Chapter 3: Transformations Groups, Orbits, And Spaces Of Orbits

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Abstract: This chapter focuses on of group actions on spaces, group orbits, and spaces of orbits. Then we discuss mathematical symmetric objects of various kinds. April 27, 2018

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1. Introduction

2. Definitions and the stabilizer-orbit theorem

Let X be any set (possibly infinite). Recall the definition from chapter 1:

♣LEFT G-ACTIONS WERE INTRODUCED AT THE BEGINNING OF CHAPTER 1 ♣

Definition 1: A *permutation* of X is a 1-1 and onto mapping $X \to X$. The set S_X of all permutations forms a group under composition.

Definition 2a: A *transformation group* on X is a subgroup of S_X.

This is an important notion so let's put it another way:

Definition 2b:

A *G-action on a set X* is a map $\phi : G \times X \to X$ compatible with the group multiplication law as follows:

A *left-action* satisfies:

$$\phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x) \tag{2.1}$$

A *right-action* satisfies

$$\phi(g_1, \phi(g_2, x)) = \phi(g_2 g_1, x) \tag{2.2}$$

In addition in both cases we require that

$$\phi(1_G, x) = x \tag{2.3}$$ eq:identfix

for all $x \in X$.

A set X equipped with a (left or right) G-action is said to be a G-set.

Remarks:

1. If ϕ is a left-action then it is natural to write $g \cdot x$ for $\phi(g, x)$. In that case we have

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x. \tag{2.4}$$

   Similarly, if ϕ is a right-action then it is better to use the notation $\phi(g, x) = x \cdot g$ so that

$$(x \cdot g_2) \cdot g_1 = x \cdot (g_2 g_1). \tag{2.5}$$

2. If ϕ is a left-action then $\tilde{\phi}(g, x) := \phi(g^{-1}, x)$ is a right-action, and vice versa. Thus there is no essential difference between a left- and right-action. However, in computations with nonabelian groups it is extremely important to be consistent and careful about which choice one makes.

3. A given set X can admit more than one action by the same group G. In that case, to avoid confusion one should take care to distinguish the G-actions unless a particular one is understood. For example one could write $\phi_g(x) = \phi(g, x)$ and speak of ϕ_g, rather than write $g \cdot x$.

There is some important terminology one should master when working with G-actions:

Definitions:

1. A group action is *effective* or *faithful* if for any $g \neq 1$ there is *some* x such that $g \cdot x \neq x$. Equivalently, the only $g \in G$ such that ϕ_g is the identity transformation is $g = 1_G$.

2. A group action is *transitive* if for any pair $x, y \in X$ there is some g with $y = g \cdot x$.

3. A point $x \in X$ is a *fixed point* of G if there exists an element $g \in G$ with $g \neq 1$ such that $g \cdot x = x$.

4. Given a point $x \in X$ the set of group elements:

$$\mathrm{Stab}_G(x) := \{g \in G : g \cdot x = x\} \tag{2.6}$$

   is called the *isotropy group at x*. It is also called the *stabilizer group* of x. It is often denoted G^x. The reader should show that $G^x \subset G$ is in fact a subgroup.

5. Given a group element $g \in G$ the *fixed point set* of G is the set

$$\mathrm{Fix}_X(g) := \{x \in X : g \cdot x = x\} \tag{2.7}$$

   The fixed point set of g is often denoted by X^g.

6. A group action is *free* if for any $g \neq 1$ then for *every* x, we have $g \cdot x \neq x$. That is, for every x the stabilizer group G^x is the trivial subgroup $\{1_G\}$. Equivalently, for every $g \neq 1$ the set $\mathrm{Fix}(g)$ is the empty set.

7. The *orbit of G through a point* x is the set of points $y \in X$ which can be reached by the action of G:

$$O_G(x) = \{y : \exists g \quad \text{such that} \quad y = g \cdot x\} \tag{2.8}$$

Remarks:

1. If we have a G-action on X then we can define an equivalence relation on X by defining $x \sim y$ if there is a $g \in G$ such that $y = g \cdot x$. (Check this is an equivalence relation!) The orbits of G are then exactly the equivalence classes of under this equivalence relation.

2. The group action restricts to a transitive group action on any orbit.

3. If x, y are in the same orbit then the isotropy groups G^x and G^y are conjugate subgroups in G. Therefore, to a given orbit, we can assign a definite *conjugacy class* of subgroups.

Point 3 above motivates the

Definition If G acts on X a *stratum* is a set of G-orbits such that the conjugacy class of the stabilizer groups is the same. The set of strata is sometimes denoted $X \parallel G$.

Exercise

Recall that a group action of G on X can be viewed as a homomorphism $\phi : G \to S_X$. Show that the action is effective iff the homomorphism is injective.

Exercise

Suppose X is a G-set.

a.) Show that the subset H of elements which act ineffectively, i.e. the set of $h \in G$ such that $\phi(h, x) = x$ for all $x \in X$ is a normal subgroup of G.

b.) Show that G/H acts effectively on X.

Exercise

Let G act on a set X.

a.) Show that the stabilizer group at x, denoted G^x above, is in fact, a subgroup of G.

b.) Show that the G action is free iff the stabilizer group at every $x \in X$ is the trivial subgroup $\{1_G\}$.

c.) Suppose that $y = g \cdot x$. Show that G^y and G^x are conjugate subgroups in G.

