrm{Ind}:\mathrm{Rep}(H)\to\mathrm{Rep}(G). \tag{12.8}$$

(We denoted this by Ind_H^G before but H, G will be fixed in what follows so we simplify the notation.) On the other hand, there is an obvious functor going the other way, since any G-rep W is *a foriori* an H-rep, by restriction. Let us denote this "restriction functor"

$$R:\mathrm{Rep}(G)\to\mathrm{Rep}(H) \tag{12.9}$$

How are these two maps related? The answer is that they are "adjoints" of each other! This is the statement of Frobenius reciprocity:

$$\mathrm{Hom}_G(W,\mathrm{Ind}(V)) = \mathrm{Hom}_H(R(W),V) \tag{12.10}$$ eq:frobrecip

We can restate the result in another way which is illuminating because it helps to answer the question: How is $\mathrm{Ind}_H^G(V)$ decomposed in terms of irreducible representations of G? Let W_α denote the distinct irreps of G. Then Schur's lemma tells us that

$$\mathrm{Ind}_H^G(V)\cong\oplus_\alpha W_\alpha\otimes\mathrm{Hom}_G(W_\alpha,\mathrm{Ind}_H^G(V)) \tag{12.11}$$ eq:schurlemma

But now Frobenius reciprocity (12.10) allows us to rewrite this as

$$\mathrm{Ind}_H^G(V) \cong \oplus_\alpha W_\alpha \otimes \mathrm{Hom}_H(R(W_\alpha), V) \tag{12.12}$$

eq:frobrecipii

where the sum runs over the unitary irreps W_α of G, with multiplicity one.

The statement (12.12) can be a very useful simplification of (12.11) if H is "much smaller" than G. For example, G could be nonabelian, while H is abelian. But the representation theory for abelian groups is much easier! Similarly, G could be noncompact, while H is compact. etc.

Proof of Frobenius reciprocity:

In order to prove (12.12) we note that it is equivalent (see the exercise below) to the statement that the character of $\mathrm{Ind}_H^G(V)$ is given by

$$\chi(g) = \sum_{x \in G/H} \hat{\chi}(x^{-1} g x) \tag{12.13}$$

eq:charind

where x runs over a set of representatives and $\hat{\chi}$ is the character χ_V for H when the argument is in H and zero otherwise.

On the other hand, (12.13) can be understood in a very geometrical way. Think of the homogeneous vector bundle $G \times_H V$ as a collection of points $g_j H$, $j = 1, \dots, n$ with a copy of V sitting over each point. Now, choose a representative $g_j \in G$ for each coset. Having chosen representatives g_j for the distinct cosets, we may write:

$$g \cdot g_j = g_{g\cdot j} h(g, j) \tag{12.14}$$

eq:permcos

where $j \mapsto g \cdot j$ is just a permutation of the integers $1, \dots, n$, or more invariantly, a permutation of the points in G/H.

Now let us define a basis for the induced representation by introducing a basis v_a for the H-rep V and the equivariant functions determined by:

$$\psi_{i,a}(g_j) := v_a \delta_{i,j} \tag{12.15}$$

Geometrically, this is a section whose support is located at the point $g_i H$. The equivariant function is then given by

$$\psi_{i,a}(g_j h) := \rho(h^{-1}) v_a \delta_{i,j} \tag{12.16}$$

Now let us compute the action of $g \in G$ in this basis:

$$\begin{aligned} (g \cdot \psi_{i,a})(g_j) &= \psi_{i,a}(g^{-1} g_j) \\ &= \psi_{i,a}(g_{g^{-1}\cdot j} h(g^{-1}, j)) \\ &= \delta_{i, g^{-1}\cdot j} \rho(h(g^{-1}, j)^{-1}) \cdot v_a \end{aligned} \tag{12.17}$$

Fortunately, we are only interested in the trace of this G-action. The first key point is that *only the fixed points of the g-action on G/H contribute.* Note that the RHS above is supported at $j = g \cdot i$, but if we are taking the trace we must have $i = j$. But in

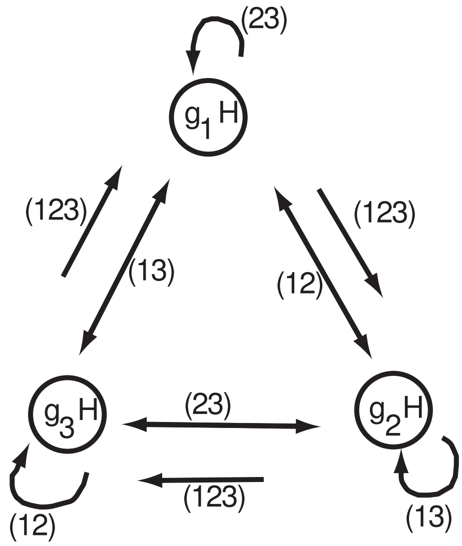

Figure 4: The left action of $G = S_3$ on G/H. In fact, this picture should be considered as a picture of a category, in this case, a groupoid. fig:indrepi

this case $gg_i = g_i h(g,i)$ and hence $g^{-1} g_i = g_i h(g,i)^{-1}$ so for fixed points we can simplify $h(g^{-1}, i) = h(g,i)^{-1}$, and hence when we take the trace the contribution of a fixed point $gg_iH = g_iH$ is the trace in the H-rep of $h(g,i) = g_i^{-1} g g_i$, as was to be shown ♠

Remark: The *Ind* map does not extend to a ring homomorphism of representation rings.

Example A simple example from finite group theory nicely illustrates the general idea. Let $G = S_3$ be the permutation group. Let $H = \{1, (12)\} \cong \mathbb{Z}_2$ be a $\mathbb{Z}_2$ subgroup. G/H consists of 3 points. The left action of G on this space is illustrated in (12.5).

There are two irreducible representations of H, the trivial and the sign representation. These are both 1-dimensional. Call them $V(\epsilon)$, with $\epsilon = \pm$. Accordingly, we are looking at a line bundle over G/H and the vector space of sections of $G \times_H V(\epsilon)$ is 3-dimensional. A natural basis for the space of sections is given by the functions which are "δ-functions supported at each of the three points":

$$\begin{aligned} s^i(g_jH) &= \delta_{ij} \\ g_1H &= (13)H = \{(13), (123)\} \\ g_2H &= (23)H = \{(23), (132)\} \\ g_3H &= (12)H = \{1, (12)\} \end{aligned} \tag{12.18}$$ eq:basis

These sections correspond to equivariant functions on the total space. The space of all functions $F : G \to \mathbb{R}$ is a six-dimensional vector space. The equivariance condition:

$$
\begin{aligned}
F(12) &= \epsilon F(1) \\
F(123) &= \epsilon F(13) \\
F(132) &= \epsilon F(23)
\end{aligned}
\tag{12.19}
$$

eq:equicond

cuts this six-dimensional space down to a three-dimensional space.

We can choose a basis of equivariant functions by *choosing* 3 representatives g_1, g_2, g_3 for the cosets in G/H and setting $F^i(g_j) = \delta_{ij}$. Using such a basis the representation of the group is easily expressed as a permutation representation.

In our example of $G = S_3$ it is prudent to choose $g_1 = (13), g_2 = (23), g_3 = (12)$ so that

$$
\begin{aligned}
(12)g_1 &= g_2(12) \\
(12)g_2 &= g_1(12) \\
(12)g_3 &= g_3(12) \\
(13)g_1 &= g_3(12) \\
(13)g_2 &= g_2(12) \\
(13)g_3 &= g_1(12) \\
(23)g_1 &= g_1(12) \\
(23)g_2 &= g_3(12) \\
(23)g_3 &= g_2(12)
\end{aligned}
\tag{12.20}
$$

eq:multout

From this one easily gets the induced representation

$$
\rho_{ind}(12) = \begin{pmatrix} 0 & \epsilon & 0 \\ \epsilon & 0 & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \tag{12.21}
$$

$$
\rho_{ind}(132) = \begin{pmatrix} 0 & 0 & \epsilon \\ \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \end{pmatrix} \tag{12.22}
$$

and so forth.

Now let us look at Frobenius reciprocity. The irreducible representations of G are $W(\epsilon)$ defined by $(ij) \to \epsilon$, and W_2 defined by the symmetries of the equilateral triangle, embedded into $O(2)$:

$$
\begin{aligned}
\rho_2(12) &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\
\rho_2(123) &= \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}
\end{aligned}
\tag{12.23}
$$

eq:rhotwo

As $H = \mathbb{Z}_2$ representations, we have $W(\epsilon) \cong V(\epsilon)$ and

$$W_2 \cong V(+1) \oplus V(-1) \tag{12.24}$$

Therefore

$$\begin{aligned} \mathrm{Hom}_H(W(\epsilon), V(\epsilon')) &= \delta_{\epsilon,\epsilon'}\mathbb{R} \\ \mathrm{Hom}_H(W_2, V(\epsilon)) &= \mathbb{R} \end{aligned} \tag{12.25}$$

By Frobenius reciprocity we have

$$\mathrm{Ind}_{S_2}^{S_3}(V(\epsilon)) = W(\epsilon) \oplus W_2 \tag{12.26}$$ eq:frobexple

Let us check this by computing characters from the geometric perspective.

The induced representation consists of functions on G/H valued in $V(\epsilon)$. The action of g acts as a permutation representation on the support of the functions.

Therefore, we can compute the character of g in the induced representation by looking at its action on fixed points:

$$\chi_{\rho_{ind}}(g) = \sum_{Fix(g)} \chi_{V(\epsilon)}(h(g,i)) \tag{12.27}$$

To find out how to decompose the representation $\mathrm{Ind}_{S_2}^{S_3}(V(\epsilon))$ in terms of $G = S_3$ irreps it suffices to compute the character for $g = (12)$ and $g = (123)$. Now, $g = (12)$ has exactly one fixed point, namely g_3H and $h(g,3) = (12)$ for this element. Therefore,

$$\chi_{\rho_{ind}}(12) = \chi_V(12) = \epsilon \tag{12.28}$$

On the other hand, $g = (123)$ clearly has no fixed points, and therefore the character is zero. It follows immediately that we have the decomposition (12.26).

****

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INCORPORATE:

Now let us turn to induced representations:

Let G be a group and H a subgroup. Suppose that $\rho : H \to \mathrm{End}(V)$ is a representation of the *subgroup* H. Then, as we have seen $\mathrm{Map}(G,V)$ is canonically a $G \times H$-space. To keep the notation under control we denote a general function in $\mathrm{Map}(G,V)$ by Ψ. Then the left-action of $G \times H$ defined by declaring that for $(g,h) \in G \times H$ and $\Psi \in \mathrm{Map}(G,V)$ the new function $\phi((g,h),\Psi) \in \mathrm{Map}(G,V)$ is the function $G \to V$ defined by:

$$\phi((g,h),\Psi)(g_0) := \rho(h) \cdot \Psi(g^{-1}g_0h) \tag{12.29}$$

for all $g_0 \in G$. Now, we can consider the subspace of functions *fixed by the action of* $1 \times H$. That is, we consider the H-equivariant functions which satisfy

$$\boxed{\Psi(gh^{-1}) = \rho(h)\Psi(g)} \tag{12.30}$$ eq:EquivFun

for every $g \in G$ and $h \in H$. Put differently: There are two natural left-actions on $\mathrm{Map}(G, V)$ and we consider the subspace where they are equal. Note that the space of such functions is a linear subspace of $\mathrm{Map}(G, V)$. We will denote it by $\mathrm{Ind}_H^G(V)$. Moreover, it is still a representation of G since if Ψ is equivariant so is $(g, 1) \cdot \Psi$.

The subspace $\mathrm{Ind}_H^G(V) \subset \mathrm{Map}(G, V)$ of H-equivariant functions, i.e. functions satisfying (12.30) is called the *induced representation of G, induced by the representation V of the subgroup H*. This is an important construction with a beautiful underlying geometrical interpretation. In physics it yields:

1. The irreducible unitary representations of space groups in condensed matter physics.

2. The irreducible unitary representations of the Poincaré group in QFT.

Example: Let us take $V = \mathbb{C}$ with the trivial representation of H, i.e. $\rho(h) = 1$. Then the induced representation is the vector space of functions on G which are invariant under right-multiplication by H. This is precisely the vector space of $\mathbb{C}$-valued functions on the homogeneous space G/H. For example, the invariant Wigner functions $D^j_{m_L,m_R}$ under right-action by the diagonal $U(1)$ subgroup of $SU(2)$ are $D^j_{m_L,0}(g)$. These descend to functions on $SU(2)/U(1) \cong S^2$ known (for j integral) as the *spherical harmonics*. The case of V a trivial representation generalizes in a beautiful way: When (ρ, V) is nontrivial the induced representation is interpreted not as a space of functions on G/H but rather as a vector space of sections of a homogeneous vector bundle over G/H determined by the data (ρ, V). See **** below.

Exercise
Prove (6.20).

Exercise
Let G be the symmetric group on $\{1, 2, 3\}$ and let $H = \{1, (12)\}$. Choose a representation of H with $V \cong \mathbb{C}$ and $\rho(\sigma) = +1$ or $\rho(\sigma) = -1$. ♣Physics 619, ch. 5, 2002 for more ♣

a.) Show that in either case, the induced representation $\mathrm{Ind}_H^G(V)$ is a three-dimensional vector space.

b.) Choose a basis for $\mathrm{Ind}_H^G(V)$ and compute the representation matrices of the elements of S_3 explicitly.

Exercise *Relation to the Peter-Weyl theorem*

Take H to be the trivial group and V the trivial representation and explain the relation of (12.12) to the Peter-Weyl theorem.

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13. Representations Of $SU(2)$

♣TAKEN FROM GMP 2010 BUT ALL MATERIAL ASSUMING INDUCED REPRESENTATIONS, BUNDLE THEORY, AND BOREL-WEIL-BOTT IS SUPPRESSED ♣

Let us now consider the space of $\mathbb{C}^\infty$ sections $\Gamma(\mathcal{L}_k)$ as a representation of $SU(2)$. The right-action by $U(1)$ is given by

$$g \cdot e^{i\chi} := g e^{-i\chi\sigma^3} \tag{13.1}$$

where on the RHS we have ordinary matrix multiplication. According to our general principle, the sections of V_k are the same as equivariant functions $\psi : SU(2) \to \mathbb{C}$ such that

$$\psi(g e^{-i\chi\sigma^3}) = e^{-ik\chi}\psi(g) \tag{13.2}$$

eq:eqvfunct

In order to analyze (13.2) we must recall the Peter-Weyl theorem for $SU(2)$:

$$L^2(SU(2)) \cong \oplus_{j\in\frac{1}{2}\mathbb{Z}_+} \overline{\mathcal{D}^j} \otimes \mathcal{D}^j \tag{13.3}$$

as a representation of the $G_{\rm left} \times G_{\rm right}$. An orthonormal basis of functions is given by the matrix elements of $SU(2)$ in the unitary irreducible representations $\mathcal{D}^j$. Recall these are labeled by $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$ of dimension $n = 2j+1 = 1, 2, 3, 4, 5, \ldots$.

Using our parametrization of $SU(2)$ in terms of Euler angles from Chapter 5 we have:

$$\begin{aligned}\langle j, m_L \mid T^{(j)}(g) \mid j, m_R\rangle &\equiv D^j_{m_L m_R}(\phi,\theta,\psi) \\ &= \langle j, m_L | T^{(j)}(e^{-i\frac{1}{2}\phi\sigma^3}) T^{(j)}(e^{-i\frac{1}{2}\theta\sigma^2}) T^{(j)}(e^{-i\frac{1}{2}\psi\sigma^3}) | j, m_R\rangle \\ &= e^{-im_L\phi} P^j_{m_L,m_R}(\cos\theta) e^{im_R\psi}\end{aligned} \tag{13.4}$$

eq:sutmtrx

Here $P^j_{m_L,m_R}(x)$ is an associated Legendre polynomial and we have chosen the slightly unconventional representation of *antihermitian* generators:

$$J^k = -\frac{i}{2}\sigma^k \qquad\qquad [J^i, J^j] = \epsilon^{ijk} J^k \tag{13.5}$$

and diagonalized $J^3|j,m\rangle = im|j,m\rangle$.

Thus subspace of $L^2(SU(2))$ satisfying the equivariance condition (13.2) is that spanned by the D-functions with $k = -2m_R$.

Thus, an ON basis of functions satisfying the equivariance condition is $D^j_{m_L,-k/2}$, as j, m_L range over all possible values. Clearly, we must have $j \geq |k|/2$, and for fixed j the span of functions $m_L = -j, -j+1, \ldots, +j$ form a representation of $SU(2)$ and hence: From this description it is clear that the induced representation is infinite dimensional, and is given by a sum of representations of $SU(2)$,

$$\mathrm{Ind}_{U(1)}^{SU(2)}(V_k) \cong \oplus_{j'\geq \frac{1}{2}|k|}\overline{\mathcal{D}^{j'}} \tag{13.6}$$ eq:indrepsut

where the sum is over $j' = \frac{1}{2}|k|, \frac{1}{2}|k|+1, \frac{1}{2}|k|+2, \dots$.

Remarks

• To explain the complex conjugate in (13.6) recall that under the left regular representation we have

$$\begin{aligned}(L(g_0)\cdot D^j_{m_L,m_R})(g) &:= D^j_{m_L,m_R})(g_0^{-1}g) \\ &= \langle j, m_L | T^{(j)}(g_0^{-1}) T^{(j)}(g) | j, m_R\rangle \\ &= \sum_{m'} \left(D^j_{m'm_L}(g_0)\right)^* D^j_{m'm_R}(g)\end{aligned} \tag{13.7}$$

• Let us check the Frobenius reciprocity statement:

$$\mathrm{Hom}_{SU(2)}(\mathcal{D}^{j'}, \mathrm{Ind}(V_k)) = \mathrm{Hom}_{U(1)}(R^*\mathcal{D}^{j'}, V_k) \tag{13.8}$$

The RHS is one-dimensional exactly for those values of j' given by $2j' - |k| \in 2\mathbb{Z}_+$.

• In the special case $k=0$ we are talking about sections of the trivial line bundle. But these are just complex valued L^2 functions on S^2. Indeed, in this case $m_R = 0$ and the functions become independent of ψ and hence well-defined functions on S^2 rather than sections of nontrivial line bundles. In this case the functions are known as spherical harmonics and usually denoted:

$$D^\ell_{m,0} = Y_{\ell,m}(\theta,\phi) \tag{13.9}$$

• If we were studying the quantum mechanics of a charged particle of electric charge $+1$ moving on a sphere surrounding a magnetic monopole of magnetic charge k then the Hilbert space of the particle would be $\Gamma(\mathcal{L}_k) = \mathrm{Ind}_{U(1)}^{SU(2)}(V_k)$. Thus, we will sometimes refer to vectors in this representation space as "states" or "wavefunctions."

Now for $k \geq 0$ let

$$\mathcal{H}_k := \{\psi : \psi(\lambda u, \lambda v) = \lambda^k \psi(u,v)\} \subset \mathbb{C}[u,v] \tag{13.10}$$

be the vector space of homogeneous polynomials in two variables u, v of degree k.

Now, there is a natural basis of functions in $\mathcal{H}_k$, given by by the monomials $\psi_\ell(u,v) = u^\ell v^{k-\ell}$, $\ell = 0, 1, \dots, \ell$. The matrix elements in this basis can be computed from

$$\begin{aligned}(g\cdot\psi_\ell)(u,v) &= \psi_\ell(g^{-1}\begin{pmatrix} u \\ v\end{pmatrix}) \\ &= \psi_\ell((\mathrm{det}g)^{-1}(g_{22}u - g_{12}v, -g_{21}u + g_{11}v)) \\ &= (\mathrm{det}g)^{-k}(g_{22}u - g_{12}v)^\ell(-g_{21}u + g_{11}v)^{k-\ell}\end{aligned} \tag{13.11}$$ eq:mtrxelem

By expanding out using the binomial theorem and collecting terms one obtains explicit matrix elements. We will do this momentarily.

What we have learned so far is that the holomorphically induced representation from B to $SL(2,\mathbb{C})$ for V_k is the $k+1$-dimensional representation which can canonically be viewed as the space of homogeneous polynomials of degree k. [*** Redundant??? ***]

Now, since $SU(2) \subset GL(2,\mathbb{C})$ we can restrict the functions in $\mathrm{HolInd}_B^{GL(2,\mathbb{C})}(V_k)$ to $SU(2)$. Since $SU(2) \cap B = T \cong U(1)$ they will be functions in the induced representation $\mathrm{Ind}_{U(1)}^{SU(2)}(V_k)$. The monomial basis functions become

$$\psi_\ell(g) = \alpha^\ell \beta^{k-\ell} \qquad \ell = 0, 1, \dots, k \tag{13.12}$$ eq:monbasis

Now, let us try to identify this with one of the Wigner functions in $\mathrm{Ind}_{U(1)}^{SU(2)}(V_k)$.

In order to identify (13.12) with Wigner functions let us restrict the $GL(2,\mathbb{C})$ representation $\mathcal{H}_k$ for $k \geq 0$ to $SU(2)$. The representation remains irreducible. If we define j by $k = 2j$ then it is convenient to rename the basis slightly to

$$\tilde{f}_m(u,v) \equiv u^{j+m} v^{j-m} \tag{13.13}$$ eq:hombasp

for $m = -j, -j+1, -j+2, \cdots, j-1, j$. Note that m increases in steps of $+1$ and hence $j \pm m$ is always an integer even though j, m might be half-integer.