Exercise

a.) Show that whenever G acts on a set X one can canonically define a groupoid: The objects are the points $x \in X$. The morphisms are pairs (g, x), to be thought of as arrows $x \stackrel{g}{\to} g \cdot x$. Thus, $X_0 = X$ and $X_1 = G \times X$.

b.) What is the automorphism group of an object $x \in X$.

This groupoid is commonly denoted as $X//G$.

2.0.1 The stabilizer-orbit theorem

There is a beautiful relation between orbits and isotropy groups:

Theorem [Stabilizer-Orbit Theorem]: Each left-coset of G^x in G is in 1-1 correspondence with the points in the G-orbit of x:

$$\psi : Orb_G(x) \to G/G^x \tag{2.9}$$

for a $1-1$ map ψ.

Proof: Suppose y is in a G-orbit of x. Then $\exists g$ such that $y = g \cdot x$. Define $\psi(y) \equiv g \cdot G^x$. You need to check that ψ is actually well-defined.

$$y = g' \cdot x \qquad \to \qquad \exists h \in G^x \qquad g' = g \cdot h \qquad \to \qquad g'G^x = ghG^x = gG^x \tag{2.10}$$

Conversely, given a coset $g \cdot G^x$ we may define

$$\psi^{-1}(gG^x) \equiv g \cdot x \tag{2.11}$$

Again, we must check that this is well-defined. Since it inverts ψ, ψ is 1-1. ♠

Corollary: If G acts transitively on a space X then there is a $1-1$ correspondence between X and the set of cosets of H in G where H is the isotropy group of any point $x \in X$. That is, at least as sets: $X = G/H$. The isotropy groups for points in G/H are the conjugate subgroups of H in G.

Remark: Sets of the type G/H are called *homogeneous spaces*. This theorem is the beginning of an important connection between the *algebraic* notions of subgroups and cosets to the *geometric* notions of orbits and fixed points. Below we will show that if G, H are topological groups then, in some cases, G/H are beautifully symmetric topological spaces, and if G, H are Lie groups then, in some cases, G/H are beautifully symmetric manifolds.

Exercise *The Lemma that is not Burnside's*

Suppose a finite group G acts on a finite set X as a transformation group. A common notation for the set of points fixed by g is X^g. Show that the number of distinct orbits is the averaged number of fixed points:

$$|\{orbits\}| = \frac{1}{|G|}\sum_g |X^g| \tag{2.12}$$

For the answer see. [1]

Exercise *Jordan's theorem*

Suppose G is finite and acts transitively on a finite set X with more than one point. Show that there is an element $g \in G$ with no fixed points on X. [2]

2.1 First examples

The concept of a G-action on a set is an extremely important concept, so let us consider a number of examples:

Examples

[1] *Answer*: Write

$$\sum_{g\in G} |X^g| = |\{(x,g)|g\cdot x = x\}| = \sum_{x\in X} |G^x| \tag{2.13}$$

Now use the stabilizer-orbit theorem to write $|G^x| = |G|/|\mathcal{O}_G(x)|$. Now in the sum

$$\sum_{x\in X} \frac{1}{|\mathcal{O}_G(x)|} \tag{2.14}$$

the contribution of each distinct orbit is exactly 1.

[2] Hint: Note that $X = G/H$ for some H and apply the Burnside lemma.

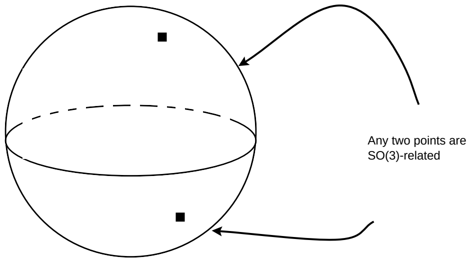

Figure 1: Transitive action of $SO(3, \mathbb{R})$ on the sphere. `fig:orbii`

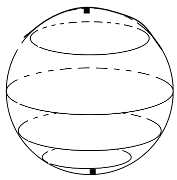

Figure 2: Orbits of $SO(2, \mathbb{R})$ on the two sphere. `fig:orbiii`

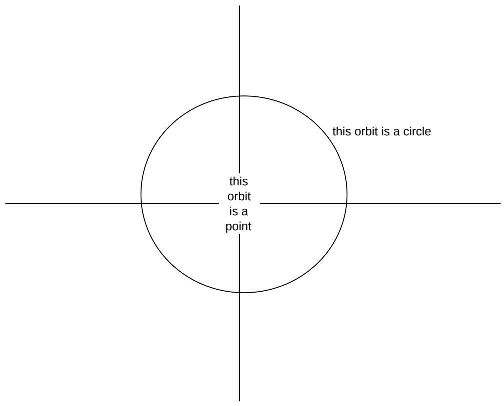

Figure 3: Notice not all orbits have the same dimensionality. There are two qualitatively different kinds of orbits of $SO(2, \mathbb{R})$. `fig:sotwo`

1. Let $X = \{1, \cdots n\}$, so $S_X = S_n$ as before. The action is effective and transitive, but not free. Indeed, the fixed point of any $j \in X$ is just the permutations that permute everything else, and hence $S_X^j \cong S_{n-1}$. Note that different j have different stabilizer subgroups isomorphic to S_{n-1}, but they are all conjugate.

2. *Group actions on the plane.* The group $G = GL(2, \mathbb{R})$ acts on the plane $X = \mathbb{R}^2$ by linear transformation. The action is effective: If $g \neq 1$ then it moves some vector a nonzero amount! There are only two orbits: Note that if $\vec{x} = 0$ then it remains zero under linear transformation, so $\{\vec{0}\}$ is an orbit. On the other hand, if $\vec{x}, \vec{y}$ are both nonzero then some linear transformation certainly will map $\vec{x}$ to $\vec{y}$. The action is therefore not transitive, and not free.

3. Similarly, $GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$. If we act with a matrix on a column vector we get

a left action. If we act on a row vector we get a right action. Either way, there are two orbits.

4. We can restrict the $GL(2,\mathbb{R})$ action on $\mathbb{R}^2$ to the action of the subgroup $G = SO(2,\mathbb{R})$. This completely changes the picture. The action is:
$$R(\phi) : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.15}$$
The group action is effective. It is not free, and it is not transitive. There are now infinitely many orbits of $SO(2)$, and they are all distinguished by the invariant value of x^2+y^2 on the orbit. From the viewpoint of topology, there are two distinct "kinds" of orbits acting on $\mathbb{R}^2$. One has trivial isotropy group and one has isotropy group $SO(2)$. See Figure 3. These give two strata.

5. Orbits of $O(2)$. The two-dimensions orthogonal group $O(2,\mathbb{R})$ can be written as a semidirect product
$$O(2) = SO(2) \rtimes \mathbb{Z}_2 \tag{2.16}$$
where $\mathbb{Z}_2$ acts on $SO(2)$ by taking $R(\theta) \to R(-\theta)$. The group has two components which can be written as
$$O(2) = SO(2) \amalg P \cdot SO(2) \tag{2.17}$$
where P is not canonical and can be taken to be reflection in any line through the origin. The orbits of $SO(2)$ and $O(2)$ are the same.

6. Similarly, $SO(3,\mathbb{R})$ acts on $X = \mathbb{R}^3$. It is effective, not transitive, and not fixed-point-free. We can restrict the action to a sphere of any radius S^2_R. The action is then transitive on the sphere, The isotropy group of any point $x \in S^2_R$ is the subgroup of rotations about the axis through that point. That subgroup is isomorphic to $SO(2,\mathbb{R})$, but as x varies the particular subgroup varies. For example, with usual conventions, if x is on the x^3-axis then the subgroup is the subgroup of matrices of the form
$$\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.18}$$
but if x is on the x^1-axis the subgroup is the subgroup of matrices of the form
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \tag{2.19}$$
and so on. There are two strata: Those with G^x congruent to $SO(2,\mathbb{R})$ and those with $G^x = SO(3,\mathbb{R})$.

7. By contrast consider a fixed $SO(2,\mathbb{R})$ subgroup of $SO(3,\mathbb{R})$, say, the subgroup defined by rotations around the z-axis. This subgroup also acts on the sphere - but <u>*not*</u> transitively. The G-orbits are shown in Figure 2.

8. If $G = \mathbb{Z}_2$ acts linearly on $\mathbb{R}^{n+1}$ (i.e. $V = \mathbb{R}^{n+1}$ is a representation of $\mathbb{Z}_2$) then we can choose coordinates so that the nontrivial element $\sigma \in G$ acts by

$$\sigma \cdot (x^1, \dots, x^{n+1}) = (x^1, \dots, x^p, -x^{p+1}, \cdots, -x^{p+q}) \tag{2.20}$$

eq:Z2-sphere

where $p + q = n + 1$. Note that this action preserves the equation of the sphere $\sum_i (x^i)^2 - 1 = 0$ and hence descends to a $\mathbb{Z}_2$-action on the sphere S^n. The case $p = 0, q = n + 1$ is the antipodal map, but there are many other natural actions of $\mathbb{Z}_2$ on S^n.

9. Let the group be $G = \mathbb{C}^*$. This acts on $X = \mathbb{C}^n$ by scaling all the coordinates. The set of orbits is not a nice manifold, but the set of orbits of the action on $\tilde{X} = \mathbb{C}^n - \{0\}$ is a good manifold (see below). It is called $\mathbb{CP}^{n-1}$. Consider a set of integers $(q_1, \dots, q_n) \in \mathbb{Z}^n$. Then for each such set of integers there is a $\mathbb{C}^*$-action on $\mathbb{CP}^{n-1}$ defined by

$$\mu \cdot [X^1 : \cdots : X^n] := [\mu^{q_1} X^1 : \cdots : \mu^{q_n} X^n] \tag{2.21}$$

eq:Cstar-CPn

for $\mu \in \mathbb{C}^*$. (Check it is well-defined!)

10. The group $G = SL(2, \mathbb{R})$ acts on the complex upper half plane:

$$\mathcal{H} = \{\tau | \mathrm{Im}\tau > 0\} \tag{2.22}$$

via

$$g \cdot \tau := \frac{a\tau + b}{c\tau + d} \tag{2.23}$$

eq:FracLin

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.24}$$

11. *Actions of* $\mathbb{Z}$. Let us consider $\mathbb{Z}$ to be the free group with one generator g_0. Then, given <u>any</u> invertible map $f : X \to X$ we can define a group action of $\mathbb{Z}$ on X by

$$g_0^n \cdot x = \begin{cases} \underbrace{f \circ \cdots \circ f}_{n \quad \text{times}}(x) & n > 0 \\ x & n = 0 \\ \underbrace{f^{-1} \circ \cdots \circ f^{-1}}_{|n| \quad \text{times}}(x) & n < 0 \end{cases} \tag{2.25}$$

eq:Z-action

Conversely, any $\mathbb{Z}$-action must be of this form since we can define $f(x) := g_0 \cdot x$.