Now, for

$$g = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \qquad |\alpha|^2 + |\beta|^2 = 1 \tag{13.14}$$

we continue the computation of the matrix elements (13.11) for this representation of $SU(2)$:

$$\begin{aligned} (g \cdot \tilde{f}_m)(u,v) &:= \tilde{f}_m(\bar{\alpha} u + \bar{\beta} v, -\beta u + \alpha v) \\ &= (\bar{\alpha} u + \bar{\beta} v)^{j+m} (-\beta u + \alpha v)^{j-m} \\ &:= \sum_{m'} \tilde{D}^j_{m'm}(g) \tilde{f}_{m'} \end{aligned} \tag{13.15}$$ eq:expbas

Note that for $g = \exp[-\frac{i}{2}\phi\sigma^3] = \exp[\phi J^3]$ (with our unconventional normalization of J^3) we have

$$g \cdot \tilde{f}_m = e^{im\phi} \tilde{f}_m \tag{13.16}$$

More generally, we can derive an explicit formulat for the matrix elements $\tilde{D}^j_{m'm}(g)$ as functions on $SU(2)$ by expanding out the two factors in (13.15) using the binomial theorem and collecting terms:

$$\tilde{D}^j_{m'm}(g) = \sum_{s+t=j+m'} \binom{j+m}{s} \binom{j-m}{t} \bar{\alpha}^s \alpha^{j-m-t} \bar{\beta}^{j+m-s} (-\beta)^t \tag{13.17}$$ eq:xplcttldee

Warning: This differs from standard Wigner functions by normalization factors. These are important when dealing with unitary representations. We return to unitarity below. For the moment we merely remark that $\tilde{D}^j_{m_L m}$ is proportional to $D^j_{m_L m}$.

Now, we found earlier that $\mathrm{Ind}_{U(1)}^{SU(2)}(V_k)$ is spanned by matrix elements with $m_R = -k/2$. In equation (13.17) this means we should put $m = -j$ which forces $s = 0$ and hence

$$\tilde{D}^j_{m,-j} = (-1)^{j+m} \binom{2j}{j+m} \alpha^{j-m} \beta^{j+m} \tag{13.18}$$

Up to a normalization constatn this is the explicit basis (13.12) we set out to identify.

Remarks

• If we consider a charged particle confined to a sphere and coupled to a Dirac monopole of charge k we will see that quantization of the system leads to a Hilbert space

$$\Gamma(\mathcal{L}_k) = \mathrm{Ind}_{U(1)}^{SU(2)}(V_k). \tag{13.19}$$

Since $SU(2)$ is a global symmetry of the quantum system the Hilbert space must be a representation of $SU(2)$. Above we have given the decomposition of this Hilbert space in terms of irreps.

• $SU(2)/U(1)$ is also a Kahler manifold and therefore we will be able to introduce the supersymmetric quantum mechanics with this target space with $N = 2$ supersymmetry. If the particle is coupled to a gauge field on a line bundle of first chern class k (i.e. if the particle is charged, with charge k) then the above representation has the physical interpretation as the space of BPS states.

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13.0.1 Matrix Reps Of $\mathfrak{su}(2)$

As we have mentioned above, given a representation of a Lie group we can recover a representation of the corresponding Lie algebra. The representations corresponding to $\mathcal{H}^n$ with $n = 2j$ are the following:

Working to first order in ϵ we have:

$$T(e^{\epsilon X}) \cdot f = T(1 + \epsilon X) \cdot f = (T(1) + \epsilon T(X)) \cdot f = f + \epsilon (T(X) \cdot f)$$

for any $X \in su(2)$. Taking $X = X_3 = \frac{i}{2}\sigma_3$ and evaluating the definition of the function on the LHS evaluated on (u, v) we get:

$$f(u - i\frac{1}{2}\epsilon u, v - i\frac{1}{2}\epsilon v) = f - \epsilon \frac{i}{2}\left(u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v}\right) f$$

so we conclude:

$$T(X_3) \cdot f = -i\frac{1}{2}\left(u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v}\right) f$$

and, in particular,

$$iT(X^3) \cdot \tilde{f}_{j,m} = m \tilde{f}_{j,m}$$

Similarly we have

$$\begin{aligned} f + \epsilon T(X^1) \cdot f &= T(e^{\epsilon \frac{i}{2}\sigma^1}) \cdot f \\ &= f - \epsilon \frac{i}{2}\left(v \frac{\partial}{\partial u} + u \frac{\partial}{\partial v}\right) f \end{aligned} \tag{13.20}$$

since for $g = e^{\epsilon \frac{i}{2}\sigma^1}$ we have $\alpha = 1 + \mathcal{O}(\epsilon^2), \beta = -\frac{i}{2}\epsilon$. In just the same way we get:

$$\begin{aligned} f + \epsilon T(X^2) \cdot f &= T(e^{\epsilon \frac{i}{2}\sigma^2}) \cdot f \\ &= f - \epsilon \frac{1}{2}\left(v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}\right) f \end{aligned} \tag{13.21}$$

Let us define operators:

$$J^{\pm} \equiv iT(X^1) \pm T(X^2)$$

These act as

$$\begin{aligned} J^+ \cdot \tilde{f}_{j,m} &= u\frac{\partial}{\partial v}\tilde{f}_{j,m} \\ &= (j-m)\tilde{f}_{j,m+1} \\ J^- \cdot \tilde{f}_{j,m} &= v\frac{\partial}{\partial u}\tilde{f}_{j,m} \\ &= (j+m)\tilde{f}_{j,m-1} \end{aligned} \tag{13.22}$$

Or, as matrices:

$$J^+ = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2j \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{13.23}$$

eq:sutwomtrx

$$J^- = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 2j & 0 & 0 & 0 & \cdots \\ 0 & 2j-1 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{13.24}$$

Exercise

Check that this is a representation of $su(2)$. It is different from what one finds in most textbooks on quantum mechanics because it is not a unitary matrix representation. See the next section.

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13.1 Unitary structure

In Chapter 7 we observed that we can define an Hermitian metric on the holomorphic line bundle H^k such that the norm-squared of a section is

$$\parallel s \parallel^2 = \frac{1}{(1+|z|^2)^k}|s(z)|^2 \tag{13.25}$$

in the trivializations on $\mathcal{U}_\pm$. Therefore, we can define a unitary structure on the space of holomorphic sections by declaring:

$$\langle s, s\rangle := \int_{\mathbb{C}P^1} \parallel s \parallel^2 \mathrm{vol}\,(\mathbb{C}P^1) \tag{13.26}$$

where

$$\mathrm{vol}\,(\mathbb{C}P^1) = \frac{i}{2\pi}\frac{dzd\bar{z}}{(1+|z|^2)^2} \tag{13.27}$$

is the unit volume form. We claim that the action of $SU(2)$ on $H^0(\mathbb{C}P^1; H^k)$ we described above is unitary with respect to this metric.

To see this let us first note that we can identify the space $\mathcal{H}_k$ of homogeneous polynomials in two variables of degree k with the space $\mathcal{P}_k$ of polynomials in a single variable of degree $\leq k$ as follows:

If $\psi(u,v) \in \mathcal{H}_k$ then we map it to

$$s(z) := u^{-k}\psi(u,v) \tag{13.28}$$

Note that $s(z)$ will be a polynomial in the variable $z = v/u$ of degree $\leq k$.

If we require that the map $\mathcal{H}_k \to \mathcal{P}_k$ is an intertwiner of $SU(2)$ representations then we arrive at the formula for the $SU(2)$ action on $\mathcal{P}_k$: It takes a polynomial $s(z)$ to

$$(g\cdot s)(z) := (\bar{\beta}z + \bar{\alpha})^k s\left(\frac{\alpha z - \beta}{\bar{\beta}z + \bar{\alpha}}\right) \tag{13.29}$$

Now, the map

$$z \to z' = \frac{\alpha z - \beta}{\bar{\beta}z + \bar{\alpha}} \tag{13.30}$$

preserves the volume form $\mathrm{vol}\,(\mathbb{C}P^1)$, and a small computation using the identity

$$\frac{1}{(1+|z'|^2)} = \frac{|\bar{\beta}z + \bar{\alpha}|^2}{(1+|z|^2)} \tag{13.31}$$

shows that

$$\langle g\cdot s, g\cdot s\rangle = \langle s, s\rangle \tag{13.32}$$

In this way $H^0(\mathbb{C}P^1; H^k)$ becomes a Hilbert space on which $SU(2)$ acts unitarily.

The ON basis corresponding to $\tilde{f}_m$ is

$$\psi_{j,m} = \sqrt{\frac{(2j+1)!}{(j+m)!(j-m)!}}z^{j-m} \tag{13.33}$$

and can be taken to be $|j, m\rangle$ in our definition above of the Wigner functions $D^j_{m_L,m_R}$.

Exercise
Find the proportionality factor between $\tilde{D}^j_{m_L,m_R}$ and $D^j_{m_L,m_R}$.

****
****
UNITARIZE REP OF J^+ AND J^-
****
****

13.2 Lie algebra representations and differential operators

TO BE WRITTEN: Explain that

1. Generators are represented by differential operators.

2. The usual differential equations and recursion relations on special functions follow from this point of view.

13.3 Coherent state formalism

The above discussion of representations of $SU(2)$ is closely related to the subject of *coherent states.*

One reference for this material is

A. Perelomov, *Generalized Coherent States and Their Applications*, Springer Verlag 1986

Another reference is:

K. Fuji, Introduction to coherent states and quantum information theory, arXiv:quant-ph/0112090

One way to motivate this is to derive the representation of the Lie algebra in terms of differential operators.

If

$$J^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{13.34}$$

then actiong on $\mathcal{P}_k$ we represent

$$J^+ = -\frac{\partial}{\partial z} \tag{13.35}$$

It is then natural to define coherent states:

$$|\zeta\rangle := \frac{1}{(1+|\zeta|^2)^j} \exp(\zeta J_+)|j, -j\rangle \tag{13.36}$$

One can show that

1.

$$|\zeta\rangle = \sum_m u_m(\zeta)|j, m\rangle \tag{13.37}$$

with

$$u_m(\zeta) = \sqrt{\frac{(2j)!}{(j+m)!(j-m)!}} \frac{\zeta^{j+m}}{(1+|\zeta|^2)^j} \tag{13.38}$$

******* IS SQRT CORRECT? ********

2. The inner product of two such polynomials is

$$\langle \xi | \eta \rangle = \frac{(1+\bar{\xi}\eta)^{2j}}{((1+|\xi|^2)^j (1+|\eta|^2)^j} \tag{13.39}$$

In particular $|\zeta\rangle$ is of unit norm.

3. If we regard polynomials in z of degree $\leq 2j$ as an irrep of $SU(2)$ as above then we may identify the state $|\zeta\rangle$ with the polynomial: ???

$$f(z) = \frac{(1+\zeta z)^{2j}}{(1+|\zeta|^2)^j} \tag{13.40}$$

Examples:

1. The wavefunction of the oxygen molecule!

2. Wavefunctions of electrons in the presence of a magnetic monopole will take values in this bundle.

3. Nice application of this coherent state formalism to the QHE in paper by Haldane.

Remarks:

1. The coherent states are good for taking the semiclassical limit of large j. Then the support of $|\zeta\rangle$ is sharply peaked at a point on $\mathbb{C}P^1$. **** SHOW *****

2.

3. Relate infinite dimensional induced rep to the Schwinger representation in terms of oscillators. *** What is this??? ***

14. Orbits of the Lorentz group and relativistic wave equations

14.1 Orbits, Representations, and Differential Equations

♣OVERLAP HERE WITH PETER-WEYL THEOREM IN CHAPTER 4 ON REPRESENTATIONS. MOVE THIS? ♣

Now let us turn to how these orbits are related to some important differential equations in field theory.

As we have seen, since $O(1,1)$ acts on $\mathbb{M}^{1,1}$, it follows that it acts on the space of fields $\mathrm{Map}(\mathbb{M}^{1,1}, \kappa)$, where $\kappa = \mathbb{R}$ or $\mathbb{C}$ for a real or complex-valued scalar field. For a *scalar* field recall the action of $A \in O(1,1)$ on the space of solutions is

$$(A \cdot \Psi)(x) := \tilde{\Psi}(x) := \Psi(A^{-1}x) \tag{14.1}$$

Note the A^{-1} in the argument of Ψ in the second equality. This is necessary to get a left-action of the group on the space of fields. If we use Ax then we get a right-action.

♣For the case of *complex* scalar fields when A is in a component of $O(1,1)$ that reverses the orientation of time we have the choice of whether to include complex conjugation in the action. This would be adding a "charge conjugation" action. ♣

Now, quite generally, if V is a representation space for G, and $\mathcal{O} \in \mathrm{End}(V)$ is an invariant linear operator, i.e. an operator which commutes with the action of G,

$$\rho(g)\mathcal{O} = \mathcal{O}\rho(g) \tag{14.2}$$

then any eigenspace, say $\{v \in V|\mathcal{O}v = \lambda v\}$ will be a sub-representation of G.

Consider the operator

$$\partial^\mu \partial_\mu = -\partial_0^2 + \partial_1^2 \tag{14.3}$$

acting on the space of scalar fields. This is an example of an invariant operator, as one confirms with a simple computation. It can be made manifest by writing

$$\partial^\mu \partial_\mu = -4\partial_+\partial_- \tag{14.4}$$

where

$$\partial_\pm := \frac{\partial}{\partial x^\pm} = \frac{1}{2}(\partial_0 \mp \partial_1) \tag{14.5}$$

The *Klein-Gordon equation* for a complex or real scalar field $\Psi(x^0, x^1)$ is

$$\left(\partial^\mu \partial_\mu + m^2\right)\Psi = 0 \tag{14.6}$$

The space of fields satisfying the KG equation is a representation space of $O(1,1)$, by our general remark above.

Now we relate the orbits of $O(1,1)$ to the representations furnished by solutions of the KG equation:

For field configurations which are Fourier transformable we can write

$$\Psi(x^0, x^1) = \int dk_0 dk_1 \hat{\psi}(k) e^{\mathrm{i}k_0 x^0 + \mathrm{i}k_1 x^1} \tag{14.7}$$

If the field Ψ is on-shell then $\hat{\psi}(k)$ must have support on set

$$\{k : (k_0)^2 - (k_1)^2 = m^2\} \tag{14.8}$$ eq:MassShell

that is:

$$(k^2 + m^2)\hat{\psi}(k) = 0 \tag{14.9}$$

Thus, the support in Fourier space is an orbit. In physics the orbit with $k_0 > 0$ is called the *mass-shell.*

For example, suppose that $m^2 > 0$. Then on the orbit with $k^0 > 0$ we have $k = m(\cosh\theta, \sinh\theta)$ and we can write the general complex-valued solution as

$$\hat{\psi}(k) = \delta(k^2 + m^2)\bar{a}, \tag{14.10}$$

where the amplitude $\bar{a}$ should be regarded as a complex-valued function on the orbit. Then a complex-valued solution on the KG equation is generated by such a function on the orbit by:

$$\begin{aligned}
\Psi(x) &= \int d^2k\delta(k^2 + m^2)\bar{a}e^{\mathrm{i}k\cdot x} \\
&= \int_{\mathbb{R}} \frac{dk_1}{2\sqrt{k_1^2 + m^2}} \bar{a}(k_1) e^{\mathrm{i}k\cdot x} \\
&= \frac{1}{2}\int_{-\infty}^{\infty} d\theta \bar{a}(\theta) e^{\mathrm{i}m(x^0\cosh\theta + x^1 \sinh\theta)}
\end{aligned} \tag{14.11}$$ eq:MomentumMod

where we chose a convention designation of the argument of the function $\bar{a}$ on the orbit.

Thus, we can identify a space of solutions to the KG equation with a space of functions on an orbit.

Now recall that the space of functions on an orbit is a special case of an induced representation. As we will soon see, induced representations are the right concept for generalization to other representations of the Lorentz group.

Remark 1: *The components of the Lorentz group:* If we use a particular orbit to generate representations we only represent the subgroup of the Lorentz group which preserves that orbit. Thus, for the orbit with $m > 0$ and $k_0 > 0$ we can include the parity-reversing component, but not the time-reversing component. Of course, there is a similar representation of $SO_0(1,1) \amalg P \cdot SO_0(1,1)$ given by the hyperbola with $k_0 < 0$. If we wish to represent the full $O(1,1)$ then we must include both orbits. Note that, if we wish to use *real* solutions of the KG equation we must include both hyperbolae and there is a reality condition relating the Fourier modes: $(\bar{a}(k))^* = a(-k)$

Remark 2: The representation can be made into a unitary representation. See below.

Exercise

Show that an invariant linear operator $\mathcal{O} \in \mathrm{End}(V)$ on a representation space V of G is a fixed point of the $G \times G$ action on $\mathrm{End}(V)$.

♣This, and the concept of an invariant operator really goes above in the general section on functions on groups. ♣

14.2 The massless case in $1+1$ dimensions

Having phrased things this way it is natural to ask what relativistic wave equations correspond to amplitudes supported on the light-like orbits.

In the massless case $m^2 = 0$ the general solution of the KG equation is easily written as

$$\Psi = \psi_L(x^+) + \psi_R(x^-) \tag{14.12}$$

Solutions of the form $f_R = 0$ are called *left-moving waves* because, as time evolves forward, the profile of the function f_L moves to the left on the x^1-axis. Similarly, solutions of the form $f_L = 0$ are called *right-moving waves.*

Remark: In the massless case there is a "more primitive" relativistic wave equation which is first order, and whose solutions are always solutions of the massless KG equation. Namely, we can consider the separate equations

$$\partial_+ \Psi = 0 \tag{14.13}$$

eq:anti-self-c

$$\partial_- \Psi = 0 \tag{14.14}$$

eq:self-dual

Note that these equations are themselves Lorentz invariant, even though the operators $\partial_\pm$ are not invariant differential operators. Solutions to (14.13) are right-moving scalar fields and solutions to (14.14) are left-moving scalar fields. Such on-shell scalar fields are also known as *chiral scalar fields*. Equations (14.13) and (14.14) are notable in part because they play an important role in string theory and conformal field theory. It is also interesting to note that it is quite subtle to write an action principle that leads to such equations of motion. They are also called the anti-self-dual and self-dual equations of motion, respectively because, if we choose the orientation $dx \wedge dt$ then the Hodge star operation is $*dt = dx$ and $*dx = dt$ and hence $*(dx^{\pm}) = \pm dx^{\pm}$. Therefore if $\partial_{\pm}\Psi = 0$ its "fieldstrength" $F = d\Psi$ satisfies $*F = \mp F$.

The action of $P, T \in O(1,1)$ on a real scalar field is given by:

$$\begin{aligned}(P \cdot \Psi)(x,t) &:= \Psi(-x,t)\\ (T \cdot \Psi)(x,t) &:= \Psi^*(x,-t)\end{aligned} \tag{14.15}$$

The KG equation is separately P and T invariant, but the (anti)-self-dual equations are not. Nevertheless, the latter equations are PT invariant. This is a special case of the famous CPT theorem:

Roughly speaking, mathematical consistency implies that if a physical theory is invariant under group transformations in the neighborhood of the identity then it must be invariant under the transformations in the full connected component of the identity. But this does not mean the theory is invariant under disconnected components such as the components containing P and T. As a matter of fact, Nature chooses exactly that option in the standard model of the electroweak and strong interactions. However, if we also assume that there is a certain relation to "analytically continued" equations in Euclidean signature then, since PT is in the connected component of the identity of $O(2)$, such theories must in fact be PT invariant.

♣There is much more to discuss about the nature of the analytic continuation implied here. In what sense is $SO(2)$ and "analytic continuation" of $SO(1,1)$? It does not do simply to make the boost pure imaginary unless $x^{\pm} \to z, \bar{z}$. ♣

14.3 The case of d dimensions, $d > 2$

Now consider Minkowski space $\mathbb{M}^{1,d-1}$ with $d > 2$. The nature of the orbits is slightly different.

1. For $\lambda^2 > 0$ we can define

$$\mathcal{O}^{+}(\lambda) = \{x | (x^0)^2 - (\vec{x})^2 = \lambda^2 \qquad \& \qquad \mathrm{sign}(x^0) = \mathrm{sign}(\lambda)\} \tag{14.16}$$

   By the stabilizer-orbit theorem we can identify this with

$$SO_0(1,d-1)/SO(d-1) \tag{14.17}$$

   by considering the isotropy group at $(x^0 = \lambda, \vec{x} = 0)$. See Figure 5(a).