12. Let G be any group and consider the group action defined by $\phi(g, x) = x$ for all $g \in G$. This is as ineffective as a group action can be: For every x, the istropy group is all of G, and for all $g \in G$, $\mathrm{Fix}(g) = X$. In particular, this situation will arise if X consists of a single point. This example is not quite as stupid as might at first appear, once one takes the categorical viewpoint, for $pt//G$ is a very rich category indeed.

Exercise

Consider the action of $\mathbb{Z}_2$ on the sphere defined by (2.20).

a.) For which values of p, q is the action effective?

b.) For which values of p, q is the action transitive?

c.) Compute the fixed point set of the nontrivial element $\sigma \in \mathbb{Z}_2$.

d.) For which values of p, q is the action free?

Exercise

Consider the action of $G = \mathbb{C}^*$ on $\mathbb{CP}^{n-1}$ defined by (2.21).

a.) For which values of $(q_1, \ldots, q_n)$ is the action effective?

b.) For which values of $(q_1, \ldots, q_n)$ is the action transitive?

c.) What are the fixed points of the $\mathbb{C}^*$ action?

d.) What are the stabilizers at the fixed points of the $\mathbb{C}^*$ action?

Exercise

a.) Show that (2.23) above defines a left-action of $SL(2,\mathbb{R})$ on the complex upper half-plane. [3]

b.) Is the action effective?

c.) Is the action transitive?

d.) Which group elements have fixed points?

e.) What is the isotropy group of $\tau = \mathrm{i}$?

2.1.1 The Case Of $1+1$ Dimensions

Consider 1+1-dimensional Minkowski space with coordinates $x = (x^0, x^1)$ and metric given by

$$\eta := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.26}$$

i.e. the quadratic form is $(x, x) = -(x^0)^2 + (x^1)^2$. The two-dimensional Lorentz group is defined by

$$O(1,1) = \{A | A^{tr} \eta A = \eta\} \tag{2.27}$$

This group acts on $\mathbb{M}^{1,1}$ preserving the Minkowski metric.

The connected component of the identity is the group of Lorentz boosts of rapidity θ:

[3] *Hint*: Show that $\mathrm{Im}(g \cdot \tau) = \frac{\mathrm{Im}\tau}{|c\tau+d|^2}$.

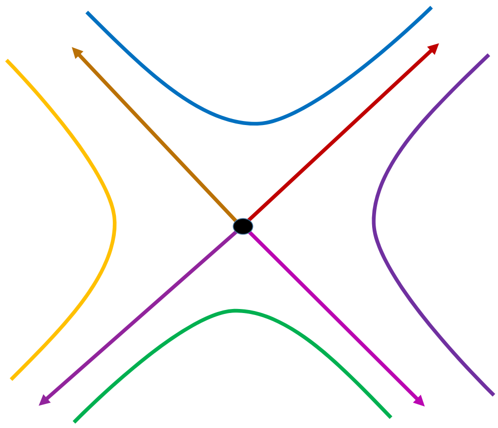

Figure 4: The distinct kinds of orbits of $SO(1,1,\mathbb{R})$ are shown in different colors. If we enlarge the group to include transformations that reverse the orientation of time and/or space then orbits of the larger group will be made out of these orbits by reflection in the space or time axis. fig:LorentzOrb

$$x^0 \to \cosh\theta\ x^0 + \sinh\theta\ x^1 \tag{2.28}$$

$$x^1 \to \sinh\theta\ x^0 + \cosh\theta\ x^1 \tag{2.29}$$

that is:

$$SO_0(1,1;\mathbb{R}) \equiv \{B(\theta) = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \vert -\infty < \theta < \infty\} \tag{2.30}$$

In the notation the S indicates we look at the determinant one subgroup and the subscript 0 means we look at the connected component of 1. This is a group since

$$B(\theta_1)B(\theta_2) = B(\theta_1+\theta_2) \tag{2.31}$$

so $SO_0(1,1) \cong \mathbb{R}$ as groups. Indeed, note that

$$B(\theta) = \exp\left[\theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \tag{2.32}$$

It is often useful to define *light cone coordinates*: [4]

$$x^{\pm} := x^0 \pm x^1 \tag{2.33}$$

[4] Some authors will define these with a $1/2$ or $1/\sqrt{2}$. One should exercise care with this choice of convention.

and the group action in these coordinates is simply:

$$x^{\pm} \to e^{\pm\theta} x^{\pm} \tag{2.34}$$

so it is obvious that $x^+ x^- = -(x, x)$ is invariant.

It follows that the orbits of the Lorentz group are, in general, hyperbolas. They are separated by different values of the Lorentz invariant $x^+ x^- = \lambda$, but this is not a complete invariant, since the sign (or vanishing) of x^+ and of x^- is also Lorentz invariant. For a real number r define

$$\mathrm{sign}(r) := \begin{cases} +1 & r > 0 \\ 0 & r = 0 \\ -1 & r < 0 \end{cases} \tag{2.35}$$

Then $(\lambda, \mathrm{sign}(x^+), \mathrm{sign}(x^-))$ is a complete invariant of the orbits. That is, given this triple of data there is a unique orbit with these properties.

It is now easy to see what the different type of orbits are. They are shown in Figure 4: They are:

♣Actually, the lightrays and hyperbolas have trivial stabilizer and hence are in the same strata. This is a problem with using strata. ♣

1. hyperbolas in the forward/backward lightcone and the left/right of the lightcone
2. 4 disjoint lightrays.
3. the origin: $x^+ = x^- = 0$.

It is now interesting to consider the orbits of the full Lorentz group $O(1,1)$ and its relation to the massless wave equations. This group has four components and is, group-theoretically

$$O(1,1) = SO_0(1,1) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2) \tag{2.36}$$

We can write, noncanonically,

$$O(1,1) = SO_0(1,1) \amalg P \cdot SO_0(1,1) \amalg T \cdot SO_0(1,1) \amalg PT \cdot SO_0(1,1) \tag{2.37}$$

with

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.38}$$

The P and T operations map various orbits of $SO_0(1,1)$ into each other: P is a reflection in the time axis and T is a reflection in the space axis. Thus the orbits of the groups $SO(1,1)$, $SO_0(1,1) \amalg PT \cdot SO_0(1,1)$, and $O(1,1)$ all differ slightly from each other.

♣Should give more details here, or form an exercise. ♣

As an example of a physical manifestation of orbits let us consider the energy-momentum dispersion relation of a particle of mass m with energy-momentum $(E, p) \in \mathbb{R}^{1,1}$.

1. Massive particles: $m^2 > 0$ have (E, p) along an orbit in the upper quadrant:

$$\mathcal{O}^+(m) = \{(m\cosh\theta, m\sinh\theta) | \theta \in \mathbb{R}\} \tag{2.39}$$

2. Massless particles move at the speed of light. In 1+1 dimensions there is an interesting distinction: Left-moving particles with positive energy have support on [5] $p_+ = \frac{1}{2}(E+p) = 0$ and $p_- = \frac{1}{2}(E-p) \neq 0$. Right-moving particles with positive energy have support on $p^- = 0$ and $p^+ \neq 0$.

3. Tachyons have $E^2 - p^2 = m^2 < 0$ and have their support on the left or right quadrant. If we try to expand a solution to the wave-equation with $e^{\mathrm{i}(k_0 x^0 + k_1 x^1)}$ then $k_0^2 = \sqrt{k_1^2 + m^2}$ and so if the spatial momentum k_1 is sufficiently small then k_0 is pure imaginary and the wave grows exponentially, signaling and instability. This tells us our theory is out of control and some important new physical input is needed.

4. A massless "particle" of zero energy and momentum.

3. Action of a topological group on a topological space

If the group G is a topological group it is said to act continuously on a topological space X when $\phi : G \times X \to X$ is a continuous map. When working with topological groups acting on topological spaces, this is generally assumed. Note that $\phi_g : X \to X$ has a continuous inverse, namely $\phi_{g^{-1}}$ and therefore ϕ_g is a homeomorphism. Therefore, a topological group action on a topological space can be defined to be a homomorphism from the group G to the group of homeomorphisms on X. [6] One often wants to work with *proper* actions: This means that the ϕ is a proper continuous map.

Warning: There is some very important, but regrettably very confusing terminology associated with topological group actions on topological spaces. When G is a discrete group there is an important notion of a *properly discontinuous* action. The general definition is that the map $G \times X \to X \times X$ given by $(g, x) \mapsto (g \cdot x, x)$ is a proper continuous map. This does not mean the function taking $x \mapsto g \cdot x$ is discontinuous! On the contrary, as stated above, it is continuous. To make matters worse, one will find inequivalent definitions of the term "properly discontinuous" in textbooks and on the internet. The subtleties melt away when G is finite or when X is locally compact. We will follow the definitions used by W.P. Thurston, since he was one of the great masters of the subject. Specifically, Definition 3.5.1 of W.P. Thurston, *Three Dimensional Geometry and Topology*, vol 1, Princeton University Press 1997 includes:

Definition: Let G be a discrete group acting continuously on a topological space X. Then

1. The action *has discrete orbits* if every $x \in X$ has a neighborhood U such that the set of group elements $g \in G$ with $g \cdot x \in U$ is finite.

2. The action *is wandering* if every $x \in X$ has a neighborhood U such that the set of group elements $g \in G$ with $g \cdot U \cap U \neq \emptyset$ is finite.

[5] Note the factors of two, so that $x^0 p_0 + x^1 p_1 = x^+ p_+ x^- p_-$. This is an example of the tricky factors of two one encounters when working with light-cone coordinates.

[6] Continuous, with a suitable topology on the group of homeomorphisms of X.

3. If X is locally compact then the action is said to be *properly discontinuous* if for every compact set $K \subset X$ the set of g with $g \cdot K \cap K \neq \emptyset$ is finite.

If $\mathcal{M}$ is a measure space then a *discrete dynamical system* is a pair of $\mathcal{M}$ together with a measure-preserving group action of $\mathbb{Z}$ on $\mathcal{M}$. According to (2.25) this means we have a map $f : \mathcal{M} \to \mathcal{M}$ such that $\mu(f(\mathcal{A})) = \mu(\mathcal{A})$ for any measurable set $\mathcal{A}$. For example, in Hamiltonian dynamics one can take $\mathcal{M}$ to be phase space equipped with a symplectic form ω The natural measure is then the Liouville measure:

$$\mu(\mathcal{A}) := \int_{\mathcal{A}} \frac{\omega^n}{n!} \tag{3.1}$$

In particular, if f is symplectic, i.e. $f^*(\omega) = \omega$, or, in equations:

$$\omega_{\mu\nu}(f(x)) \frac{\partial f^\mu}{\partial x^\lambda} \frac{\partial f^\nu}{\partial x^\rho} = \omega_{\lambda\rho}(x) \tag{3.2}$$

then the corresponding $\mathbb{Z}$-action is a dynamical system.