2. For $\mu^2 > 0$ we can define

$$\mathcal{O}^{-}(\lambda^2) = \{x | (x^0)^2 - (\vec{x})^2 = -\mu^2\} \tag{14.18}$$

   By the stabilizer-orbit theorem we can identify this with

$$SO_0(1,d-1)/SO_0(1,d-2) \tag{14.19}$$

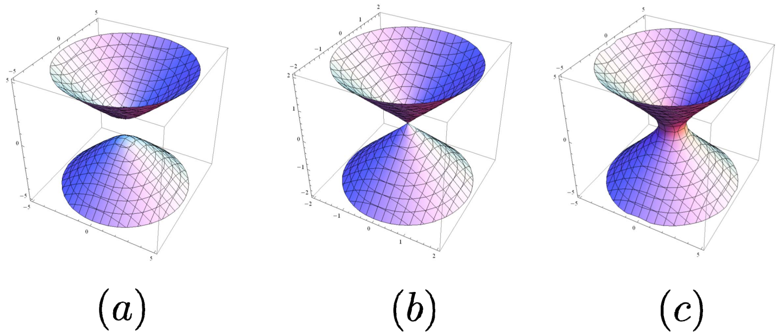

Figure 5: Illustrating orbits of the connected component of the identity in $O(1,3)$. In (a) the top and bottom hyperboloids are separate orbits, and if we include time-reversing transformations the orbits are unions of the two hyperboloids. In (b) there are three orbits shown with $x^0 > 0$ $x^0 < 0$ (the future and past, or forward and backward light cones), and the orbit consisting of the single point. In (c), once x^2 has been specified, there is just one orbit, for $d > 2$. fig:LorentzOrb

by considering the isotropy group at $x = (x^0 = 0, x^1 = 0, \dots, x^{d-2} = 0, x^{d-1} = \mu)$. The sign of μ does not distinguish different orbits for $d > 2$ because the sphere S^{d-2} is connected. See Figure 5(c).

3.
$$\mathcal{O}^{\pm} = \{x | x^2 = 0 \qquad \& \qquad \mathrm{sign}(x^0) = \pm 1\} \tag{14.20}$$

Vectors in this orbit are of the form $(x^0, |x^0|\hat{n})$ where $\hat{n} \in S^{d-2} \subset \mathbb{R}^{d-1}$ and the sign of x^0 is invariant under the action of the identity component of $O(1,3)$. (Show this!). Note that, for $d = 2$ the sphere S^0 has two disconnected components, leading to left- and right-movers. But for $d > 2$ there is only one component. We can think of $\hat{n} \in S^{d-2}$ as parametrizing the directions of light-rays. That is, the point where the light ray hits the celestial sphere. In one spatial dimension, a light ray either moves left or right, and this is a Lorentz-invariant concept. In $d-1 > 1$ spatial dimensions, we can rotate any direction of light ray into any other. See Figure 5(b). One can show that these orbits too are homogeneous spaces: [6]

$$\mathcal{O}^{\pm} \cong SO_0(1, d-1)/\mathcal{I} \tag{14.21}$$

[6]The isotropy group of a light ray is $\mathcal{I} \cong ISO(d-2)$, where $ISO(d-2)$ is the Euclidean group on $\mathbb{R}^{d-2}$. The easiest way to show this is to use the Lie algebra of $so(1, d-1)$ and work with light-cone coordinates. Choosing a direction of the light ray along the x^{d-1} axis and introducing light-cone coordinates $x^{\pm} := x^0 \pm x^{d-1}$, and transverse coordinates x^i, $i = 1, \dots, d-2$ if the lightray satisfies $x^- = 0$ then we have unbroken generators M^{+i} and M^{ij}.

4. The final orbit is of course $\{x = 0\}$.

As in $1+1$ dimensions we can identify a representation space of the Lorentz group associated with the space of solutions to the KG equation with functions on various orbits. The formulae are essentially the same. For example for the orbit $\mathcal{O}^+(m)$ a function $\bar{a} : \mathcal{O}^+(m) \to \mathbb{C}$ determines a solution:

$$\begin{aligned}\Psi(x) &= \int d^d k \delta(k^2 + m^2) \bar{a} e^{ik\cdot x} \\ &= \int_{\mathbb{R}^{d-1}} \frac{d^{d-1}\vec{k}}{2\sqrt{\vec{k}^2 + m^2}} \bar{a}(\vec{k}) e^{ik\cdot x}\end{aligned} \tag{14.22}$$

eq:MomentumMod

However, for scalar fields, there is no analog of the left- and right-chiral boson. There *are* analogs involving interesting first order equations such as the Dirac equation and the (anti-) self-dual equations for fields with spin.

Quite generally, we can define an inner product on the space of complex-valued solutions of the KG equation such that the action of the Lorentz group is unitary. Observe that, given any two complex-valued solutions Ψ_1, Ψ_2 the current

$$j_\mu := -i \left(\Psi_1^* \partial_\mu \Psi_2 - (\partial_\mu \Psi_1)^* \Psi_2 \right) \tag{14.23}$$

Note that, if Ψ_1 and Ψ_2 both satisfy the KG equation then

$$\partial^\mu j_\mu = 0 \tag{14.24}$$

is conserved. Therefore, if we choose a spatial slice with normal vector n^μ and induced volume form vol the inner product

$$(\Psi_1, \Psi_2) := \int_\Sigma n^\mu j_\mu \mathrm{vol} \tag{14.25}$$

is independent of the choice. So, fixing a Lorentz frame and taking Σ to be the slace at a fixed time we have

$$(\Psi_1, \Psi_2) := -i \int_{\mathbb{R}^{d-1}} \left(\Psi_1^* \partial_0 \Psi_2 - (\partial_0 \Psi_1)^* \Psi_2 \right) d^{d-1}\vec{x} \tag{14.26}$$

This is clearly not positive definite on the space of all solutions but does become positive definite when restricted to the space of complex solutions associated with a single orbit. Indeed, substituting the expansion in momentum space (14.22) we get

$$(\Psi_1, \Psi_2) = \int_{\mathbb{R}^{d-1}} \frac{d^{d-1}\vec{k}}{2\sqrt{\vec{k}^2 + m^2}} (\bar{a}_1(\vec{k}))^* (\bar{a}_2(\vec{k})) \tag{14.27}$$

Having phrased things this way, it is clear that there is an interesting generalization: We can choose other representations of $H = SO(d-1)$ and consider the induced representations of the Lorentz group. This indeed leads to the unitary representations, corresponding to particles with nontrivial spin.

♣Incorporate some of the following remarks: ♣

15. Characters and the Decomposition of a Representation to its Irreps

Given a group G there are some fundamental problems one wants to solve:

- Construct the irreducible representations.
- Given a representation, how can you tell if it is reducible?
- Given a reducible representation, what is its reduction to irreducible representations?

I am not aware of any systematic procedure for deriving the list of irreps of any group. However, if - somehow - the list is known then the answers to 2 and 3 are readily provided through the use of characters.

15.1 Some basic definitions

Definition 1 Let (T,V) be a representation of G. The functions on G defined by

$$\chi_{(T,V)}(g) \equiv \mathrm{Tr}_V T(g) \equiv \sum_i T_{ii}(g) \tag{15.1}$$

are called the *characters* of the representation T.

Note: $\chi_{(T,V)}$ is independent of basis. Moreover, if $(T,V) \cong (T',V')$ then

$$\chi_{(T,V)} = \chi_{(T',V')} \tag{15.2}$$

Exercise
Show that for a unitary representation

$$\chi(g^{-1}) = \chi(g)^* \tag{15.3}$$

A second motivation for looking at characters is the following: matrix elements are complicated. Sometimes one can extract all the information one is seeking by considering this simpler and more basic set of functions on G associated to a representation.

Recall that for any element of $g \in G$ the set

$$C(g) = \{h \in G : \exists k, g = khk^{-1}\} \tag{15.4}$$

is called the *conjugacy class* of g.

Definition A function on G that only depends on the conjugacy class is called a *class function*. In particular, characters are class functions. The space of class functions is a subspace of R_G.

Theorem. The χ_μ form a basis for the vector space of class functions in R_G.

Proof: This is a corollary of the previous section: Any function can be expanded in

$$F(g) = \sum \hat{F}^\mu_{ij} T^\mu_{ij}(g) \tag{15.5}$$

Therefore, it suffices to compute the average of $T^\mu_{ij}(g)$ over a conjugacy class:

$$(T^\mu_{ij})^{average}(g) \equiv \frac{1}{|G|} \sum_{h\in G} T^{(\mu)}_{ij}(hgh^{-1}) \tag{15.6}$$

Expanding this out and doing the sum on h using the ON relations shows that

$$(T^\mu_{ij})^{average}(g) = n_\mu \chi_\mu(g) \tag{15.7}$$

On the other hand, if F is a class function then

$$F^{\rm average} = F \tag{15.8}$$

so

$$F(g) = \sum_{ij} \hat{F}^\mu_{ij} n_\mu \chi_\mu(g) \tag{15.9}$$

So that χ_μ form a basis of class functions. ♠

Now, there is another obvious basis of the space of class functions: Denote the distinct conjugacy classes by $C_i, i = 1, \dots r$. For each C_i can define a class function δ_{C_i} in R_G to be the characteristic function of C_i. That is it is 1 on the class C_i and zero on all other group elements. Comparing these two ways of finding the dimension we get:

Theorem The number of conjugacy classes of G is the same as the number of irreducible representations of G.

Since the number of representations and conjugacy classes are the same we can define a *character table*:

$$\begin{array}{cccccc} & m_1C_1 & m_2C_2 & \cdots & \cdots & m_rC_r \\ \chi_1 & \chi_1(C_1) & \cdots & \cdots & \cdots & \chi_1(C_r) \\ \chi_2 & \chi_2(C_1) & \cdots & \cdots & \cdots & \chi_2(C_r) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \chi_r & \chi_r(C_1) & \cdots & \cdots & \cdots & \chi_r(C_r) \end{array} \tag{15.10}$$

Here m_i denotes the order of C_i.

As we will see, character tables are a very useful summary of important information about the representation theory of a group.

Exercise

In a previous exercise we computed the average number of fixed points of a finite group G acting on a finite set X.

Rederive the same result by interpreting the number of elements of X fixed by g as a character $\chi(g)$ of the representation given by the functions on X.

15.2 Orthogonality relations on the characters

Let χ_μ be the character of $T^{(\mu)}$. The orthogonality relations for matrix elements implies:

$$\frac{1}{|G|}\sum_{g\in G}\chi_\nu(g^{-1})\chi_\mu(g) \equiv (\chi_\nu,\chi_\mu) = \delta_{\mu\nu} \tag{15.11}$$

To prove this, simply put $i=\ell$ and $s=m$ in (9.9) and sum on i and s.

Now, we argued before - abstractly - that *any* finite dimensional representation (T,V) of G is completely reducible:

$$T \cong a_1 T^{(1)} \oplus a_2 T^{(2)} \oplus \cdots \oplus a_r T^{(r)} = \oplus_\mu a_\mu T^{(\mu)} \tag{15.12}$$ eq:reducible

The a_i are nonnegative integers, they measure the number of times a rep appears and are referred to as the "multiplicities."

From the orthogonality relation we get:

$$a_\mu = (\chi_{(T,V)},\chi_\mu) \tag{15.13}$$ eq:multipl

Thus we arrive at a key fact:

Theorem

Two representations are equivalent iff they have the same character.

If you have two reps and you want to see whether they are inequivalent then an efficient test is to compute the characters.

The orthogonality relations on characters can be stated more beautifully if we use the fact that characters are class functions:

Theorem Orthogonality Relations for characters Denote the distinct conjugacy classes $C_i \in \mathcal{C}$, and the (distinct) characters of the distinct irreducible unitary representations χ_μ. Then

$$\frac{1}{|G|}\sum_{C_i\in\mathcal{C}} m_i \chi_\mu(C_i)\chi_\nu(C_i)^* = \delta_{\mu\nu} \tag{15.14}$$ eq:orthogrel

where $m_i = |C_i|$ is the order of the conjugacy class C_i. We also have

$$\sum_\mu \chi_\mu(C_i)^*\chi_\mu(C_j) = \frac{|G|}{m_i}\delta_{ij} \tag{15.15}$$ eq:orthogrelii

Proof: Let us first prove (15.14). From unitarity we have $\chi_\mu(g^{-1}) = (\chi_\mu(g))^*$. Since χ_μ are class functions the sum over the elements in the group can be written as a sum over the conjugacy classes $C_i \in \mathcal{C}$.

The equation (15.14) can be interpreted as the statement that the $r \times r$ matrix

$$S_{\mu i} := \sqrt{\frac{m_i}{|G|}}\chi_\mu(C_i) \qquad \mu = 1, \dots, r \quad i = 1, \dots, r \tag{15.16}$$

satisfies

$$\sum_{i=1}^{r} S_{\mu i} S^*_{\nu i} = \delta_{\mu\nu} \tag{15.17}$$

Therefore, $S_{\mu i}$ is a unitary matrix. The left-inverse is the same as the right-inverse, and hence we obtain (15.15). ♠

Exercise

Show that right-multiplication of the character table by a diagonal matrix produces a unitary matrix.

Exercise

Using the orthogonality relations on matrix elements, derive the more general relation on characters:

$$\frac{1}{|G|}\sum_{g\in G} \chi_\mu(g)\chi_\nu(g^{-1}h) = \frac{\delta_{\mu\nu}}{n_\mu}\chi^{(\nu)}(h) \tag{15.18}$$ eq:convolchar

We will interpret this more conceptually later.

Exercise

A more direct proof of (15.15) goes as follows. Consider the operator $L(g_1) \otimes R(g_2)$ acting on the regular representation. We will compute

$$\mathrm{Tr}_{R_G}[L(g_1) \otimes R(g_2)] \tag{15.19}$$

in two bases.

First consider the basis ϕ^μ_{ij} of matrix elements. We have:

$$L(g_1) \otimes R(g_2) \cdot \phi^\mu_{ij} = \sum_{i',j'} T^\mu_{j'j}(g_2) T^\mu_{i'i}(g_1^{-1}) \phi^\mu_{i'j'} \tag{15.20}$$

so the trace in this basis is just:

$$\mathrm{Tr}_{R_G}[L(g_1) \otimes R(g_2)] = \sum_\mu \chi_\mu(g_1)^* \chi_\mu(g_2) \tag{15.21}$$

On the other hand, we can use the delta-function basis: δ_g. Note that

$$L(g_1) \otimes R(g_2) \cdot \delta_g = \delta_{g_1 g g_2^{-1}} \tag{15.22}$$

So in the delta function basis we get a contribution of $+1$ to the trace iff $g = g_1 g g_2^{-1}$, that is iff $g_2 = g^{-1} g_1 g$, that is iff g_1 and g_2 are conjugate, otherwise we get zero.

Now show that

$$\#\{g : g_2 = g^{-1} g_1 g\} = |C(g_1)| = m_1 \tag{15.23}$$ eq:weight

by considering $Z(g_1)$, the centralizer subgroup of g_1 in G and identifying the cosets $G/Z(g_1)$ with the elements of the conjugacy class $C(g_1)$.

So we get the second orthogonality relation. ♠

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16. Some examples of representations of discrete groups and their character tables

16.1 S_3

The group S_3 has three obvious irreducible representations:

$\mathbf{1}_+$. $1 \to 1$ $(ij) \to 1$ $(123) \to 1$, etc.

$\mathbf{1}_-$. $1 \to 1$ $(ij) \to -1$ $(123) \to +1$, etc. This is the sign representation, or the ϵ homomorphism to $\mathbb{Z}_2$ mentioned in lecture 1.

2. Symmetries of the triangle.

$$\begin{aligned} 1 &\to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ (23) &\to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ (123) &\to \begin{pmatrix} \cos(2\pi/3) & \sin(2\pi/3) \\ -\sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} \end{aligned} \tag{16.1}$$

These reps are irreducible. Note that by applying group elements in the third representation to any vector we obtain a set of vectors which spans $\mathbb{R}^2$.

Indeed we can check:

$$6 = 1^2 + 1^2 + 2^2 \tag{16.2}$$

to conclude that there are no other irreps.

It is now straightforward to write out the character table for S_3.

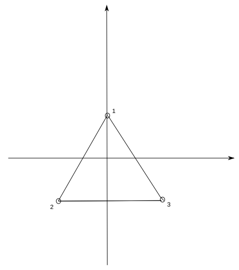

Figure 6: The symmetries of the equilateral triangle give the 2-dimensional irrep of S_3. fig:triangle

$$\begin{array}{cccc} & [1] & 3[(12)] & 2[(123)] \\ \chi_{\mathbf{1}_+}: & 1 & 1 & 1 \\ \chi_{\mathbf{1}_-}: & 1 & -1 & 1 \\ \chi_{2}: & 2 & 0 & -1 \end{array} \tag{16.3}$$

Here is an example of the use of characters to decompose a representation:

Now let us see how we can use the orthogonality relations on characters to find the decomposition of a reducible representation.

Example 1 Consider the 3×3 rep generated by the ♮ action of the permutation group S_3 on $\mathbb{R}^3$. We'll compute the characters by choosing one representative from each conjugacy class:

$$1 \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (12) \rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (132) \rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{16.4}$$

From these representatives the character of $V = \mathbb{R}^3$ is easily calculated:

$$\chi_V(1) = 3 \qquad \chi_V([(12)]) = 1 \qquad \chi_V([(132)] = 0 \tag{16.5}$$

Using the orthogonality relations we compute

$$a_{\mathbf{1}_+} = (\chi_{\mathbf{1}_+}, \chi) = \frac{1}{6}3 + \frac{3}{6}1 + \frac{2}{6}0 = 1 \tag{16.6}$$

$$a_{\mathbf{1}_-} = (\chi_{\mathbf{1}_-}, \chi) = \frac{1}{6}3 + \frac{3}{6}(-1) \cdot 1 + \frac{2}{6}0 = 0 \tag{16.7}$$

$$a_{\mathbf{2}} = (\chi_{\mathbf{2}}, \chi) = \frac{1}{6}3 \cdot 2 + \frac{3}{6}0 \cdot 1 + \frac{2}{6}(-1) \cdot 0 = 1 \tag{16.8}$$

Therefore:

$$\chi_V = \chi_{\mathbf{1}_+} + \chi_{\mathbf{2}} \tag{16.9}$$

showing the decomposition of $\mathbb{R}^3$ into irreps.

Example 2 Consider S_3 acting by permuting the various factors in the tensor space $V \otimes V \otimes V$ for any vector space V. Now, if $\dim V = d$ then we have

$$\begin{aligned} \chi([1]) &= d^3 \\ \chi([(ab)]) &= d^2 \\ \chi[(abc)] &= d \end{aligned} \tag{16.10}$$

eq:vcubed

So we can compute

$$a_{\mathbf{1}_+} = (\chi_{\mathbf{1}_+}, \chi) = \frac{1}{6}d^3 + \frac{3}{6}d^2 + \frac{2}{6}d = \frac{1}{6}d(d+1)(d+2) \tag{16.11}$$

$$a_{\mathbf{1}_-} = (\chi_{\mathbf{1}_-}, \chi) = \frac{1}{6}d^3 + \frac{3}{6}(-1)\cdot d^2 + \frac{2}{6}d = \frac{1}{6}d(d-1)(d-2) \tag{16.12}$$

$$a_{\mathbf{2}} = (\chi_{\mathbf{2}}, \chi) = \frac{1}{6}2d^3 + \frac{3}{6}0\cdot d^2 + \frac{2}{6}(-1)\cdot d = \frac{1}{3}d(d^2-1) \tag{16.13}$$

Thus, *as a representation of* S_3, we have

$$V^{\otimes 3} \cong \frac{d(d+1)(d+2)}{6}\mathbf{1}_+ \oplus \frac{d(d-1)(d-2)}{6}\mathbf{1}_- \oplus \frac{d(d+1)(d-1)}{3}\mathbf{2} \tag{16.14}$$

Note that the first two dimensions are those of S^3V and Λ^3V, respectively, and that the dimensions add up correctly.

We are going to develop this idea in much more detail when we discuss Schur-Weyl duality in section ****

Remarks

• Notice that this does not tell us *how* to block-diagonalize the matrices. We will see how to do that later.

Exercise

Repeat example 2 to decompose $V^{\otimes 2}$ as a representation of S_2.

Exercise

Write out the unitary matrix $S_{\mu i}$ for $G = S_3$.

Exercise

a.) Suppose we tried to define a representation of S_3 by taking $(12) \to 1$ and $(23) \to -1$. What goes wrong?

b.) Show that for any n there are only two one-dimensional representations of S_n

---

16.2 Dihedral groups

The irreps of the dihedral group D_n are easily written down.

Recall that D_n is the group generated by x, y subject to the relations:

$$\begin{aligned} x^2 &= 1 \\ y^n &= 1 \\ xyx^{-1} &= y^{-1} \end{aligned} \tag{16.15}$$

eq:rels

We can define two-dimensional complex representations $T^{(j)}$ by:

$$\begin{aligned} T^{(j)} : y &\to \begin{pmatrix} \omega^j & \\ & \omega^{-j} \end{pmatrix} \\ x &\to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{16.16}$$

eq:twodee

where $\omega = e^{2\pi i/n}$. Clearly, these satisfy the defining relations of D_n.

Note that $T^{(j)} = T^{(j+n)}$, and moreover, $T^{(j)} \cong T^{(n-j)}$ are equivalent representations. Note that $(T^{(j)})^* = T^{(-j)}$ and hence these are all real representations. Indeed, by a similarity transformation we can make y map to a rotation matrix.

Suppose n is odd. Then the representations $T^{(j)}$ are irreducible and inequivalent for $j = 1, \dots, (n-1)/2$. For $j = 0$ the representation is reducible to $\mathbf{1}_+$ given by $y \to 1, x \to 1$ and $\mathbf{1}_-$ given by $y \to 1, x \to -1$. Altogether we have

$$2n = \frac{n-1}{2} \cdot 2^2 + 1^2 + 1^2 \tag{16.17}$$

so these are the only irreps.

Suppose n is even. Then the representations $T^{(j)}$ are irreducible and inequivalent for $j = 1, \dots, \frac{n}{2} - 1$. For $j = 0$ the representation is reducible to $\mathbf{1}_{++}$ given by $y \to 1, x \to 1$ and $\mathbf{1}_{+-}$ given by $y \to 1, x \to -1$. For $j = n/2$ the representation is reducible to $\mathbf{1}_{-+}$ given by $y \to -1, x \to 1$ and $\mathbf{1}_{--}$ given by $y \to -1, x \to -1$. Altogether we have

$$2n = (\frac{n}{2} - 1) \cdot 2^2 + 1^2 + 1^2 + 1^2 + 1^2 \tag{16.18}$$

so we have all the representations.

The character tables are:

For $n = 2k + 1$ odd:

	$[1]$	$2k[x]$	$2[y]$	$\cdots$	$2[y^k]$
$\mathbf{1}_+$	1	1	1	$\cdots$	1
$\mathbf{1}_-$	1	-1	1	$\cdots$	1
$\mathbf{2}^{(\mathbf{j})}$	2	0	$\omega^j + \omega^{-j}$	$\cdots$	$\omega^{jk} + \omega^{-jk}$

where $j = 1, \ldots, k$.
For $n = 2k$ even we have
MISSING TABLE
where $j = 1, \ldots, k-1$
**** SHOULD apply these tables to solve some problem ***

17. The decomposition of tensor products

A very common problem one faces is the following:

Suppose we have two representations V_1, V_2. Then the tensor product $V_1 \otimes V_2$ is also a representation.

Question: *If we know how to decompose V_1, V_2 into irreps, how does $V_1 \otimes V_2$ fall apart into its irreducible pieces?*

An example of this general question arises in the quantum mechanical addition of angular momenta. For this reason it is sometimes called the "Clebsch-Gordon problem."

(**** Following should be moved to place where we talk about complete reducibility ****)

If V is a completely decomposable representation (always true for compact groups) then we can write

$$V \cong \oplus_{\mu=1}^{r} \oplus_{i=1}^{a_\mu} T^{(\mu)} := \oplus_\mu a_\mu T^{(\mu)} \tag{17.1}$$

For a fixed μ let let

$$V^{(\mu)} := \oplus_{i=1}^{a_\mu} T^{(\mu)} \tag{17.2}$$

It contains the representation μ with the correct degeneracy in V. It is called the *isotypical component* belonging to μ. Note that it can be written as

$$V^{(\mu)} = \mathbb{C}^{a_\mu} \otimes T^{(\mu)} \tag{17.3}$$

eq:isotpyp

where $\mathbb{C}^{a_\mu}$ is understood to be the trivial representation of G. Thus by Schur's lemma, the intertwiners of $V^{(\mu)}$ with itself are of the form $K \otimes 1$ where $K \in \mathrm{End}(\mathbb{C}^{a_\mu})$ is arbitrary.

We abbreviate $\mathbb{C}^{a_\mu} \otimes T^{(\mu)}$ to $a_\mu T^\mu$ and with this understood we write the decomposition into isotypical components as:

$$V = \oplus a_\mu T^\mu \tag{17.4}$$

Now suppose:

$$V_1 = \oplus a_\mu T^\mu \qquad V_2 = \oplus b_\nu T^\nu \tag{17.5}$$

then

$$V_1 \otimes V_2 = \oplus_{\mu,\nu} a_\mu b_\nu T^\mu \otimes T^\nu \tag{17.6}$$

so in finding the isotypical decomposition of $V_1 \otimes V_2$ it suffices to find the decomposition of products of irreps.

Definition *Fusion Coefficients* The isotypical decomposition of the product of irreps is of the form:

$$T^{(\mu)} \otimes T^{(\nu)} = \oplus_\lambda \left(\overbrace{T^{(\lambda)} \oplus \cdots \oplus T^{(\lambda)}}^{N_{\mu\nu}^\lambda \ times} \right) \tag{17.7}$$

$$= \oplus_\lambda N_{\mu\nu}^\lambda T^{(\lambda)}$$

eq:decomp

The multiplicities $N_{\mu\nu}^\lambda$ are known as *fusion coefficients.*

Exercise
Show that

$$\chi_{T_1 \otimes T_2}(g) = \chi_{T_1}(g)\chi_{T_2}(g) \tag{17.8}$$

Now let us put this observation to use and solve our problem. Taking the trace of (17.7) we get:

$$\chi_\mu(g)\chi_\nu(g) = \sum_\lambda N_{\mu\nu}^\lambda \chi_\lambda(g) \tag{17.9}$$

eq:chardec

Now, taking the inner product we get:

$$N_{\mu\nu}^\lambda = (\chi_\lambda, \chi_\mu\chi_\nu) \tag{17.10}$$

eq:fusion

We can write this out as:

$$N_{\mu\nu}^\lambda = \frac{1}{|G|} \sum_{g\in G} \chi_\mu(g)\chi_\nu(g)\chi_\lambda(g^{-1}) \tag{17.11}$$

eq:fusioni

This can be written in a different way, which we will explain conceptually in the next section. For simplicity choose unitary irreps, and recall that the orthogonality relations on characters is equivalent to the statement that

$$S_{\mu i} := \sqrt{\frac{m_i}{|G|}} \chi_\mu(C_i) \qquad \mu = 1, \dots, r \quad i = 1, \dots, r \tag{17.12}$$

is a unitary matrix. Notice that if we take $\mu = 0$ to correspond to the trivial 1-dimensional representation then $S_{0i} = \sqrt{\frac{m_i}{|G|}}$. Thus we can write

$$N_{\mu\nu}^\lambda = \sum_i \frac{S_{\mu i} S_{\nu i} S_{\lambda i}^*}{S_{0i}} \tag{17.13}$$

eq:fusionii

This is a prototype of a celebrated result in conformal field theory known as the "Verlinde formula."

Equations (17.10)(17.11)(17.13) give a very handy way to get the numbers $N_{\mu\nu}^\lambda$. Note that, by their very definition the coefficients $N_{\mu\nu}^\lambda$ are nonnegative integers, although this is hardly obvious from, say, (17.13).

18. Algebras

We first need the general abstract notion: **Definition** An *algebra* over a field k is a vector space V over k with a notion of multiplication of two vectors

♣First part with definitions was copied into chapter 2. No need to repeat here. ♣

$$V \times V \to V \tag{18.1}$$

denoted:

$$v_1, v_2 \in V \to v_1 \cdot v_2 \in V \tag{18.2}$$

which has a ring structure compatible with the scalar multiplication by the field. Concretely, this means we have axioms:

i.) $(v_1 + v_2) \cdot v_3 = v_1 \cdot v_3 + v_2 \cdot v_3$

ii.) $v_1 \cdot (v_2 + v_3) = v_1 \cdot v_2 + v_1 \cdot v_3$

iii.) $\alpha(v_1 \cdot v_2) = (\alpha v_1) \cdot v_2 = v_1 \cdot (\alpha v_2), \qquad \forall \alpha \in k.$

The algebra is *unital*, i.e., it has a unit, if $\exists 1_V \in V$ (*not* to be confused with the multiplicative unit $1 \in k$ of the ground field) such that:

iv.) $1_V \cdot v = v \cdot 1_V = v$

If, in addition, the product satisfies:

$$(v_1 \cdot v_2) \cdot v_3 = v_1 \cdot (v_2 \cdot v_3) \tag{18.3}$$

for all $v_1, v_2, v_3 \in V$ then V is called an *associative algebra.* In general, a *nonassociative algebra* means a not-necessarily associative algebra. In any algebra we can introduce the *associator*

$$[v_1, v_2, v_3] := (v_1 \cdot v_2) \cdot v_3 - v_1 \cdot (v_2 \cdot v_3) \tag{18.4}$$

eq:associator

Note that it is trilinear.

In general, if $\{v_i\}$ is a basis for the algebra then the *structure constants* are defined by

$$v_i \cdot v_j = \sum_k c_{ij}^k v_k \tag{18.5}$$

A *representation* of an algebra V is a vector space W and a map $T : V \to \mathrm{End}(W)$ so that

$$\begin{aligned} T(\alpha v + \beta w) &= \alpha T(v) + \beta T(w) \\ T(v_1 \cdot v_2) &= T(v_1) \circ T(v_2) \end{aligned} \tag{18.6}$$

eq:repalgebra

Example. $M_n(k)$ is a vector space over k of dimension n^2. It is also an associative algebra because matrix multiplication defines an algebraic structure. An important fact is that the only irreducible representation of the algebra $M_n(k)$ is the natural one acting on $W = k^{\oplus n}$. This leads to a concept called "Morita equivalence" of algebras.

Exercise

a.) Write out a basis and structure constants for the algebra $M_n(k)$.

b.) Show that the set of $n \times n$ matrices over an associative algebra, A, denoted $M_n(A)$, is itself an associative algebra.

Exercise

a.) If A is an algebra, then it is a module over itself, via the left-regular representation (LRR). $a \to L(a)$ where

$$L(a) \cdot b := ab \tag{18.7}$$

Show that if we choose a basis a_i then the structure constants

$$a_i a_j = c_{ij}{}^k a_k \tag{18.8}$$

define the matrix elements of the LRR:

$$(L(a_i))_j{}^k = c_{ij}^k \tag{18.9}$$

An algebra is said to be *semisimple* if these operators are diagonalizable.

b.) If A is an algebra, then it is a bimodule over $A \otimes A^o$ where A^o is the opposite algebra.

18.1 Coalgebras, Bialgebras, and Frobenius algebras

Definition A *Frobenius algebra* A is an associative algebra over a field k with a trace $\theta : A \to k$ such that the quadratic form $A \otimes A \to k$ given by $a \otimes b \to \theta(ab)$ defines a nondegenerate bilinear form on A.

Matrix algebras over a field are examples of Frobenius algebras where the nondegenerate bilinear form is the trace map. Frobenius algebras have nice properties with their duals.

The dual space of an algebra V^* has a structure known as a *coalgebra*. The dual of the multiplication is a comultiplication

$$\Delta : V^* \to V^* \otimes V^* \tag{18.10}$$

and the dual of associativity says $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$.

The dual of the unit is the counit $\epsilon : V^* \to k$. The dual of the axioms for the unit are

$$(\epsilon \otimes 1)\Delta = (1 \otimes \epsilon)\Delta = 1. \tag{18.11}$$

DIAGRAMS.

A *bialgebra* is an algebra which is simultaneously a coalgebra so that the structures are all compatible, that is, the comultiplication Δ and the counit ϵ are algebra homomorphisms.

Finally, a *Hopf* algebra is a bialgebra with an extra bit of structure $S : H \to H$. It is a generalization of the algebra of functions on a group and is a starting point for the theory of quantum groups.

Exercise

a.) In the literature one sometimes sees a Frobenius algebra defined as an algebra A with a nondegenerate bilinear form

$$\sigma : A \otimes A \to k \tag{18.12}$$ eq:frob

such that $\sigma(xy, z) = \sigma(y, zx)$. Show that this is equivalent to our definition.

b.) Show that if (A, θ) is a Frobenius algebra then the dual algebra A^* is a left A-module which is isomorphic to A as a left A-module.

18.2 When are two algebras equivalent? Introduction to Hochschild cohomology

Deformation - which are trivial? Nontrivial deformations measured by Hochschild coho. View as a flatness condition.

19. The group ring and group algebra

The group algebra is a useful tool for addressing the fundamental questions at the beginning of section ****

1. Useful for producing explicit reduction of matrix reps by the method of projection operators.

2. Useful for producing the representations of the symmetric group S_n.

3. Important and beautiful mathematical structure.

The regular representation is an algebra. Indeed, the space of complex-valued functions $Fun(X)$ on *any* space X is an algebra under pointwise multiplication:

$$(f_1 \cdot f_2)(x) := f_1(x) f_2(x) \tag{19.1}$$

Note that this is a commutative algebra. In particular, we can apply this remark to $X = G$.

Now, the *group ring*, usually denoted $\mathbb{Z}[G]$ is, as an abelian group the free abelian group generatred by G. That is, it is the space of formal linear combinations

$$\mathbb{Z}[G] := \{x = \sum_{g \in G} x_g g : x_g \in \mathbb{Z}\} \tag{19.2}$$

The ring structure is defined by taking

$$\delta_{g_1} *_2 \delta_{g_2} = \delta_{g_1 \cdot g_2} \tag{19.3}$$

and extending by linearity. Thus, we have

$$\begin{aligned} \left(\sum_{g\in G} x(g)g\right) *_2 \left(\sum_{g\in G} y(g)g\right) &:= \sum_{g,h\in G} x(g)y(h) \quad \overbrace{g\cdot h}^{\text{group multiplication}} \\ &= \sum_{k\in G}\left[\sum_{g,h\in G:gh=k} x(g)y(h)\right] k \end{aligned} \tag{19.4}$$

eq:groupalgmul

Replacing $\mathbb{Z} \to \mathbb{C}$ (more properly, taking the tensor product with $\mathbb{C}$) gives the group algebra $\mathbb{C}[G]$.

As a vector space $\mathbb{C}[G]$ is naturally dual to the regular representation: $R_G \cong \mathbb{C}[G]^*$. The natural pairing is

$$\langle f, x\rangle := \sum_{g\in G} x_g f(g) \tag{19.5}$$

Of course, there is a natural basis for $\mathbb{C}[G]$, namely the elements g themselves, and the dual basis is the basis of delta functions δ_g. Thus the isomorphism takes $f : G \to \mathbb{C}$ to $\sum_g f(g)g$.

Now we see that there are *two* algebra structures on R_G and $\mathbb{C}[G]$.

First of all, pointwise multiplication defines an algebra structure on $\mathbb{C}[G]$ since

$$\begin{aligned} f_1 *_1 f_2(g) &\equiv f_1(g)f_2(g) \Rightarrow \\ \delta_{g_1} *_1 \delta_{g_2} &= \delta_{g_1} \qquad \text{if} g_1 = g_2 \\ &= 0 \qquad\qquad \text{otherwise} \end{aligned} \tag{19.6}$$

eq:pointwise

which implies

$$g_1 *_1 g_2 = \begin{cases} g_1 & g_1 = g_2 \\ 0 & g_1 \neq g_2 \end{cases} \tag{19.7}$$

eq:pointwiseii

However, the multiplication (19.7) is *not* what is usually understood when one speaks of the group algebra, rather what is meant is the multiplication (19.4).

Under the isomorphism $R_G \cong \mathbb{C}[G]$ the product $*_2$ must correspond to a *second* algebra structure on R_G, and indeed there is one - called the *convolution product*.

$$(f_1 *_2 f_2)(g) := \sum_{h\in G} f_1(h)f_2(h^{-1}g) \tag{19.8}$$

eq:covolution

Note that $*_2$ is therefore, in general, a *noncommutative* algebra product, whereas $*_1$ is a *commutative* algebra structure.

Example $\mathbb{Z}_2 = \{1, \sigma\}$. The group algebra is

$$R_{\mathbb{Z}_2} = \{a \cdot 1 + b \cdot \sigma : a, b \in \mathbb{C}\} \tag{19.9}$$

Now the algebra structure is:

$$(a \cdot 1 + b \cdot \sigma) \cdot (a' \cdot 1 + b' \cdot \sigma) = (aa' + bb')1 + (ba' + ab')\sigma \tag{19.10}$$

These are the "double-numbers," a lesser known cousin of the complex numbers.

Exercise
Which elements of $R_{\mathbb{Z}_2}$ are invertible?

19.1 Projection operators in the group algebra

The group algebra is of course a representation of $G_{\text{left}} \times G_{\text{right}}$ acting as a left action:

$$T(g_1, g_2) \cdot x = g_1 x g_2^{-1} \tag{19.11}$$

for $x \in \mathbb{C}[G]$.

We know from the Peter-Weyl theorem that this representation is highly reducible. We can now construct explicit projection operators which project onto the irreducible components. We assume that the distinct irreps $T_{ij}^{(\mu)}$ are somehow known.

Let us define:

$$P_{ij}^{\mu} \in \mathbb{C}[G] \qquad 1 \le i, j \le n_\mu \tag{19.12}$$ eq:idemps

by

$$P_{ij}^{\mu} := \frac{n_\mu}{|G|} \sum_{g \in G} \hat{T}_{ij}^{(\mu)}(g) g \tag{19.13}$$ eq:projops

Recall that the dual representation is the transpose inverse of the original one so $\hat{T}_{ij}^{(\mu)}(g) = T_{ji}^{(\mu)}(g^{-1})$

These elements satisfy the simple product law:

$$P_{ij}^{\mu} P_{i'j'}^{\mu'} = \delta_{\mu\mu'} \delta_{ji'} P_{ij'}^{\mu} \tag{19.14}$$ eq:prodlaw

They are therefore *orthogonal projection operators* in the group algebra acting on itself.

Moreover, again using similar manipulations one can show that:

$$g \cdot P_{ij}^{\mu} = \sum_s T_{si}^{(\mu)}(g) P_{sj}^{\mu} \tag{19.15}$$ eq:geerep

$$P_{ij}^{\mu} \cdot g^{-1} = \sum_s \hat{T}_{sj}^{(\mu)}(g) P_{is}^{\mu} \tag{19.16}$$ eq:geerepii

and thus the projection operators P^{μ}_{ij} explicitly decompose $\mathbb{C}[G]$ into the irreps $T^{(\mu)} \otimes \hat{T}^{(\mu)}$ of $G_{\rm left} \times G_{\rm right}$. This will be useful below in showing how to decompose representations into irreps.

Exercise

Prove (19.14)(19.15) and (19.16) using the orthogonality conditions The manipulations are very similar to those used to prove theorem 10.1.

19.2 The center of the group algebra and the subalgebra of class functions

As we have noted, the group algebra $\mathbb{C}[G]$ is a noncommutative algebra. Correspondingly, R_G is a noncommutative algebra under the convolution product.

A little thought shows that the *center* of the group algebra $Z[\mathbb{C}[G]]$ is spanned by the elements $c_i = \sum_{g \in C_i} g$ where we sum over the conjugacy class C_i of g. This simply follows from

$$g c_i g^{-1} = \sum_{h \in C_i} g h g^{-1} = c_i \tag{19.17}$$

Correspondingly, the center of R_G under the convolution product are the class functions.

Indeed, we can now interpret the exercise with equation (15.18) above as saying that the characters satisfy:

$$\chi_\mu *_2 \chi_\nu = \frac{\delta_{\mu\nu}}{n_\mu} \chi_\mu \tag{19.18}$$

The product is commutative, and in fact, diagonal in this basis. On the other hand, the convolution product is nondiagonal in the conjugacy class basis:

$$\delta_{C_i} *_2 \delta_{C_j} = \sum_k M^k_{ij} \delta_{C_k} \tag{19.19}$$

It is interesting to compare with the pointwise product, where the reverse is true

$$\chi_\mu *_1 \chi_\nu = \sum_\lambda N^\lambda_{\mu\nu} \chi_\lambda \tag{19.20}$$

while

$$\delta_{C_i} *_1 \delta_{C_j} = \delta_{ij} C_i \tag{19.21}$$

So the unitary matrix $S_{\mu i}$ is a kind of Fourier transform which exchanges the product for which the characters of representations or characteristic functions of conjugacy classes is diagonalized:

$$\chi_\mu = \sum_i \sqrt{\frac{|G|}{m_i}} S_{i\mu} \delta_{C_i} \tag{19.22}$$

Exercise

Show that the structure constants in the basis δ_{C_i} in the convolution product can be written as:

$$M_{ij}^{k} = \frac{S_{0i}S_{0j}}{S_{0k}} \sum_{\mu} \frac{S_{i\mu}^{*}S_{j\mu}^{*}S_{k\mu}}{n_{\mu}} \tag{19.23}$$

eq:convolusc

Exercise

An algebra is always canonically a representation of itself in the left-regular representation. The representation matrices are given by the structure constants, thus

$$L(\chi_{\mu})_{\nu}{}^{\lambda} = N_{\mu\nu}^{\lambda} \tag{19.24}$$

shows that the fusion coefficients are the structure constants of the pointwise multiplication algebra of class functions.

Note that since the algebra of class functions is commutative the matrices $L(\chi_{\mu})$ can be simultaneously diagonalized.

Show that the unitary matrix $S_{i\mu}$ is in fact the matrix which diagonalizes the fusion coefficients.

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20. Interlude: 2D Topological Field Theory and Frobenius algebras

The group algebra illustrates the notions of bialgebra and Frobenius algebra very nicely. We would like to explain briefly the relation of these algebraic structures to two-dimensional *topological field theory.*

The axioms of topological field theory give a caricature of what one wants from a field theory: Spaces of states and transition amplitudes. By stripping away the many complications of "real physics" one is left with a very simple structure. Nevertheless, the resulting structure is elegant, it is related to beautiful algebraic structures which, at least in two dimensions, which have surprisingly useful consequences. This is one case where one can truly "solve the theory."