One important result is the

Theorem [Poincaré recurrence theorem]. If $f : \mathcal{M} \to \mathcal{M}$ is a measure-preserving map and has bounded orbits then in any open U set there are points x such that for infinitely many n, $g^n x \in U$.

The proof is based on the idea that if this were not true then the volume of $\cup f^n(U)$ would be infinite, but that cannot be for a volume preserving map with bounded orbits. ♣Should give a better proof! ♣

There are various ways of expressing how "chaotic" a map is. A dynamical system is said to be *mixing* if for all pairs of (measurable) sets $\mathcal{A}, \mathcal{B} \subset \mathcal{M}$ we have

$$\lim_{n\to\infty} \mu(\mathcal{A}_n \cap \mathcal{B}) = \frac{\mu(\mathcal{A})\mu(\mathcal{B})}{\mu(\mathcal{M})} \tag{3.3}$$

where $\mathcal{A}_n = f^n(\mathcal{A})$. If $\mu(\mathcal{B}) \neq 0$ this means

$$\lim_{n\to\infty} \frac{\mu(\mathcal{A}_n \cap \mathcal{B})}{\mu(\mathcal{B})} = \frac{\mu(\mathcal{A})}{\mu(\mathcal{M})} \tag{3.4}$$

so that the "weight" of the set $\mathcal{A}$ is equally distributed over any $\mathcal{B}$. We will give an example of a dynamical system which is mixing below.

Let us examine some special cases of discrete dynamical systems:

1. First, take $X = \mathbb{R}^2$ and let

$$f(x_1, x_2) = (\lambda x_1, \lambda^{-1} x_2) \tag{3.5}$$ eq:lambda-act

where λ is a positive real number greater than 1. For $n > 0$, g_0^n stretches in the x_1 direction and flattens in the x_2 direction. For $n < 0$ the situation is reversed. This action clearly does *not* have discrete orbits, since the origin $(0,0)$ is a fixed point for

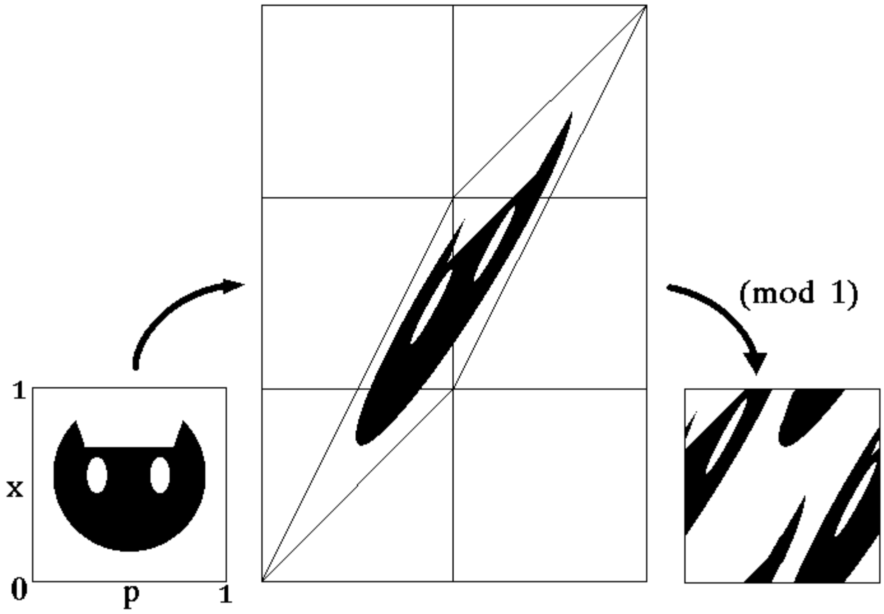

Figure 5: The famous Arnold cat map. The picture ultimately comes from the book V. Arnold and A. Avez, *Ergodic Problems in Classical Mechanics*. fig:ArnoldCatM

the entire group. This action can come up, for example in symplectic geometry: Note that this is a symplectic action with Poisson bracket $\{x_1, x_2\} = 1$, i.e. symplectic form $\omega = dx_1 \wedge dx_2$. It also comes up in some string theory models of cosmological singularities, where we view x_1, x_2 as light-cone coordinates in $1 + 1$-dimensional Minkowski space.

2. Next, consider $X = \mathbb{R}^2 - \{0, 0\}$. Now we can consider (3.5) to be a symplectic action with Poisson bracket for $\{x_1, x_2\} = 1$ just by restriction. However, having excised the point $(0, 0)$ we are now free to define the symplectic structure $\{x_1, x_2\} = x_1 x_2$, i.e.

$$\omega = \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \tag{3.6}$$

and since $x_1 x_2$ is preserved the action (3.5) is still symplectic. Now the action has discrete orbits and is in fact wandering: Consider a neighborhood of some point. We may assume the projection on the x_1 axis will be an open interval (a, b). Then there are only a finite number of solutions to $a < \lambda^k b < b$ and only a finite number of solutions to $a < \lambda^k a < b$. On the other hand, the action is *not* properly discontinuous.