Of course, we are really interested in more complicated theories. But the basic framework here can be adapted to any field theory. What changes is the geometric category under consideration.

20.1 Geometric Categories

What are the most primitive things we want from a physical theory in d spacetime dimensions?

In a physical theory one often decomposes spacetime into space and time as in 7. If space is a $(d-1)$-dimensional manifold time Y then, in quantum mechanics, we associate to it a vector space of states $\mathcal{H}(Y_d)$.

Figure 7: A spacetime $X_d = Y \times \mathbb{R}$. Y is $(d-1)$-dimensional space. `fig:tfti`

In a generic physical theory the vector space has a lot of extra structure: It is a Hilbert space, there are natural operators acting on this Hilbert space such as the Hamiltonian. The spectrum of the Hamiltonian and other physical observables depends on a great deal of data. Certainly they depend on the metric on spacetime since a nonzero energy defines a length scale

$$L = \frac{\hbar c}{E} \tag{20.1}$$

In topological field theory one ignores most of this structure, and focuses on the dependence of $\mathcal{H}(Y)$ on the topology of Y. For simplicity, we will initially assume Y is compact.

So: We have an association:

$(d-1)$-manifolds Y to vector spaces: $Y \to \mathcal{H}(Y)$, such that homeomorphic manifolds map to isomorphic vector spaces.

Now, we also want to incorporate some form of locality, at the most primitive level. Thus, if we take disjoint unions

$$\mathcal{H}(Y_1 \amalg Y_2) = \mathcal{H}(Y_1) \otimes \mathcal{H}(Y_2) \tag{20.2}$$ `eq:disunion`

Note that (20.2) implies that we should assign to $\mathcal{H}(\emptyset)$ the field of definition of our vector space. For simplicity we will take $\mathcal{H}(\emptyset) = \mathbb{C}$, although one could use other ground fields.

Figure 8: Time development in X_d gives a linear map. `fig:tftitime`

In addition, in physics we want to speak of transition amplitudes. If there is a spacetime X_d interpolating between two time-slices, then mathematically, we say there is a cobordism between Y and Y'. That is, a *cobordism* from Y to Y' is a d-manifold with boundary and a partition of its boundary

$$\partial X_d = (\partial X_d)_{\rm in} \cup (\partial X_d)_{\rm out} \tag{20.3}$$

so that there is a homeomorphism $(\partial X_d)_{\rm in} \cong Y$ and $(\partial X_d)_{\rm out} \cong Y'$.

Here we will be considering *oriented* manifolds. Then an oriented cobordism from Y_1 to Y_2 must have

$$\partial M = Y_2 - Y_1 \tag{20.4}$$

where the minus sign indicates the opposite orientation on Y_1. (Recall that an orientation on M determines one on ∂M. We will adopt the convention "outward normal first" - ONF - "One Never Forgets.").

If X_d is an oriented cobordism from Y to Y' then the Feynman path integral assigns a linear transformation

$$F(X_d) : \mathcal{H}(Y) \to \mathcal{H}(Y'). \tag{20.5}$$

Again, in the general case, the amplitudes depend on much more than just the topology of X_d, but in topological field theory they are supposed only to depend on the topology. More precisely, if $X_d \cong X'_d$ are homeomorphic by a homeomorphism $= 1$ on the boundary of the cobordism, then

$$F(X_d) = F(X'_d) \tag{20.6}$$

One key aspect of the path integral we want to capture - again a consequence of locality - is the idea of summing over a complete set of intermediate states. In the path integral formalism we can formulate the sum over all paths of field configurations from t_0 to t_2 by composing the amplitude for all paths from t_0 to t_1 and then from t_1 to t_2, where $t_0 < t_1 < t_2$, and then summing over all intermediate field configurations at t_1. We refer to this property as the "gluing property." The gluing property is particularly obvious in the path integral formulation of field theories.

Figure 9: Gluing cobordisms fig:tftii

In topological field theory this is formalized as:

If M is a cobordism from Y_1 to Y_2 with

$$\partial M = Y_2 - Y_1 \tag{20.7}$$

and M' is another oriented cobordism from Y_2 to Y_3

$$\partial M' = Y_3 - Y_2 \tag{20.8}$$

then we can compose $M' \circ M$ as in 9 to get a cobordism from Y_1 to Y_3.

Naturally enough we want the associated linear maps to compose:

$$F(M \circ M') = F(M') \circ F(M) : \mathcal{H}(Y_1) \to \mathcal{H}(Y_3) \tag{20.9}$$

What we are describing, in mathematical terms is a functor between categories.

Definition A *functor* between categories C_1, C_2 is an association between categories which preserves structure. Thus there is a map on objects: $F : Obj(C_1) \to Obj(C_2)$. Moreover, we have a

a.) *Covariant functor* if $F : Arr(C_1) \to Arr(C_2)$ such that $F(a \to b)$ is an arrow from $F(a)$ to $F(b)$ for all $a, b \in Obj(C_1)$ and

$$F(\phi_2 \circ \phi_1) = F(\phi_2) \circ F(\phi_1) \tag{20.10}$$

b.) *Contravariant functor* if $F : Arr(C_1) \to Arr(C_2)$ such that $F(a \to b)$ is an arrow from $F(b)$ to $F(a)$ for all $a, b \in Obj(C_1)$ and

$$F(\phi_2 \circ \phi_1) = F(\phi_1) \circ F(\phi_2) \tag{20.11}$$

The above axioms can be concisely summarized by defining a cobordism category whose objects are homeomorphism classes of oriented $(d-1)$-manifolds and whose morphisms are oriented cobordisms, two morphisms being identified if they differ by a homeomorphism which is an identity on the boundary. Let us call this $\mathbf{Cob}(d)$.

Definition A d-dimensional topological field theory is a functor from the category $\mathbf{Cob}(d)$ to the category **VECT** of vector spaces and linear transformations.

Actually, these are both *tensor categories* and we want a functor of tensor categories.

At this point we begin to see how we can incorporate other field theories. We can change the geometric category to include manifolds with Riemannian metric, spin structure, etc.

20.2 Some general properties

Let us deduce some simple general facts following from the above simple remarks.

First note that if M is closed then it can be regarded as a cobordism from $\emptyset$ to $\emptyset$. Therefore $F(M)$ must be a linear map from $\mathbb{C}$ to $\mathbb{C}$. But any such linear map is canonically associated to a complex number. We define the *partition function* of M, $Z(M)$ to be this complex number.

There is one cobordism which is distinguished, namely $Y \times [0,1]$. This corresponds to a linear map $P : \mathcal{H}(Y) \to \mathcal{H}(Y)$. Indeed, physically it is just the operator

$$\exp[-TH] \tag{20.12}$$

where H is the Hamiltonian, and T is the Euclidean time interval (in some presumed metric).

Evidently, by the axioms of topological field theory, $P^2 = P$ and therefore we can decompose

$$\mathcal{H}(Y) = P\mathcal{H}(Y) \oplus (1-P)\mathcal{H}(Y) \tag{20.13}$$ eq:decompp

All possible transitions are zero on the second summand since, topologically, we can always insert such a cylinder. It follows that it is natural to assume that

$$F(Y \times [0,1]) = Id_{\mathcal{H}(Y)} \tag{20.14}$$ eq:identmap

One can think of this as the statement that the Hamiltonian is zero.

Figure 10: Deforming the cylinder. fig:dfscyl

Now, the cobordism (20.14) is closely related to the cobordism $\emptyset \to Y \cup -Y$ thus defining a map

$$\delta : \mathbb{C} \to \mathcal{H}(Y) \otimes \mathcal{H}(-Y) \tag{20.15}$$

and also to a cobordism $Y \cup -Y \to \emptyset$ thus defining a quadratic form:

$$Q : \mathcal{H}(Y) \otimes \mathcal{H}(-Y) \to \mathbb{C} \tag{20.16}$$

Figure 11: Composing $\delta \otimes 1$ and $1 \otimes Q$ fig:scobord

Let us now compose these cobordisms we get the identity map as in 11. It then follows from some linear algebra that Q is a *nondegenerate* pairing, so we have an isomorphism to the linear dual space:

$$\mathcal{H}(-Y) \cong \mathcal{H}(Y)^*, \tag{20.17}$$

under which Q is just the dual pairing. Moreover $\delta(1)$ has the explicit formula:

$$\delta : 1 \to \sum \phi_\mu \otimes \phi^\mu \tag{20.18}$$

where we can let ϕ_μ be any basis for $\mathcal{H}(Y)$, and then define the dual basis by $Q(\phi_\mu, \phi^\nu) = \delta_\mu{}^\nu$.

Exercise

Prove this formula. Write $\delta(1) = \sum \alpha^{\mu\nu} \phi_\mu \otimes \phi_\nu$ and compute the process in 9.

Figure 12: Composing Q with δ gives the dimension: $\dim\mathcal{H}(Y) = Z(Y \times S^1)$. fig:dimcirc

Now consider the diagram in 12. Geometrically this is a morphism from $\emptyset$ to $\emptyset$, so the functor maps this to a linear map from $\mathbb{C}$ to $\mathbb{C}$. Such a linear map is completely determined by a complex number, and this complex number is the composition $Q\delta$. From our formula for $\delta(1)$ we see that the value $Z(Y \times S^1)$ is just the dimension $\dim\mathcal{H}(Y)$. Thus, in topological field theories, the vector spaces are necessarily finite dimensional.

Note that if we change the category to the category of manifolds with Riemannian structure and we take the product Riemannian structure on $Y \times S^1$ then

$$Z(Y \times S^1) = \mathrm{Tr} e^{-\beta H} \tag{20.19}$$ eq:traccrc

where β is the radius of the circle and H is the Hamiltonian. If in addition we enrich the category to include spin structures then there will be several kinds of traces.

Exercise

Show that a 1-dimensional TFT is completely specified by choosing a single vector space V and a vector in that space.

Exercise

Show that any cobordism of $\emptyset$ to Y defines a *state* in the space $\mathcal{H}(Y)$. This is a primitive version of the notion of the "Hartle-Hawking" state in quantum gravity. It is also related to the state/operator correspondence in conformal field theory.

20.3 Two dimensional closed topological field theory and Commutative Frobenius Algebras

Some beautiful extra structure shows up when we consider the case $d = 2$, due to the relatively simple nature of the topology of 1-manifolds and 2-manifolds. To begin, we restrict attention to closed $(d-1)$-manifolds, that is, we consider a theory of closed strings.

In this case, the spatial $(d-1)$ manifolds are necessarily of the form $S^1 \cup S^1 \cup \cdots \cup S^1$, i.e. disjoint unions of n S^1's.

If we assign $\mathcal{H}(S^1) = \mathcal{C}$ then

$$\mathcal{H}(S^1 \amalg S^1 \amalg \cdots \amalg S^1) = \mathcal{C}^{\otimes n} \tag{20.20}$$

Thus, we have a single basic vector space.

Figure 13: Typical Riemann surface amplitude

fig:twodeei

Transition amplitudes can be pictured as in 13. We get

$$F(\Sigma) : \mathcal{C}^{\otimes n} \to \mathcal{C}^{\otimes m} \tag{20.21}$$

where Σ is a Riemann surface. There are n ingoing circles and m outgoing circles.

Now, topology dictates that the vector space $\mathcal{C}$ in fact must carry some interesting extra structure.

Figure 14: The sphere with 3 holes

fig:product

In 14 we see that the sphere with three holes defines a product

$$m : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \tag{20.22}$$

With the other orientation we get a co-product:

$$\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C} \tag{20.23}$$

In 15 we see that there is a trace:

$$\theta : \mathcal{C} \to \mathbb{C} \tag{20.24}$$

In 16 we see that there is a map $\mathbb{C} \to \mathcal{C}$. This is completely determined by its value on $1 \in \mathbb{C}$. From the diagram in 17 we see that the image of 1 must be in fact a unit for the muiltiplication.

Figure 15: the trace map fig:trace

Figure 16: The disk defines the unit. fig:unit

Figure 17: Showing that $1_{\mathcal{C}}$ really is the unit. fig:uniti

Moreover, from 18 we see the multiplication is associative.

Finally, we can make a diffeomorphism of the disk with 2 holes, holding the outer circle fixed and rotating the inner two circles. This shows that the product must be *commutative*.

Any oriented surface can be decomposed into the basic building blocks we have used above. However, the same surface can be decomposed in many different ways. When we have different decompositions we get algebraic relations on the basic data $m, \Delta, \theta_{\mathcal{C}}$. At this point you might well ask: "Can we get more elaborate relations on the algebraic data by cutting up complicated surfaces in different ways?" The beautiful answer is: "No, the above relations are the only independent relations." The algebraic structure we have discovered is just that of a *Frobenius algebra* of section *** !

The Sewing Theorem. To give a 2d topological field theory is equivalent to giving a commutative associative finite dimensional Frobenius algebra.

The proof (which is not difficult) depends on Morse theory and will not be given here. For a detailed discussion see, for example,

Figure 18: Associativity

fig:associativ

Figure 19: Commutativity. In the supercase we will have graded commutativity: $\phi_1\phi_2 = (-1)^{\deg\phi_1 \deg\phi_2}\phi_2\phi_1$.

fig:commutativ

G. Segal and G. Moore, "D-branes and K-theory in 2D topological field theory" hep-th/0609042; To appear in *Mirror Symmetry II*, Clay Mathematics Institute Monograph, by P. Aspinwall et. al.

20.4 Boundary conditions

Now let us enrich our theory by allowing the time-slices Y to be manifolds with boundary. There will be a set of boundary conditions $\mathcal{B}_0$, and we will attach an element of $\mathcal{B}_0$ to each boundary component of ∂Y.

A cobordism X from Y_0 to Y_1 will thus have two kinds of boundaries:

$$\partial X = Y_0 \cup Y_1 \cup \partial_{\mathrm{cstr}} X \tag{20.25}$$

where $\partial_{\mathrm{cstr}} X$ is the time-evolution of the spatial boundaries. We will call this the "constrained boundary."

Figure 20: Morphism space for open strings: $\mathcal{O}_{ab}$.

fig:openo

In $d = 2$, in this enlarged geometric category the initial and final state-spaces are associated with circles, as before, and now also with intervals. The boundary of each interval carries a label $a, b, c, \ldots$ from the set $\mathcal{B}_0$.

Definition: We denote the space $\mathcal{O}_{ab}$ for the space associated to the interval $[0,1]$ with label b at 0 and a at 1.

In the theory of D-branes, the intervals are open strings ending on submanifolds of spacetime. That is why we call the time-evolution of these boundaries the "constrained boundaries" – because the ends are constrained to live in the D-brane worldvolume.

Figure 21: Basic cobordism of open strings. fig:openi

Figure 22: A disk defining an element $1_a \in \mathcal{O}_{aa}$ fig:dsdki

Figure 23: The trace element: $\theta_a : \mathcal{O}_{aa} \to \mathbb{C}$. fig:tftiii

As in the closed case, the cobordism $[0,1] \times [0,1]$ defines $P_{ab} : \mathcal{O}_{ab} \to \mathcal{O}_{ab}$, and we can assume WLOG that it is $P_{ab} = 1$.

Now consider the cobordism in 21. This clearly gives us a bilinear map

$$\mathcal{O}_{ab} \times \mathcal{O}_{bc} \to \mathcal{O}_{ac} \tag{20.26}$$

As in the closed case we see that these maps satisfy an associativity law. Moreover, as in the closed case, 1_a is an identity for the multiplication.

Comparing with the definition of a category we see that we should interpret $\mathcal{B}_0$ as the space of objects in a category $\mathcal{B}$, whose morphism spaces $Hom(b,a) = \mathcal{O}_{ab}$. Note that the morphism spaces are vector-spaces, so we have a $\mathbb{C}$-linear category. In fact, this category has a very special property as we learn from considering the S-shaped cobordism (the open string analog of 11). We learn that $\theta_a : \mathcal{O}_{aa} \to \mathbb{C}$ defines a nondegenerate inner product:

$$Q_a(\psi_1, \psi_2) = \theta_a(\psi_1 \psi_2) \tag{20.27}$$

Thus, the $\mathcal{O}_{aa}$ are Frobenius algebras.

Moreover, using the S-shaped cobordism analogous to 11 we learn that $\mathcal{O}_{ab}$ is dual to $\mathcal{O}_{ba}$. In fact we have

$$\begin{aligned} \mathcal{O}_{ab} \otimes \mathcal{O}_{ba} \to \mathcal{O}_{aa} \overset{\theta_a}{\to} \mathbb{C} \\ \mathcal{O}_{ba} \otimes \mathcal{O}_{ab} \to \mathcal{O}_{bb} \overset{\theta_b}{\to} \mathbb{C} \end{aligned} \tag{20.28}$$

eq:dualpair

are perfect pairings with

$$\theta_a(\psi_1 \psi_2) = \theta_b(\psi_2 \psi_1) \tag{20.29}$$

eq:cyclic

for $\psi_1 \in \mathcal{O}_{ab}, \psi_2 \in \mathcal{O}_{ba}$.

Definition A *Frobenius category* is a $\mathbb{C}$-linear category in which there is a perfect pairing of $Hom(a,b)$ with $Hom(b,a)$ for all $a, b \in Ob(C)$ by a pairing which factorizes through the composition in either order.

Remark: It is important to note that the argument for commutativity fails in the open case: The algebras $\mathcal{O}_{aa}$ are in general *noncommutative.*

Figure 24: The open-closed transition maps

fig:openclosed

So, to give an open and closed TFT involves giving a Frobenius cateogry. But the open and closed strings must be related to each other. The essential new information is a pair of linear maps

$$\begin{aligned} \iota_a : \mathcal{C} \to \mathcal{O}_{aa} \\ \iota^a : \mathcal{O}_{aa} \to \mathcal{C} \end{aligned} \tag{20.30}$$

eq:linmaps

corresponding to the open-closed string transitions of 24.

By drawing pictures we can readily discover the following necessary algebraic conditions:

1. ι_a is an algebra homomorphism

Figure 25: Factorization of the open string loop on closed string exchange. Also known as the "Cardy condition." fig:cardycond

$$\iota_a(\phi_1\phi_2) = \iota_a(\phi_1)\iota_a(\phi_2) \tag{20.31}$$ eq:centerh

2. The identity is preserved

$$\iota_a(1_{\mathcal{C}}) = 1_a \tag{20.32}$$ eq:ident

3. Moreover, ι_a is central in the sense that

$$\iota_a(\phi)\psi = \psi\iota_b(\phi) \tag{20.33}$$ eq:center

for all $\phi \in \mathcal{C}$ and $\psi \in \mathcal{O}_{ab}$
4. ι_a and ι^a are adjoints:

$$\theta_{\mathcal{C}}(\iota^a(\psi)\phi) = \theta_a(\psi\iota_a(\phi)) \tag{20.34}$$ eq:adjoint

for all $\psi \in \mathcal{O}_{aa}$.

5. The "Cardy conditions."[7] Define $\pi_b{}^a : \mathcal{O}_{aa} \to \mathcal{O}_{bb}$ as follows. Since $\mathcal{O}_{ab}$ and $\mathcal{O}_{ba}$ are in duality (using θ_a or θ_b), if we let ψ_μ be a basis for $\mathcal{O}_{ba}$ then there is a dual basis ψ^μ for $\mathcal{O}_{ab}$. Then we define

$$\pi_b{}^a(\psi) = \sum_\mu \psi_\mu\psi\psi^\mu, \tag{20.35}$$ eq:dblttw

and we have the "Cardy condition":

$$\pi_b{}^a = \iota_b \circ \iota^a. \tag{20.36}$$ eq:cardycon

This is illustrated in 25.

Exercise

[7]These are actually generalization of the conditions stated by Cardy. One recovers his conditions by taking the trace. Of course, the factorization of the double twist diagram in the closed string channel is an observation going back to the earliest days of string theory.

Draw pictures associated to the other algebraic conditions given above.

Theorem Open-Closed Sewing Theorem. The above conditions are the complete set of sewing constraints on the algebraic data.

This is proved in the reference cited above.

Example: Let G be a finite group. In section 8 we saw that the class functions on G form a commutative Frobenius algebra. Now we take our category of boundary conditions $\mathcal{B}$ to be the category of finite dimensional complex representations V of G, with $\mathcal{O}_{VV} = End(V)$ and $\theta_V : \mathcal{O}_{VV} \to \mathbb{C}$ given by

$$\theta_V(\psi) = \frac{1}{|G|}\mathrm{Tr}(\psi) \tag{20.37}$$

Exercise

Write out the full set of open-closed string data for the example of a finite group and check the sewing conditions.

---

21. Applications of the Projection Operators

Let us return to the projection operators in $\mathbb{C}[G]$. In this section we will assume - as we may - that the irreps T^μ are unitary so

$$\hat{T}^{(\mu)}_{ij}(g) = (T^{(\mu)}_{ij}(g))^* \tag{21.1}$$

Thus we have

$$P^\mu_{ij} = \frac{n_\mu}{|G|}\sum_{g\in G}(T^{(\mu)}_{ij}(g))^* g \tag{21.2}$$ eq:projopsa

$$P^\mu_{ij}P^{\mu'}_{i'j'} = \delta_{\mu\mu'}\delta_{ji'}P^\mu_{ij'} \tag{21.3}$$ eq:prodlawa

$$g\cdot P^\mu_{ij} = \sum_s T^{(\mu)}_{si}(g)P^\mu_{sj} \tag{21.4}$$ eq:geerepa

$$P^\mu_{ij}\cdot g^{-1} = \sum_s (T^{(\mu)}_{sj}(g))^* P^\mu_{is} \tag{21.5}$$ eq:geerepiia

21.1 Decomposition of a representation into its isotypical parts

While characters tell us the irreps into which a representation can be decomposed, they do not tell us *how* to decompose the representation explicitly. To do this, we need the method of projection operators. We assume that the matrix elements $T^{(\mu)}_{ij}$ are known.

For simplicity we consider finite groups $\mid G \mid < \infty$, but the same techniques extend to compact groups.

Now, let $T(g)$ be any representation on V. Extending by linearity we obtain a representation of the algebra R_G. Define operators on V by applying T to (19.12) to get [8]

$$\mathcal{P}^{\mu}_{ij} = T(P^{\mu}_{ij}) = \frac{n_\mu}{|G|}\sum_{g\epsilon G}(T^{(\mu)}_{ij}(g))^* T(g) \tag{21.6}$$

Now we imediately get the properties (19.14), (19.15), (19.16) for the operators $\mathcal{P}^{\mu}_{ij}$. Therefore, if $\mathcal{P}^{lk}_{\mu}\vec{v} = \vec{w}_l \neq 0$ then

$$\{\vec{w}_j\} = \left\{\mathcal{P}^{jk}_{\mu}\vec{v}\right\}_{j=1,\cdots n_\mu} \tag{21.7}$$

span an invariant subspace of V transforming in the rep $T^{(\mu)}$ according to the representation T^{μ}. Using these operators we can in principle decompose a representation into its irreps.

Exercise *Calculating projection operators*

Now we return to the example of the 3-dimensional reducible representation of S_3 acting on $\mathbb{R}^3$ in section *** above.

Referring to the three irreducible representations of S_3, called $1_+, 1_-, 2$ in section **** show that:

$$\begin{aligned}
P_{1_+} &= \frac{1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} \qquad P_{1_+}^2 = P_{1_+}\\
P_{1_-} &= \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\\
P_{\mathbf{2}}^{1,1} &= \begin{pmatrix}0&0&0\\0&1/2&-1/2\\0&-1/2&1/2\end{pmatrix} \qquad (P_{\mathbf{2}}^{1,1})^2 = P_{\mathbf{2}}^{1,1}\\
P_{\mathbf{2}}^{2,1} &= \begin{pmatrix}0&1/\sqrt{3}&-1/\sqrt{3}\\0&-1/2\sqrt{3}&1/2\sqrt{3}\\0&-1/2\sqrt{3}&1/2\sqrt{3}\end{pmatrix}
\end{aligned} \tag{21.8}$$

[8] We choose $T^{(\mu)}_{ij}$ = unitary matrices for simplicity.

Therefore,

$$P_{\mathbf{1}+}\begin{pmatrix}x\\y\\z\end{pmatrix} = (\frac{x+y+z}{3})\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{21.9}$$

so:

$$V_1 = Span\left\{\vec{V}_1 = (1,1,1)\right\} = \{(x,x,x) : x \in \mathbb{R}\} \tag{21.10}$$

Similarly $P_{\mathbf{2}}$ projects onto the orthogonal subspace

$$P_{\mathbf{2}}^{1,1}\begin{pmatrix}x\\y\\z\end{pmatrix} = \frac{1}{2}(y-z)\begin{pmatrix}0\\1\\-1\end{pmatrix} \qquad P_{\mathbf{2}}^{2,1}\begin{pmatrix}x\\y\\z\end{pmatrix} = \frac{\sqrt{3}}{2}(y-z)\begin{pmatrix}2\\-1\\-1\end{pmatrix} \tag{21.11}$$

So:

$$V_2 = Span\left\{\vec{V}_2 = (0,1,-1)\vec{V}_3 = (2,-1,-1)\right\} \tag{21.12}$$

are the two invariant subspaces. Check

$$S = \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & -1\\ 2 & -1 & -1\end{pmatrix} \tag{21.13}$$

Conjugates the rep into block form. Thus, the decomposition into irreps is:

$$\mathbb{R}^3 \cong V_{\mathbf{1}+} \oplus V_{\mathbf{2}} \tag{21.14}$$ eq:expldec

---

An important shortcut: In general the characters of irreps are much easier to calculate than the matrix elements. Therefore the following route to decomposition is an important shortcut:

If

$$V \cong \oplus_\mu a_\mu T^{(\mu)} \tag{21.15}$$

then we can *easily* project onto the isotypical component as follows: Note that for unitary $T(g)$, $(\mathcal{P}_\mu^{ij})^* = \mathcal{P}_\mu^{ij}$, so

$$\mathcal{P}^\mu := \sum_{i=1}^{n_\mu} \mathcal{P}_{ii}^\mu \tag{21.16}$$ eq:charporj

form a system of *orthogonal, Hermitian, projection operators*:

$$\begin{aligned}(\mathcal{P}_\mu)^2 &= \mathcal{P}_\mu\\ (\mathcal{P}_\mu)^\dagger &= \mathcal{P}_\mu\\ \mathcal{P}_\mu\mathcal{P}_\nu &= 0 \qquad \mu \neq \nu\end{aligned} \tag{21.17}$$ eq:projopsaa

They project onto the invariant subspaces $V^{(\mu)}$.

The point is: from the definition (21.16) we get:

$$\mathcal{P}_\mu = \frac{n_\mu}{|G|} \sum_{g \in G} \overline{\chi_\mu(g)} T(g) \tag{21.18}$$

eq:calcpmu

so, we need only know the characters to compute the $\mathcal{P}_\mu$.

Example: In the above example, one easily computes

$$\mathcal{P}_{\mathbf{2}} = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{21.19}$$

21.2 Block diagonalization of Hermitian operators

Suppose H is an $N \times N$ Hermitian operator acting on the inner product space $\mathbb{C}^N$ with standard innder product. Suppose we have a group representation T of G with $\mathbb{C}^N$ as carrier space and that moreover, G is a symmetry of H in the sense that

$$\forall g \in G \qquad [T(g), H] = 0 \tag{21.20}$$

Then

$$[\mathcal{P}^\mu_{ij}, H] = 0 \tag{21.21}$$

Then, the projection operators P_λ onto the different eigenspaces of H commute with the different projection operators $\mathcal{P}^\mu_{ij}$ and in particular with $\mathcal{P}^\mu$.

Therefore, in this situation, by reducing $\mathbb{C}^N$ to the irreps of G, we have partially block-diagonalized H.

Of course, this is very useful in quantum mechanics where one wants to diagonalize Hermitian operators acting on wavefunctions.

21.2.1 Projecting quantum wavefunctions

Suppose G acts on X as a tranformation group, and $\{\psi_a\}$ is a collection of functions on X transforming according to some representation T of G. (For example, energy eigenfunctions in a Schrödinger problem.)

Then:

$$\psi^{ij}_{a,\mu}(x) \equiv \frac{n_\mu}{\mid G \mid} \sum_{h \epsilon G} (T^\mu_{ij}(h))^* \psi_a(h^{-1} \cdot x) \tag{21.22}$$

for fixed μ, j, a, letting $i = 1, \ldots, n_\mu$ be a collection of functions transforming according to the irrep T^μ: This is a special case of the general statement we made above. Here, again, is the explicit calculation:

$$
\begin{aligned}
\psi_a^{ij}(g^{-1}\cdot x) &= \frac{n_\mu}{\mid G\mid}\sum_{h\epsilon G}(T^\mu_{ij}(h))^*\psi_a(h^{-1}\cdot g^{-1}\cdot x) \\
&= \frac{n_\mu}{\mid G\mid}\sum_{h\epsilon G}(T^\mu_{ij}(g^{-1}h))^*\psi_a(h^{-1}\cdot x) \\
&= \sum_{s=1}^{n_\mu}(T^\mu_{is}(g^{-1}))^*\psi_a^{sj}(x) \\
&= \sum_{s=1}^{n_\mu}T^\mu_{si}(g)\psi_a^{sj}(x) \qquad \text{if } T^\mu \text{ is unitary.}
\end{aligned}
\tag{21.23}
$$

Example 2 As a somewhat trivial special case of the above consider $\mathbb{Z}$ acting on $\mathbb{R}$ $x \to -x$ therefore acts on $Fun(\mathbb{R})$. Decompose into irreps:

$\psi =$ function on $\mathbb{R}$

$$
\begin{aligned}
\psi^1(x) &= \frac{1}{2}(\psi(x)+\psi(-x)) \\
\psi^2(x) &= \frac{1}{2}(\psi(x)-\psi(-x))
\end{aligned}
\tag{21.24}
$$

transform acccording to the two irreps of $\mathbb{Z}_2$.

21.2.2 Finding normal modes in classical mechanics

Consider a classical mechanical system with degrees of freedom

$$
\vec{q} = (q^1,\dots,q^n) \tag{21.25}
$$

Suppose we have a generalized harmonic oscillator so that the kinetic and potential energies are:

$$
T = \frac{1}{2}m_{ij}\dot{q}^i\dot{q}^j \qquad\qquad V = \frac{1}{2}U_{ij}q^iq^j \tag{21.26}
$$

where U_{ij} is independent of q^i. We can obtain solutions of the classical equations of motion by taking

$$
q^i(t) = \mathrm{Re}(\gamma^i e^{i\omega t}) \tag{21.27}
$$

where

$$
(-\omega^2 m_{ij} + U_{ij})\gamma^j = 0 \tag{21.28}
$$

If a group G acts as $(T(g)\cdot q)^i = T(g)^{ij}q^j$ and

$$
[T(g),m] = [T(g),U] = 0 \tag{21.29}
$$

then we can partially diagonalize $H = -\omega^2 m + U$ by choosing $\vec{\gamma}$ to be in irreps of G.

Consider n beads on a ring connected by springs of the same force constant and hence the same frequency ω as in 26.

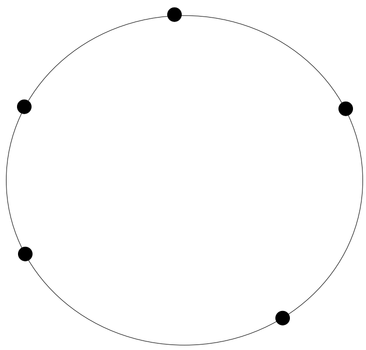

Figure 26: A system of beads and springs. fig:normalmode

The Lagrangian is

$$L = \frac{1}{2}m(\dot{q}^i)^2 - \frac{1}{2}k\sum_i (q^i - q^{i+1})^2 \tag{21.30}$$

where we understand the superscript on q^i to modulo n: $q^{i+n} = q^i$.

We will consider the system where the angle coordinates q^i have been lifted to the universal cover, i.e. $q^i \in \mathbb{R}$ so that we can consider $\vec{q}$ to be in a vector space. [9]

The normal mode problem is

$$(-m\omega^2 + kA)\gamma = 0 \tag{21.31}$$

where

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & \cdots & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 \\ -1 & 0 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix} \tag{21.32}$$

Thus, we reduce the problem of finding normal modes to the problem of diagonalizing this matrix.

The problem has an obvious $\mathbb{Z}_n$ symmetry where a generator $T(\omega)$ takes

$$T(\omega) : q^1 e_1 + \cdots q^n e_n \to q^1 e_2 + \cdots + q^n e_{n-1} + q^1 e_n \tag{21.33}$$

Here e_i is the standard basis for $\mathbb{C}^n$ so $T(\omega^\ell)e_i = e_{i+\ell}$ and again the subscript is understood modulo n.

We know the irreps of $\mathbb{Z}_n$: $T^\mu(\omega^j) = \omega^{j\mu}$, $\mu = 0, \ldots, n-1$. The projection operators are thus

$$\mathcal{P}^\mu = \frac{1}{n}\sum_{\ell=0}^{n-1} \omega^{-\mu\ell} T(\omega^\ell) \tag{21.34}$$

[9] This alters the problem physically. We should add a constraint which says, roughly speaking that $\sum(q^{i+1} - q^i) = 2\pi$. We will ignore this constraint in what follows.

In terms of matrix units e_{ij} we have

$$\mathcal{P}^\mu = \frac{1}{n}\sum_{i,\ell}\omega^{-\mu\ell}e_{i+\ell,i} \tag{21.35}$$

On the other hand,

$$A = 2 - \sum_i (e_{i,i+1} + e_{i+1,i}) \tag{21.36}$$

Now we carry out the multiplication:

$$A\mathcal{P}^\mu = 2\mathcal{P}^\mu - \sum_{i,\ell}\omega^{-\mu\ell}(e_{i+\ell,i+1} + e_{i+\ell,i-1}) \tag{21.37}$$

After some relabeling of the subscripts we easily find

$$\begin{aligned} A\mathcal{P}^\mu &= (2 - \omega^{-\mu} - \omega^{\mu})\mathcal{P}^\mu \\ &= (2\sin\frac{\pi\mu}{n})^2\mathcal{P}^\mu \end{aligned} \tag{21.38}$$

Similarly,

$$\mathcal{P}^\mu\vec{q} = \frac{1}{n}\sum_{\ell,s}\omega^{-\mu\ell}q^s e_{s+\ell} = \frac{1}{n}\sum_s \omega^{\mu s}q^s\left(\sum_j \omega^{-\mu j}e_j\right) \tag{21.39}$$

So the normal mode frequencies are

$$\omega_\mu^2 = 4\frac{k}{m}(2\sin\frac{\pi\mu}{n})^2 \tag{21.40}$$

and the normal modes are:

$$\vec{q}_{(\mu)} = \mathrm{Re}(\alpha_\mu e^{i\omega_\mu t}\sum_j \omega^{-\mu j}e_j) \tag{21.41}$$

Note that there is a *degeneracy* in the mode frequencies

$$\omega_\mu = \omega_{n-\mu}. \tag{21.42}$$

eq:funnydegen

It is a general principle that when you have unexplained degeneracies you should search for a further symmetry in the problem.

Indeed, in this case, we have overlooked a further symmetry:

$$q^i \to q^{n-i} \tag{21.43}$$

This leads to D_n symmetry of the problem which explains the degeneracy (21.42).

Exercise

Using the character tables above for D_n construct the normal modes which are in representations of D_n.

---

22. Representations of the Symmetric Group

22.1 Conjugacy classes in S_n

Recall from chapter one that the conjugacy classes in S_n are labeled by partitions of n: A cycle decomposition has the form

$$(1)^{\nu_1}(2)^{\nu_2}\cdots(n)^{\nu_n} \tag{22.1}$$

where ν_j is the number of cycles of length j. Clearly since we must account for all n letters being permuted:

$$n = \nu_1 + 2\nu_2 + \cdots + n\nu_n = \sum j\nu_j \tag{22.2}$$

To every cycle decomposition we can associate a partition as follows:

$$\begin{aligned}
\lambda_1 &:= \nu_1 + \nu_2 + \cdots + \nu_n \\
\lambda_2 &:= \nu_2 + \cdots + \nu_n \\
\lambda_3 &:= \nu_3 + \cdots + \nu_n \\
&\cdots\cdots \\
\lambda_n &:= \nu_n
\end{aligned} \tag{22.3}$$

eq:lambdaTOnu

Note

1. $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$
2. $\sum \lambda_i = n$
3. Given a partition of n we can recover the cycle decomposition $\prod (j)^{\nu_j}$ since the transformation (22.3) is invertible over the integers: $\nu_j = \lambda_j - \lambda_{j+1}$.

As we saw when discussing symmetric functions, to a partition of n we can uniquely associate a picture, called a *Young diagram*, consisting of rows of λ_i boxes, for a total of n boxes.

classes, and hence the irreps of S_n, are in 1-1 correspondence with the partitions of n and hence in 1-1 correspondence with Y

22.2 Young tableaux

Young diagrams can be used to define projection operators onto the irreducible representations in the group algebra $\mathbb{C}[S_n]$ using the following procedure. We only give the barest minimal description here. See the references at the end of this section for a full account.

Define a *Young tableau* to be a Young diagram in which each of the integers $1, 2, \ldots, n$ has been placed into a box. These integers can be placed in any order. A *standard tableau* is one in which the integers are increasing in each row, left to right, and in each column, top to bottom. For any given partition there are $n!$ different Young tableau.

Let T denote a Young tableau. Then we define two subgroups of S_n, $R(T)$ and $C(T)$. $R(T)$ consists of permutations in S_n which only permute numbers within each row of T.

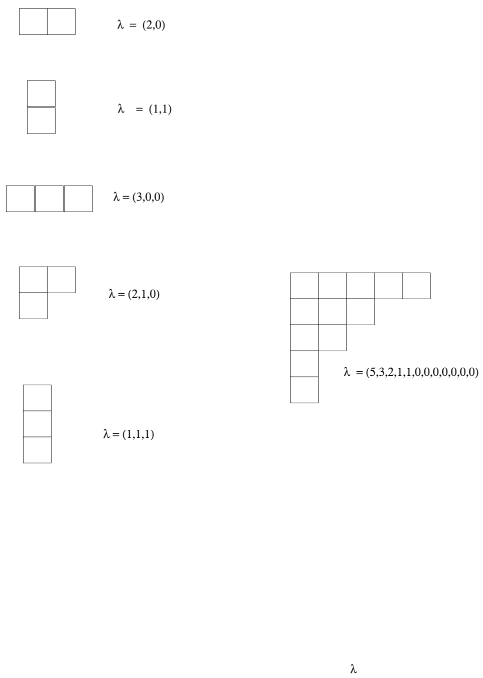

Figure 27: Examples of Young diagrams. fig:youngone

Similarly, $C(T)$ consists of permutations which only permute numbers within each column of T.

For a Young tableau T define the following elements in the group ring $\mathbb{Z}[S_n]$:

$$P := \sum_{p \in R(T)} p \tag{22.4}$$

$$Q := \sum_{q \in C(T)} \epsilon(q) q \tag{22.5}$$

$$P(T) := PQ = \sum_{p \in R(T), q \in C(T)} \epsilon(q) pq \tag{22.6}$$

Then we have the following rather nontrivial statements:

1. $P(T)^2 = c_T P(T)$ where c_T is an integer.

2. $P(T)P(T') = 0$ if T and T' correspond to different partitions.

3. If T, T' are two different standard tableaux corresponding to the same partition then $P(T)P(T') = 0$.

4. $P(T)$ projects onto an irreducible representation of S_n. That is $\mathbb{C}[S_n] \cdot P(T)$ transforms as an irreducible representation $R(T)$ of S_n in the left-regular representation. All of the irreducible representations are equivalent to one of these. Representations $R(T)$ for different partitions are inequivalent, while two representations for the same partition are equivalent.

5. The integer c_T in $P(T)^2 = c_T P(T)$ is given by

$$c_T = \frac{n!}{\text{dim} R(T)} \tag{22.7}$$

dim$R(T)$ can also be characterized as the number of standard tableaux corresponding to the underlying partition.

6. Another formula for dim$R(T)$ is the *hook length formula.* For a box in a Young diagram define its hook length to be the number of squares to the right in its row, and underneath in its column, counting that box only once. Then

$$\dim R(T) = \frac{n!}{\prod \text{hooklengths}} \tag{22.8}$$

Figure 28: Four standard Young tableaux for S_3. fig:exmplesthr

22.2.1 Example 1: $G = S_3$

There are 3 partitions, 3 Young diagrams, and 4 standard tableaux shown in 28.

Clearly $P(T_1) = \sum_{p \in S_3} p$ and $P(T_1)^2 = 6P(T_1)$. Moreover $\mathbb{Z}[S_3] \cdot P(T_1)$ is one-dimensional. This is the trivial representation.

Similarly, $P(T_4) = \sum \epsilon(q) q$ spans the one-dimensional sign representation.

There are two standard tableaux corresponding to $\lambda = (2,1)$ with

$$P(T_2) = (1 + (12))(1 - (13)) = 1 + (12) - (13) - (132) \tag{22.9}$$

eq:projteetwo

$$P(T_3) = (1 + (13))(1 - (12)) = 1 - (12) + (13) - (123) \tag{22.10}$$

One checks that $P(T_2)^2 = 3P(T_2)$ and $P(T_3)^2 = 3P(T_3)$ and $P(T_2)P(T_3) = 0$. These are consistent with the above statements about c_T and the hooklength formula.

Now consider $\mathbb{Z}[S_3] \cdot P(T_2)$. We compute

$$\begin{aligned}
(12) \cdot P(T_2) &= P(T_2) := v_1 \\
(13) \cdot P(T_2) &= -1 + (13) - (23) + (123) := v_2 \\
(23) \cdot P(T_2) &= -(12) + (23) - (123) + (132) = -v_1 - v_2 \\
(123) \cdot P(T_2) &= -1 + (13) - (23) + (123) = v_2 \\
(132) \cdot P(T_2) &= -(12) + (23) - (123) + (132) = -v_1 - v_2
\end{aligned} \tag{22.11}$$

Thus the space is two-dimensional.