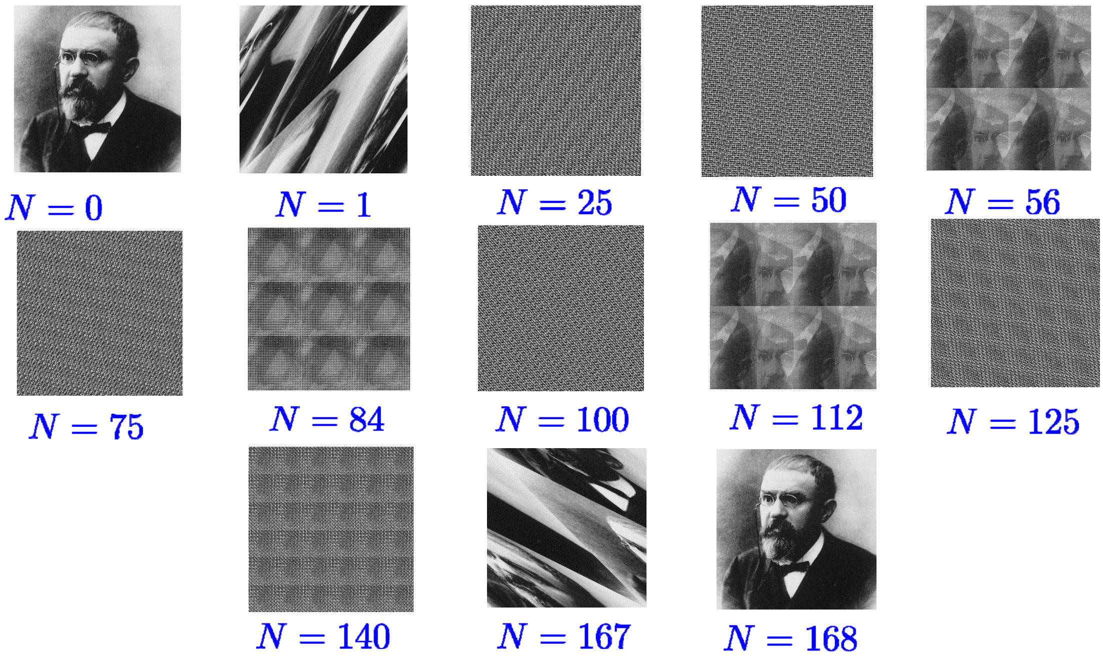

Figure 6: Illustrating Poincaré recurrence for the discrete cat map with $N \times N$ pixels where $N = 294$. Thanks to Andrew Moore for writing the code.

fig:PoincareRe

To see this, consider the compact set K which is a closed line segment from $(1, 0)$ to $(0, 1)$. For any $n \neq 0$ there is a solution to

$$\begin{aligned} x + y &= 1 \\ \lambda^n x + \lambda^{-n} y &= 1 \end{aligned} \tag{3.7}$$

with $0 < x < 1$ and $0 < y < 1$. Therefore $g_0^n \cdot K \cap K \neq 0$ for all n.

3. *The Arnold cat map*: Consider the torus as $T^2 = [0, 1]^2 / \sim$ with $x \sim x + 1$ and $y \sim y + 1$. Consider the transformation $f : T^2 \to T^2$ defined by

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.8}$$

Note again that this is a symplectic transformation. This is known as *Arnold's cat map*, and is famous in dynamical systems theory and in discussions of chaos. Figure 5 shows one iteration of this map. It is shown in Arnold-Avez that the map is mixing.

4. The cat map can also be used to give a dramatic illustration of Poincaré recurrence. A computer screen will have a finite number of pixels. Let us say it has $N \times N$ pixels.

♣Should comment that this is clearly NOT mixing because periodicity means the limit doesn't really exist. ♣

If these are black or white there will be $2^{N\times N}$ possible images. [7] Now, consider the discrete version of the cat map: We take

$$x, y \in \frac{1}{N}\{0, 1, \ldots, N-1\} \tag{3.9}$$

and apply the above transformation. Since there are only $2^{N\times N}$ images the transformation must be periodic. See Figure 6 for $N = 294$. It is rather astonishing that the period is only 164. Due to some special number-theoretic aspects of this example (related to Fibonacci numbers) one can give an exact formula for a period (not necessarily the fundamental period), which in this case turns out to confirm $N = 164$. Using this formula it turns out that the period is $\leq 3N$. [8]

5. In general, suppose that a $\mathbb{Z}$-action on a manifold M is generated by a differentiable map $g_0 \cdot x = f(x)$, where $f : M \to M$, and suppose that f has a fixed-point $f(x_0) = x_0$. Then, near the fixed point we can write, in local coordinates:

$$f(x_0 + \delta x) = f(x_0) + df(\delta x) = x_0 + df(\delta x) + \mathcal{O}((\delta x)^2) \tag{3.10}$$

where we identify an infinitesimal deviation with a tangent vector. In general $df : T_{x_0}X \to T_{x_0}X$ is just a real linear transformation and cannot be diagonalized. Suppose, however, that it can be diagonalized. Then there is a basis of $T_{x_0}X$ such that df has the matrix representation $A = \mathrm{Diag}\{\alpha_1, \ldots, \alpha_n\}$ for an n-dimensional manifold. If we consider orbits that begin close to x_0 then choose coordinates so that $x_0 = 0 \in \mathbb{R}^n$ and we have $x_{n+1} \cong (1 + A)x_n$. In the directions with $\alpha_j < 0$ the orbits contract to the fixed point. In the directions $\alpha_j > 0$ they expand away from the fixed point (and soon go beyond the linear approximation). If $\alpha_j = 0$ then we need to go to higher order to determine if the fixed point is isolated.