From this it is easy to compute: $(12) \cdot v_2 = -v_1 - v_2$, $(13)v_2 = v_1$ and $(23)v_2 = (123)P(T_2) = v_2$ and so in this basis we have a representation generated by

$$\rho(12) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \qquad \rho(13) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \rho(23) = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \tag{22.12}$$

One easily checks that the character is the same as that we computed above for the **2**.

Of course, similar statements hold for $P(T_3)$, which the reader should check as an exercise.

Figure 29: 10 standard Young tableaux for S_4. fig:exmplesthr

22.2.2 Example 2: $G = S_4$

References

1. For further discussion of the above material see the books by Miller, Hammermesh, Curtis+Reiner, Fulton+Harris *Representation Theory.*

2. There is a second, elegant method for constructing the irreps of the symmetric group using induced representations. See the book by Sternberg for an account.

23. Symmetric groups and tensors: Schur-Weyl duality and the irreps of $GL(d, k)$

Consider a vector space V of dimension d.

In previous sections we have seen that

$$V^{\otimes n} \equiv V \otimes V \otimes \cdots \otimes V \tag{23.1}$$

is a representation of S_n by permuting the factors:

$$\sigma \cdot \ (w_1 \otimes w_2 \otimes \cdots \otimes w_n) = w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes \cdots \otimes w_{\sigma(n)} \tag{23.2}$$

*** ?? should we use σ^{-1} here ?? ***

Now recall that we also characterized vectors in $V^{\otimes n}$ as tensors. If V is a representation of a matrix subgroup G of $GL(d, \mathbb{R})$ or $GL(d, \mathbb{C})$ then $V^{\otimes n}$ is also a representation of G via

$$\rho(g) \cdot (w_1 \otimes w_2 \otimes \cdots \otimes w_n) = g \cdot w_1 \otimes g \cdot w_2 \otimes \cdots \otimes g \cdot w_n \tag{23.3}$$

For example, suppose that $V = k^d$ is also the fundamental representation of $GL(d,\mathbb{R})$ or $GL(d,\mathbb{C})$ for $k = \mathbb{R}$ or $k = \mathbb{C}$. Given a basis $\{v_i\}$ for V a typical element can be expanded in the basis as:

$$v = \sum_{i_1,i_2,\dots,i_n} t^{i_1,i_2,\dots,i_n} v_{i_1} \otimes \cdots \otimes v_{i_n} \tag{23.4}$$

We will often assume the summation convention where repeated indices are automatically summed. Under the action of $GL(d,k)$:

$$g \cdot v_i = g_{ji} v_j \tag{23.5}$$

we therefore have:

$$(g \cdot t)^{i_1 \cdots i_n} = t^{j_1 \cdots j_n} g_{j_1 i_1} \cdots g_{j_n i_n} \tag{23.6}$$

eq:tensor

that is, elements in $V^{\otimes n}$ transform as tensors and $V^{\otimes n}$ is *also* a representation of $GL(d,\mathbb{R})$: On the other hand

$$(\sigma \cdot t)^{i_1 \dots i_n} = t^{i_{\sigma^{-1}(1)} \cdots i_{\sigma^{-1}(n)}} \tag{23.7}$$

Now, note that the actions of $GL(d,k)$ and S_n *commute*, so that $V^{\otimes n}$ is a representation of $GL(d,k) \times S_n$.

Considered as a representation of S_n, we have complete reducibility:

$$V^{\otimes n} \cong \oplus_\lambda \mathcal{V}_\lambda \otimes R_\lambda \tag{23.8}$$

where R_λ is the irrep of S_n corresponding to the partition λ. By Schur's lemma we can characterize the multiplicity space $\mathcal{V}_\lambda$ as

$$\mathcal{V}_\lambda = \mathrm{Hom}_{S_n}(R_\lambda, V^{\otimes n}). \tag{23.9}$$

This is the vector space of S_n-equivariant maps from R_λ to $V^{\otimes n}$.

Now, because $GL(d,k)$ and S_n commute we know that $\mathcal{V}_\lambda$ will also be a representation of $GL(d,k)$.

The beautiful result of Schur and Weyl – known as the Schur-Weyl duality theorem – is

Theorem The representations $\mathcal{V}_\lambda$ are irreducible representations of $GL(d,k)$ and, up to a power of the determinant representation, all irreducible representations may be obtained in this way. Moreover, these statements hold for $U(d) \subset GL(d,\mathbb{C})$.

Proof For complete proofs see Segal, Miller, Fulton-Harris. We will just sketch the idea from Segal's lectures.

We can see fairly easily why all the representations of $U(d)$ must occur if we accept the Peter-Weyl theorem. Note that the representations which we obtain from $V^{\otimes d}$ have matrix elements which are algebraic expressions in the matrix elements a_{ij}. Indeed, they lie in $\mathbb{C}[a_{ij}]$ (the a_{ij} occur individually of course from V itself). But this forms a dense subalgebra of $L^2(U(d))$.

Now let us prove the $\mathcal{V}_\lambda$ is irreducible. Recall that the center of a matrix algebra $End(W)$ is just $\mathbb{C}$. Now we note that

$$\mathrm{End}_{S_n}(V^{\otimes n}) = ((\mathrm{End}(V))^{\otimes n})^{S_n} \tag{23.10}$$

Since $Gl(d)$ forms a dense open set in $End(V)$, the operators $T(g)$ acting on $V^{\otimes n}$ generate an algebra which is all of $\mathrm{End}_{S_n}(V^{\otimes n})$. However, this means that the operators that commute with both G and S_n, i.e.

$$\mathrm{End}_{G\times S_n}(V^{\otimes n}) \tag{23.11}$$

must lie in the center of $\mathrm{End}_{S_n}(V^{\otimes n})$. On the other hand,

$$\mathrm{End}_{S_n}(V^{\otimes n}) = \oplus_\lambda \mathrm{End}(\mathcal{V}_\lambda) \tag{23.12}$$

and

$$\mathrm{End}_{G\times S_n}(V^{\otimes n}) = \oplus_\lambda \mathrm{End}_G(\mathcal{V}_\lambda \tag{23.13}$$

Therefore $\mathrm{End}_G(\mathcal{V}_\lambda$ must be contained in the center of $\mathrm{End}(\mathcal{V}_\lambda)$, but the latter is just $\mathbb{C}$. Therefore $\mathrm{End}_G(\mathcal{V}_\lambda$ is just $\mathbb{C}$. But, by Schur's lemma, this must mean that $\mathcal{V}_\lambda$ is irreducible. ♠

We can construct $\mathcal{V}_\lambda$ explicitly by taking a Young diagram and its corresponding symmetrizer $P(T)$ (for some tableau T) and taking the image of $P(T)$ acting on $V^{\otimes n}$. This projects onto tensors of a definite symmetry type. These tensors will transform in irreducible representations of $GL(d,k)$. As representations of $GL(d,k)$ we have

$$P(T)V^{\otimes n} \cong P(T')V^{\otimes n} \tag{23.14}$$

if T and T' correspond to the same partition, i.e. the same Young diagram.

With some work it can be shown that - as representations of $GL(d,k)$ we have an orthogonal decomposition

$$V^{\otimes n} = \oplus_T P(T)V^{\otimes n} \tag{23.15}$$

where T runs over the *standard* tableaux of S_n. Of course, tableaux with columns of length $\geq d$ will project to the zero vector space and can be omitted.

Example

• If we take the partition $\lambda = (n, 0, 0, \dots)$ then $P(\lambda)$ projects to totally symmetric tensors and we get $S^n(V)$. Weyl's theorem tells us that if $V = k^d$ is the fundamental representation of $GL(d,k)$ then $S^n(V)$ is an irreducible representation. We computed before that its dimension is

$$\mathrm{dim} S^n(V) = \binom{n+d-1}{n} \tag{23.16}$$

Note in particular that we have infinitely many irreducible representations.

• If we take the partition $\lambda = (1, 1, \ldots, 1)$ then $P(\lambda)V^{\otimes n} = \Lambda^n V$ is the subspace of totally antisymmetric tensors of dimension

$$\dim\Lambda^n(V) = \binom{d}{n} \tag{23.17}$$

Note that this subspace vanishes unless $\dim V = d \geq n$.

• Let V be of dimension d and consider $V^{\otimes 3}$ as a representation of S_3. Using characters we showed above that

$$V^{\otimes 3} \cong \frac{d(d+1)(d+2)}{6}\mathbf{1}_+ \oplus \frac{d(d-1)(d-2)}{6}\mathbf{1}_- \oplus \frac{d(d+1)(d-1)}{3}\mathbf{2} \tag{23.18}$$

Clearly, the first and second summands correspond to the totally symmetric and totally antisymmetric tensors, respectively.

Applying the projector (22.9) to a tensor $t^{ijk}v_i \otimes v_j \otimes v_k$ produces a tensor with mixed symmetry:

$$P(T_2)v_i \otimes v_j \otimes v_k = v_i \otimes v_j \otimes v_k + v_j \otimes v_i \otimes v_k - v_k \otimes v_j \otimes v_i - v_k \otimes v_i \otimes v_j \tag{23.19}$$

and therefore the components are of the form:

$$t^{ijk} + t^{jik} - t^{kji} - t^{jki} \tag{23.20}$$

for arbitrary t^{ijk}. This tensor space is the space of tensors $\tilde{t}^{ijk}$ which satisfy the identities:

$$\tilde{t}^{ijk} + \tilde{t}^{jki} + \tilde{t}^{kij} = 0 \tag{23.21}$$ eq:cyclsym

together with

$$\tilde{t}^{ijk} = -\tilde{t}^{kji} \tag{23.22}$$ eq:iksymm

The first set of equations (23.21) cuts out a space of dimension $\frac{2}{3}d(d^2-1)$. The next set of equations (23.22) cuts it down by half so we get $\frac{1}{3}d(d^2-1)$ as the dimension of this space of tensors.

Similarly,

$$P(T_3)v_i \otimes v_j \otimes v_k = v_i \otimes v_j \otimes v_k - v_j \otimes v_i \otimes v_k + v_k \otimes v_j \otimes v_i - v_j \otimes v_k \otimes v_i \tag{23.23}$$

and therefore the components are of the form:

$$t^{ijk} - t^{jik} + t^{kji} - t^{kij} \tag{23.24}$$

for arbitrary t^{ijk}. Thus, this tensor space is the space of tensors $\tilde{t}^{ijk}$ which satisfy the identities:

$$\tilde{t}^{ijk} + \tilde{t}^{jki} + \tilde{t}^{kij} = 0 \tag{23.25}$$

$$\tilde{t}^{ijk} = -\tilde{t}^{jik} \tag{23.26}$$

Remarks The algebra of operators $\mathcal{A}$ generated by linear combinations of σ acting on $V^{\otimes n}$ commutes with the algebra of operators $\mathcal{B}$ generated by $g \in GL(d,k)$ acting on $V^{\otimes n}$. In fact these algebras are full commutants of each other: Any operator commuting with $\mathcal{A}$ is in $\mathcal{B}$ and vice versa.

23.1 Free fermions on a circle and Schur functions

Figure 30: Fermi sea and particle/hole excitations. fig:Fermisea

Consider a system of N free fermions on a circle $z = e^{i\theta}$, $\theta \sim \theta + 2\pi$, with Hamiltonian $H = -\sum_i \frac{d^2}{d\theta_i^2}$. The one-particle wavefunctions are

$$\psi_n(z) = z^n \tag{23.27}$$ eq:onpart

with energy n^2. Consider a state of N fermions occupying levels n_i. By Fermi statistics these levels must all be distinct and we can assume

$$n_N > n_{N-1} > \cdots > n_1 \tag{23.28}$$

The N-fermion wavefunction is the "Slater determinant"

$$\Psi_{\vec{n}}(z_1, \dots, z_N) = \det_{1\leq i,j\leq N} z_i^{n_j} \tag{23.29}$$ eq:slater

and has energy $\sum_i n_i^2$. States can be visualized as in 30.

For simplicity let us assume N is odd. Then there is a unique groundstate obtained by filling up the states $-n_F \leq n \leq n_F$ with

$$n_F = \frac{N-1}{2} \tag{23.30}$$ eq:fermil

(If N is even there are two groundstates.) This defines the Fermi sea. The ground state wavefunction is

$$\Psi_{gnd}(z_1, \dots, z_N) = \det_{1\leq i,j\leq N} z_i^{j-1-n_F} = (z_1 \dots z_N)^{-n_F} \Delta_0(z) \tag{23.31}$$ eq:grnd

The ratio of the wavefunction to groundstate wavefunction

$$\frac{\Psi_{\vec{n}}}{\Psi_{gnd}} = \frac{\det_{1\leq i,j\leq N} z_i^{n_j}}{\det_{1\leq i,j\leq N} z_i^{j-1-n_F}} \tag{23.32}$$ eq:srat

is a totally symmetric function of the z_i. If $n_1 \geq -n_F$ then it is a symmetric polynomial known as a *Schur function*.

The Schur functions form a linear basis for $\mathbb{Z}[x_1,\dots,x_N]^{S_N}$ where basis elements are associated with partitions. To define them mathematically let us return to a monomial $x^\alpha = x_1^{\alpha_1}\cdots x_N^{\alpha_N}$ with the $\alpha_i \geq 0$ and now try to skew-symmetrize:

$$\sum_{\sigma\in S_N} \epsilon(\sigma)\sigma\cdot x^\alpha \tag{23.33}$$

Note that this function is totally antisymmetric, and moreover vanishes unless the α_i are all distinct. WLOG assume that

$$\alpha_N > \alpha_{N-1} > \cdots > \alpha_1 \geq 0. \tag{23.34}$$

The "smallest" such α is

$$N-1 > N-2 > \cdots > 1 > 0 \geq 0 \tag{23.35}$$

defining a vector $\delta := (N-1, N-2, \dots, 1, 0)$ i.e. $\delta_j = j-1$, $1\leq j\leq N$. Using δ we define λ by

$$\alpha = \lambda + \delta \tag{23.36}$$

Note that $\lambda_j = \alpha_j - \delta_j \geq 0$ and moreover

$$\lambda_j + j - 1 > \lambda_{j-1} + j - 2 \qquad \Rightarrow \qquad \lambda_j \geq \lambda_{j-1} \tag{23.37}$$

Thus, λ is a partition. It is more convenient to use λ than α so we define:

$$\Delta_\lambda(x) := \sum_{\sigma\in S_N} \epsilon(\sigma)\sigma\cdot x^{\lambda+\delta} \tag{23.38}$$

Note that this can also be written as a determinant:

$$\begin{aligned}\Delta_\lambda(x) &:= \det(x_i^{\lambda_j+j-1})\\ &= \det\begin{pmatrix} x_1^{\lambda_1} & x_1^{\lambda_2+1} & \cdots & x_1^{\lambda_N+N-1}\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ x_N^{\lambda_1} & x_N^{\lambda_2+1} & \cdots & x_N^{\lambda_N+N-1}\end{pmatrix}\end{aligned} \tag{23.39}$$

eq:deltsun

Note that $\Delta_0(x)$ is the famous Vandermonde determinant:

$$\Delta_0(x) := \prod_{i>j}(x_i - x_j) \tag{23.40}$$

Exercise
Show that

$$\Delta_\lambda(\sigma\cdot x) = \epsilon(\sigma)\Delta_\lambda(x) \tag{23.41}$$

where $\epsilon(\sigma)$ is the sign homomorphism.

Now we define the *Schur function* to be the ratio:

$$\Phi_\lambda(x) := \frac{\Delta_\lambda(x)}{\Delta_0(x)} \tag{23.42}$$ eq:schurf

This is a totally symmetric function. In fact, it is a polynomial in the x_i. To see this consider it as a meromorphic function of a complex variable x_1. Note that it is in fact an entire function since the potential poles at $x_1 = x_i$ are cancelled by zeroes of the numerator. Next note that the growth at infinity is obviously x_1^m for an integer m.

Example. Put $N = 2$. Then $\lambda_2 \geq \lambda_1 \geq 0$,

$$\Delta_\lambda(x) = (x_1 x_2)^{\lambda_1}(x_2^{\lambda_2-\lambda_1+1} - x_1^{\lambda_2-\lambda_1+1}) \tag{23.43}$$ eq:deltatwo

$$\Phi_\lambda(x_1, x_2) = (x_1 x_2)^{\lambda_1} \frac{x_2^{\lambda_2-\lambda_1+1} - x_1^{\lambda_2-\lambda_1+1}}{x_2 - x_1} = (x_1 x_2)^{\lambda_2} h_{\lambda_2-\lambda_1}(x_1, x_2) \tag{23.44}$$ eq:phitwo

Returning to the free fermion example, we have

$$\frac{\Psi_{\vec{n}}}{\Psi_{\vec{0}}} = \frac{\Delta_\lambda(z)}{\Delta_0(z)} \tag{23.45}$$ eq:fration

for

$$\begin{aligned} \alpha_j &= n_j + n_F \\ &= \lambda_j + j - 1 \end{aligned}, \tag{23.46}$$

that is

$$\lambda_j = (n_j + n_F) - (j - 1) \tag{23.47}$$ eq:snet

Remarks

• The second main theorem of symmetric polynomials is:

Theorem. The Schur functions $\Phi_\lambda(x)$ form a linear integral basis for $\mathbb{Z}[x_1, \ldots, x_N]^{S_N}$.

That is, any symmetric polynomial with integral coefficients can be written as a linear combination, with integral coefficients, of the $\Phi_\lambda(x)$. Note that we are *not* forming polynomials in the Φ_λ.

• The module A_N of skew-symmetric polynomials in $x_1, \ldots, x_N$ is isomorphic to Λ_N via multiplication by Δ_0. Therefore the Δ_λ form a a $\mathbb{Z}$-basis for A_N. This close relation between completely symmetric and antisymmetric functions comes up in the theory of matrix models – integrals over space of $N \times N$ matrices. It also suggests a relation between bosons and fermions, at least in $1+1$ dimensions. That indeed proves to be the case – there is a nontrivial isomorphism known as *bosonization* which is an isomorphism of quantum field theories of bosons and fermions in 1+1 dimensions.

•

There is an elegant expression for the Schur functions Φ_λ as a determinant of a matrix whose entries involve the h_k. The expression is

$$\Phi_\lambda = \det(h_{\lambda_i - i + j}) \tag{23.48}$$ eq:phiach

where we take an $N \times N$ matrix for N variables and it is understood that $h_0 = 1$ and $h_i = 0$ for $i < 0$. Equivalently

$$\Phi_\lambda = \det(e_{\lambda'_i - i + j}) \tag{23.49}$$ eq:phiacha

For a proof see Macdonald, p.41.

Example Consider the partition $(\lambda_1, \lambda_2, 0, 0, \dots)$. Then applying (23.48) we get

$$\begin{aligned}\Phi_\lambda &= \begin{pmatrix} h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} & h_{\lambda_1+3} & h_{\lambda_1+4} & \cdots \\ h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} & h_{\lambda_2+2} & h_{\lambda_2+3} & \cdots \\ 0 & 0 & 1 & h_1 & h_2 & \cdots \\ 0 & 0 & 0 & 1 & h_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \\ &= h_{\lambda_1} h_{\lambda_2} - h_{\lambda_1+1} h_{\lambda_2-1}\end{aligned} \tag{23.50}$$ eq:phitwoa

With a little algebra one can confirm that this is indeed (23.44). Note, however, that both sides of this equality make sense for $N > 2$.

Figure 31: A matrix for defining Schur functions. fig:symsq

• Define the $\mathbb{Z} \times \mathbb{N}$ matrix Ξ whose $(p,q)^{th}$ entry is h_{q-p} where $h_0 = 1$ and $h_k = 0$ for $k < 0$. Here $-\infty < p < \infty$ labels the rows, while $q = 1, 2, \dots$ labels the columns. Thus the matrix looks like 31.

Define the integers $s_k = k - \lambda_k$. These eventually become $s_k = k$. Then Φ_λ is the determinant of the matrix formed from Ξ by keeping only the rows labeled by s_k. This leads to an upper triangular matrix is the determinant of the finite matrix above.

• The definition of Φ_λ only makes sense as a function of N such that $\lambda_i = 0$ for $i > N$. However, the relations between Φ_λ and the h_i are universal.

Exercise
Verify that (23.50) and (23.44) are equal for $N = 2$.

Exercise
a.) Let $\lambda = (n, 0, 0, 0, \dots)$. Show that $\Phi_\lambda = h_n$.
b.) Let $\lambda = (1^n, 0, 0, \dots)$. Then $\Phi_\lambda = e_n$.

Exercise
Using Cauchy's determinant identity:

$$\det\left(\frac{1}{1 - x_i y_j}\right) = \frac{\Delta_0(x_i)\Delta_0(y_j)}{\prod_{i,j}(1 - x_i y_j)} \tag{23.51}$$

Show that

$$\exp\left[\sum_k \frac{1}{k} s_k(x) s_k(y)\right] = \sum_\lambda \Phi_\lambda(x)\Phi_\lambda(y) \tag{23.52}$$

Hint: Multiply both sides by $\Delta_0(x)\Delta_0(y)$. For the answer see Hammermesh, p. 195

Exercise *Laughlin's wavefunctions*
Another natural place where symmetric wavefunctions appear is in the quantum Hall effect, where the 1-body wavefunctions in the lowest Landau level are

$$\psi_n(z) = \frac{1}{\sqrt{\pi n!}} z^n e^{-\frac{1}{2}|z|^2} \tag{23.53}$$

Laughlin introduced a fascinating set of approximate eigenfunctions of 2d *interacting* electrons in a magnetic field:

$$\Psi = \prod_{i<j}(z_i - z_j)^{2n+1} e^{-\frac{1}{2}\sum |z_i|^2} \tag{23.54}$$

Express these in terms of Schur functions.
Many other interesting trial wavefunctions in the FQHE can be generated using theorems about symmetric functions.

23.1.1 Schur functions, characters, and Schur-Weyl duality

Now it is interesting to combine Schur functions with Schur-Weyl duality.

We have seen that irreducible representations of $GL(d,\mathbb{C})$ (and of $U(d)$) can be labeled by Young diagrams with $\leq d$ rows. We called these S_λ above.

It turns out that the $\Phi_\lambda(x)$ are characters of the representations S_λ. That is if $g \in GL(d,\mathbb{C})$ can be diagonalized to $Diag\{x_1,\dots,x_d\}$ then

$$\mathrm{Tr}_{\mathcal{V}_\lambda}\rho(g) = \Phi_\lambda(x_1,\dots x_d) \tag{23.55}$$

This is known as the Weyl character formula and will be derived later.

The relation between the Φ_λ and the power functions $s_k(x)$ is a very nice application of the Schur-Weyl duality theorem.

Suppose that a group element $\sigma \in S_N$ has cycle decomposition $(1)^{\ell_1}(2)^{\ell_2}\cdots$ where $\sum j\ell_j = N$. As we have discussed this determines a conjugacy class $C[\ell]$ as well as a partition. Now, suppose $g \in GL(d,\mathbb{C})$ is diagonalizable to $\mathrm{Diag}\{x_1,\dots,x_d\}$ and let us evaluate

$$\mathrm{Tr}_{V^{\otimes N}}(\sigma g) \tag{23.56}$$ eq:keytrace

where $V = \mathbb{C}^d$ is the fundamental representation of $GL(d,\mathbb{C})$.

We will evaluate (23.56) in two ways. On the one hand, it is clear that this should just be

$$(\mathrm{Tr}(g))^{\ell_1}(\mathrm{Tr}(g^2))^{\ell_2}(\mathrm{Tr}(g^3))^{\ell_3}\cdots \tag{23.57}$$ eq:prodns

On the other hand, since

$$V^{\otimes N} \cong \oplus_\lambda S_\lambda \otimes R_\lambda \tag{23.58}$$

the trace must also be

$$\sum_\lambda \chi_\lambda(\sigma)\Phi_\lambda(x) \tag{23.59}$$

Thus, recalling the symmetric functions which are the power sum functions $s_j = \sum_i x_i^j$ we have the following identity due to Frobenius:

Let $(\ell) = (\ell_1,\ell_2,\dots,\ell_N)$ denote a tuple of nonnegative integers. Then let

$$s_{(\ell)} := s_1^{\ell_1}s_2^{\ell_2}\dots s_N^{\ell_N} \tag{23.60}$$

Then:

$$\boxed{s_{(\ell)} = \sum_{\{\lambda\}} \chi_\lambda(C[\ell])\Phi_\lambda} \tag{23.61}$$ eq:frobenius

Returning to the free fermion interpretation we see that the operator

$$\mathcal{O}(\ell) = \prod_j (\mathrm{Tr}(g^j))^{\ell_j} \tag{23.62}$$

acts on the groundstate wavefunction $\Psi_{gnd}(x)$ to produce the quantum states:

$$\mathcal{O}(\ell)\Psi_{gnd}(x) = \sum_{\lambda} \chi_{\lambda}(C[\ell])\Psi_{\vec{n}} \tag{23.63}$$ eq:transamp

where the n_i are the fermion occupation numbers corresponding to the partition associated to λ as in (23.47). Thus the characters of the symmetric group can be regarded as "transition amplitudes" in the free fermion problem!

Indeed, we can now write the character table of the symmetric group S_N:

$$\chi_{\lambda}(C[\ell]) = \left[\Delta_0(x)s_{(\ell)}(x)\right]_{\alpha_1,\dots,\alpha_d} \tag{23.64}$$ eq:charactsymm

where the subscript means we extract the coefficient of $x_1^{\alpha_1}\cdots x_d^{\alpha_d}$ and we recall that

$$\alpha_j = \lambda_j + j - 1 \tag{23.65}$$

This is the famous formula of Frobenius.

Remark: If we think of Φ_λ as a character of a representation of $U(N)$. Then the Hamiltonian H is closely related to the quadratic Casimir of that representation. That in turn is related to the solution of two-dimensional Yang-Mills theory. See Cordes et. al. eq. (4.3)

Exercise *Dimensions of irreps of S_N*

a.) Using the Frobenius formula show that the character of 1, and hence the dimension of R_λ is

$$\dim R_\lambda = \frac{N!}{\alpha_1!\cdots\alpha_N!}\prod_{i>j}(\alpha_i - \alpha_j) \tag{23.66}$$

b.) Using this formula, derive the hook-length formula.

Warning: This is hard.

Exercise *Weyl dimension formula*

Show that

$$\Phi_\lambda(1,1,\dots,1) = \prod_{1\le i<j\le N}\frac{(\lambda_i - \lambda_j + j - i)}{(j-i)} \tag{23.67}$$ eq:weyldim

We will see later that this is a special case of the Weyl dimension formula.

Hint: Put $x_j = e^{jt}$ and let $t \to 0$.

References:

There are a lot of beautiful connections here with quantum field theory. In particular there are close connections to two-dimensional Yang-Mills theory and to the phenomenon of bosonization in two dimensions. See

For much more about this see the references below.

1. Pressley and Segal, *Loop Groups*, Chapter 10.

2. M. Stone, "Schur functions, chiral bosons, and the quantum-Hall-effect edge states," Phys. Rev. **B42** 1990)8399 (I have followed this treatment.)

3. S. Cordes, G. Moore, and S. Ramgoolam, "Lectures on 2D Yang-Mills Theory, Equivariant Cohomology and Topological Field Theories," Nucl. Phys. B (Proc. Suppl 41) (1995) 184, section 4. Also available at hep-th/9411210.

4. M. Douglas, "Conformal field theory techniques for large N group theory," hep-th/9303159.

5. Papers of Jimbo, Miwa, ... (Kyoto school) on Toda theory.

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FOLLOWING MATERIAL SHOULD BE MOVED TO CHAPTER ON REPRESENTATIONS. IT WOULD MAKE MORE SENSE TO TALK ABOUT REPS OF THE SYMMETRIC GROUP FIRST.

23.2 Bosons and Fermions in 1+1 dimensions

23.2.1 Bosonization

Finally, we want to describe a truly remarkable phenomenon, that of bosonization in 1+1 dimensions.

Certain quantum field theories of bosons are equivalent to quantum field theories of fermions in 1+1 dimensions! Early versions of this idea go back to Jordan [10] The subject became important in the 1970's. Two important references are

S. Coleman, Phys. Rev. **D11**(1975)2088

S. Mandelstam, Phys. Rev. **D11** (1975)3026

The technique has broad generalizations, and plays an important role in string theory.

To get some rough idea of how this might be so, let us consider the *loop group* $S^1 \to U(1)$. For winding number zero, loop group elements can be written as:

$$z \to g(z) = \exp[i \sum_{n=-\infty}^{+\infty} j_n z^n] \tag{23.68}$$

with $j_n^* = j_{-n}$. This group acts on the one-body wavefunction by

$$z^n \to g(z) z^n \tag{23.69}$$

where $g(z)$ is in the loop group of $U(1)$.

Under such a change of one-body wavefunctions the Slater determinant changes by:

$$\begin{aligned} \Delta_0(z) &\to \det g(z_i) z_i^{j-1} \\ &= \det \begin{pmatrix} g(z_1) & g(z_1)z_1 & g(z_1)z_1^2 & \cdots \\ g(z_1) & g(z_2)z_2 & g(z_2)z_2^2 & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix} \\ &= \prod g(z_i)\Delta_0(z) \\ &= \exp[i \sum j_n s_n(z)]\Delta_0(z) \end{aligned} \tag{23.70}$$

eq:coherentone

Note that there are two Fermi levels, and if N is large they are "far apart" meaning that operators such as $\sum_i z_i^n$ for small n will not mix states near the two respective levels. Let us therefore imagine taking N large and focussing attention on one of the Fermi levels. Therefore we adjust our energy level so that the groundstate wavefunctions are

[10] P. Jordan, Z. Phys. **93**(1935)464.

$1, z^{-1}, z^{-2}, \ldots$ and we imagine that we have taken $N \to \infty$. Moreover, let us extend the $LU(1)$ action to $L\mathbb{C}^*$. Then we can separately consider the action of

$$g_-(z) = \exp[i \sum_{-\infty}^{0} \varphi_n z^n] \tag{23.71}$$

and

$$g_+(z) = \exp[i \sum_{1}^{\infty} \varphi_n z^n] \tag{23.72}$$

on the groundstate Ψ_0. Note if we think of the groundstate wavefunction as a Slater determinant then acting with $g_-(z)$ takes one column to a linear combination of lower columns and hence does not change the wavefunction. On the other hand, by (23.70) acting with $g_+(z)$ has a nontrivial action and generates all possible symmetric functions of the z_i.

In this - rather heuristic - sense, the action of the loop group "generates the whole Hilbert space of fermionic states." Moreover, by mapping antisymmetric wavefunctions to symmetric wavefunctions we now view the Hilbert space as the space of polynomials in the s_n. [11]

Now we observe the following. The ring of symmetric functions (extending scalars to $\mathbb{C}$) is the polynomial ring $\mathbb{C}[s_1, s_2, \ldots]$. We now make this space into a *Hilbert space*. We introduce the inner product on polynomials in s_i by

$$\langle f(s)|g(s)\rangle := \int \prod_{k=1}^{\infty} \frac{ds_k \wedge d\bar{s}_k}{2\pi i k} \overline{f(s)} g(s) e^{-\sum_{k=1}^{\infty} \frac{1}{k}|s_k|^2} \tag{23.73}$$ eq:innerpf

This is the coherent state representation of an infinite system of harmonic oscillators:

$$[a_k, a_j] = k\delta_{k+j,0} \qquad -\infty < j, k < \infty \tag{23.74}$$

with $a_{-j} = a_j^\dagger$. These are realized as follows: $a_k^\dagger$ is multiplication by s_k and a_k is $k\frac{\partial}{\partial s_k}$. The state

$$(a_1^\dagger)^{\ell_1}(a_2^\dagger)^{\ell_2}\cdots(a_N^\dagger)^{\ell_N}|0\rangle \tag{23.75}$$

corresponds to the symmetric function $s_{(\ell)}$.

Now one can form the quantum field:

$$\phi := i\sum_{k\neq 0} \frac{1}{k} a_k z^{-k} \tag{23.76}$$

Then the coherent states

$$|\phi(z)\rangle = \exp i \oint \phi(z) j(z)|0\rangle \tag{23.77}$$

give an action of the loopgroup on the bosonic Fock vacuum acting as

[11] This statement is rather loose. See the Pressley-Segal book for a mathematically precise treatment. We are discussing an irreducible representation of a centrally extended group.

$$\exp[i\sum_{n=1}^{\infty} j_n a_n^\dagger]|0\rangle \qquad (23.78)$$

eq:coherenttwo

thus producing a nontrivial isomorphism between a bosonic and fermionic Fock space.

To make the isomorphism between bosonic and fermionic Fock spaces more explicit we introduce a second-quantized formalism. B_{-n}^{+} creates a fermionic state with wavefunction z^n, B_n annihilates it, so we introduce:

$$\begin{aligned}\Psi(\theta) &= \sum_{n\in\mathbb{Z}} B_n e^{in\theta}\\ \Psi^\dagger(\theta) &= \sum_{n\in\mathbb{Z}} B_{-n}^{+} e^{-in\theta}.\end{aligned} \qquad (23.79)$$

eq:fermflds

The filled Fermi sea satisfies the constraints:

$$\begin{aligned} B_n|0\rangle &= 0, \qquad |n| > n_F\\ B_{-n}^{+}|0\rangle &= 0, \qquad |n| \le n_F.\end{aligned} \qquad (23.80)$$

eq:fifer

When we have decoupled systems it is appropriate to define two independent sets of Fermi fields:

$$\begin{aligned}\Psi(\theta) &= e^{i(n_F+\frac{1}{2})\theta} b(\theta) + e^{-i(n_F+\frac{1}{2})\theta}\bar{b}(\theta)\\ \Psi^\dagger(\theta) &= e^{-i(n_F+\frac{1}{2})\theta} c(\theta) + e^{i(n_F+\frac{1}{2})\theta}\bar{c}(\theta).\end{aligned} \qquad (23.81)$$

eq:bcsys

We introduce complex coordinates $z = e^{i\theta}$, and define the mode expansions:

$$\begin{aligned} b(z) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} b_n z^n\\ c(z) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} c_n z^n\\ \bar{b}(z) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} \bar{b}_n \bar{z}^n\\ \bar{c}(z) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} \bar{c}_n \bar{z}^n\end{aligned} \qquad (23.82)$$

eq:bcsysi

$b_n, c_n, \bar{b}_n, \bar{c}_n$ are only unambiguously defined for $\mid n \mid << N$. That is, we focus on operators that do not mix excitations around the two Fermi levels. The peculiar half-integral moding is chosen to agree with standard conventions in CFT. In terms of the original nonrelativistic modes we have:

$$\begin{aligned} c_n &= B_{-n_F-\epsilon+n}^{+}\\ b_n &= B_{n_F+\epsilon+n}\\ \bar{c}_n &= B_{n_F+\epsilon-n}^{+}\\ \bar{b}_n &= B_{-n_F-\epsilon-n}\end{aligned} \qquad (23.83)$$

eq:mapmodes

where $\epsilon = \frac{1}{2}$, so that

$$\{b_n, c_m\} = \delta_{n+m,0} \qquad \{\bar{b}_n, \bar{c}_m\} = \delta_{n+m,0} \tag{23.84}$$ eq:anticomms

and all other anticommutators equal zero.

We may now reinterpret the fields $b, c, ...$ Defining $z = e^{i\theta+\tau}$ we see that these may be extended to fields in two-dimensions, and that they are (anti-) chiral, that is, they satisfy the two-dimensional Euclidean Dirac equation. Upon continuation to Minkowski space we have $z \to e^{i(\theta+t)}$ and $\bar{z} \to e^{i(\theta-t)}$. We are thus discussing two relativistic massless Fermi fields in $1+1$ dimensions. The b, c fields generate a Fermionic Fock space built on the product of vacua $\mid 0\rangle_{bc} \otimes \mid \bar{0}\rangle_{\bar{b}\bar{c}}$ where $b_n \mid 0\rangle_{bc} = c_n \mid 0\rangle_{bc} = 0$ for $n > 0$. Explicitly, the b, c fields by themselves generate the Fock space:

$$\mathcal{H}_{bc} = \text{Span}\left\{\prod b_{n_i} \prod c_{m_i} \mid 0\rangle_{bc}\right\} \qquad . \tag{23.85}$$ eq:cftstspc

The space $\mathcal{H}_{bc}$ has a natural grading according to the eigenvalue of $\sum_{n\in\mathbb{Z}} : b_n c_{-n} = \oint bc$ (called "bc-number"):

$$\mathcal{H}_{bc} = \oplus_{p\in\mathbb{Z}} \mathcal{H}_{bc}^{(p)} \tag{23.86}$$ eq:grdspce

and the states obtained by moving a state from the Fermi sea to an excited level correspond to the subspace of zero bc number:

$$\mathcal{H}_{\text{chiral}} := \mathcal{H}_{bc}^{(0)} \tag{23.87}$$ eq:chrlspc

NEED TO GIVE HERE THE TRANSLATION BETWEEN a state corresponding to a Young Diagram and an explicit fermionic Fock state.

The bc system defines a conformal field theory. Indeed we may introduce

$$L_n = \sum_{m=-\infty}^{\infty} (n/2+m) c_{-m} b_{m+n} \tag{23.88}$$ eq:frmvrop

and compute that the L_n satisfy the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0} \tag{23.89}$$ eq:virasoroalg

with $c = 1$.

Now let us consider the second quantized operators corresponding to z^n. Before taking the limit these are given by:

$$\Upsilon_n = \sum z_i^n \qquad \longleftrightarrow \qquad \int d\theta \Psi^\dagger(\theta)\, e^{in\theta} \Psi(\theta). \tag{23.90}$$ eq:upsop

Now let us take the large N limit and focus on terms that do not mix the two Hilbert spaces of excitations around the two Fermi levels. Cross terms between barred and unbarred fields involve operators that mix the two Fermi levels. Since we are only interested in the case of the decoupled Fermi level excitations we may replace:

$$
\begin{aligned}
\Upsilon_n &\to \oint dz\ z^{-1-n} c\ b(z) + \oint d\bar{z}\ \bar{z}^{-1+n} \bar{c}\ \bar{b}(\bar{z}) \\
&= \sum_m c_{n-m} b_m + \bar{c}_{m-n} \bar{b}_{-m} \\
&= \alpha_n + \bar{\alpha}_{-n}
\end{aligned}
\tag{23.91}
$$
eq:upsii

where we have introduced a field $bc = i\partial_z \phi(z)$ which has expansion

$$
\begin{aligned}
\partial_z \phi(z) &= i \sum_{m\in\mathbb{Z}} \alpha_m z^{m-1} \\
[\alpha_m, \alpha_n] = [\bar{\alpha}_m, \bar{\alpha}_n] &= m\delta_{m+n,0} \\
[\alpha_m, \bar{\alpha}_n] &= 0.
\end{aligned}
\tag{23.92}
$$
eq:bosalg

In terms of α_n, we may introduce a representation of the Virasoro algebra:

$$
L_n = \frac{1}{2} \sum \alpha_{n-m} \alpha_m \tag{23.93}
$$
eq:virop

which satisfy (23.89), again with $c = 1$. Using the α we can define a vacuum $\alpha_n \mid 0\rangle = 0$ for $n \geq 0$ and a statespace

$$
\mathcal{H}_\alpha = \text{Span}\left\{ \mid \vec{k}\rangle \equiv \prod (\alpha_{-j})^{k_j} \mid 0\rangle \right\}. \tag{23.94}
$$
eq:achalph

Bosonization states that there is a natural isomorphism:

$$
\mathcal{H}_\alpha \cong \mathcal{H}_{bc}^{(0)} \tag{23.95}
$$
eq:bsnztin

We will not prove this but it can be made very plausible as follows. The Hilbert space may be graded by L_0 eigenvalue. The first few levels are:

$$
\begin{array}{lll}
L_0 = 1 & \{ b_{-\frac{1}{2}} c_{-\frac{1}{2}} \mid 0\rangle \} & \{ \alpha_{-1} \mid 0\rangle \} \\
L_0 = 2 & \{ b_{-\frac{1}{2}} c_{-\frac{3}{2}} \mid 0\rangle, b_{-\frac{3}{2}} c_{-\frac{1}{2}} \mid 0\rangle \} & \{ \alpha_{-2} \mid 0\rangle, (\alpha_{-1})^2 \mid 0\rangle \}
\end{array}
\tag{23.96}
$$
eq:firstlev

At level $L_0 = n$, the fermion states are labeled by Young diagrams Y with n boxes. At level $L_0 = n$, the Bose basis elements are labeled by partitions of n. We will label a partition of n by a vector $\vec{k} = (k_1, k_2, \dots)$ which has almost all entries zero, such that $\sum_j j k_j = n$. Bosonization states that the two bases are linearly related:

$$
\mid Y\rangle = \sum_{\vec{k}\in \text{Partitions}(n)} \langle \vec{k} \mid Y\rangle \mid \vec{k}\rangle \tag{23.97}
$$
eq:linrel

This last relation can be understood as the relation (23.61).
****

1. Need to explain this some more.

2. Look up papers of the Kyoto school, Jimbo et. al. for perhaps helpful ways of presenting this material.

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For much more about this see the references below.

1. Pressley and Segal, *Loop Groups*, Chapter 10.

2. M. Stone, "Schur functions, chiral bosons, and the quantum-Hall-effect edge states," Phys. Rev. **B42** 1990)8399 (I have followed this treatment.)

3. S. Cordes, G. Moore, and S. Ramgoolam, "Lectures on 2D Yang-Mills Theory, Equivariant Cohomology and Topological Field Theories," Nucl. Phys. B (Proc. Suppl 41) (1995) 184, section 4. Also available at hep-th/9411210.

4. M. Douglas, "Conformal field theory techniques for large N group theory," hep-th/9303159.

Next time – include

Verma modules – how the more standard treatment of SU(2) reps fits in.

basis

$$f_m(z) = \frac{z^{j+m}}{\sqrt{(j+m)!(j-m)!}} \leftrightarrow \quad | j, m > \tag{23.98}$$

for $-j \leq m \leq j$

is orthonormal wrt:

$$(f, g) = \frac{(2j+1)!}{\pi} \int d^2 z \ \ \frac{1}{(1+ \mid z \mid^2)^{2j+1}} \ \ f^*_{(z)} g(z) \tag{23.99}$$