Remark: A very important dynamical system for quantum field theory is known as the "renormalization group." It has both discrete and continuous forms. One of the discrete forms is known as the block spin method. The dynamical system evolves on the infinite-dimensional space of all possible local couplings of the field theory. Fixed point loci are known as "scale invariant theories," and they are usually conformal field theories. One then defines local quantum field theory by a scaling of couplings near a fixed point. The key miracle is that, in this infinite dimensional space of couplings all but finitely many directions are attractive ("irrelevant operators") and only a finite number of directions are repulsive ("relevant operators") so, after a finite number of choices one has predictive power, at least for renormalizable quantum field theories.

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[7] Actually, there are 256 shades of grey, so actually there are 2^{8N^2} images on a typical computer screen.

[8] F.J. Dyson and H. Falk, "Period of a Discrete Cat Mapping," Amer. Math. Monthly, vol. 99 (1992), pp.603-614

Exercise

Consider an action of a discrete group on a topological space X. Show that properly discontinuous implies wandering implies discrete orbits.

Exercise

a.) Show that the Arnold cat map is a product of a shear by one unit in the x-direction followed by a shear by one unit in the y-direction.

b.) Find a square root of the Arnold cat map.

3.1 Left and right group actions of G on itself

Let G be any group. Then G acts on itself as a transformation group, in several ways.

To define the *left action of* G *on* G we associate to each $a \in G \to$ the *mapping* $L(a)$, $L(a) : G \to G$ defined by

$$L(a) : g \mapsto ag \tag{3.11}$$

where on the RHS we use the group multiplication of G. This mapping is 1-1 and onto, so $L(a) \in S_G$.

These transformations satisfy:

$$L(ab) = L(a)L(b) \tag{3.12}$$

eq:grplaw

and moreover $L(1_G)$ is the identity mapping.

Thus we have defined a left-action of G on itself. It is effective, free, and transitive.

Indeed, we can define a map $\mathcal{L} : a \mapsto L(a)$ which is a homomorphism $\mathcal{L} : G \to S_G$. Moreover,

$$L(a) = 1 \Longleftrightarrow a = 1 \tag{3.13}$$

and hence ker$\mathcal{L} = \{1_G\}$, so the image of $\mathcal{L}$ is isomorphic to G. Thus, we have proved

Theorem ??.1 (Cayley's Theorem): Any group G is isomorphic to a subgroup of the full permutation group S_G. If $n = |G| < \infty$ then G is isomorphic to a subgroup of S_n.

Warning: For a fixed a, although $L(a)$ is a map $G \to G$, it is *not* a homomorphism! Note, for example that $L(a)$ takes 1_G to a. Do not confuse this with the fact that the mapping $\mathcal{L} : a \to L(a)$ is a homomorphism.

All of this can be repeated for right-actions: For $a \in G$ define the *right-translation operator* $R(a) : G \to G$ by $R(a) : g \mapsto g \cdot a$.

Then $\phi_a = R(a)$ defines a right-action of G on itself, and hence $\tilde{\phi}_a = R(a^{-1})$ defines a left-action of G on itself.

It should be fairly obvious that

$$R(a)L(b) = L(b)R(a) \tag{3.14}$$

for all $a, b \in G$. Thus, there is a left $G \times G$ action on G defined by:

$$\phi_{(g_1,g_2)} := L(g_1)R(g_2^{-1}) \tag{3.15}$$

eq:GGonG

Exercise

Since there is a left-action of $G \times G$ on $X = G$ there is a left-action of the diagonal subgroup $\Delta \subset G \times G$ where $\Delta = \{(g, g)|g \in G\}$ is is a subgroup isomorphic to G.

a.) Show that this action is given by $a \mapsto I(a)$, where $I(a)$ is the conjugation by a. (See, Chapter 1, Section ***)

b.) Show that the orbits of Δ are the conjugacy classes of G.

c.) What is the stabilizer subgroup of an element $g_0 \in G$?

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4. Spaces of orbits

We now shift our focus from studying orbits to studying *the space of orbits.*

In general, if G has a right-action on a set X then the set of distinct G-orbits is denoted X/G. If G has a left-action then the set of distinct G orbits is denoted $G\backslash X$.

The study of such spaces of orbits is rather vast. Some of the ways spaces of orbits enter into physics are the following:

1. Spaces of orbits such as homogeneous spaces given beautifully symmetric manifolds (or orbifolds). Thus they form a rich source of geometrical constructions and can be used in constructing spacetimes or discussing moduli spaces of solutions to equations.

2. A natural source of orbits is *Hamiltonian dynamics.* The time evolution of a dynamical system naturally defines a system of $\mathbb{R}$ or $\mathbb{Z}$ -orbits on phase space.

3. If G is a global symmetry group in a field theory, then the vacua of the theory are a union of orbits of G, and hence a union of homogeneous spaces G/H. In general H will be different in different connected components. It is referred to as the *unbroken symmetry* in the theory of spontaneous symmetry breaking.

4. Often, it is convenient to introduce redundant variables in a physical problem to make some other property of the physics (such as locality) manifest. In this case G is a *gauge symmetry* and it acts on the set of redundant variables X while the space of orbits X/G are the physically inequivalent variables. The canonical example is obtained by identifying X with the set of all gauge potentials and G with the group of gauge transformations. For example, on $\mathbb{M}^{1,3}$ the gauge potentials A_μ are redundant variables. The group $\mathcal{G} = \mathrm{Map}(\mathbb{M}^{1,3}, \mathbb{R})$ acts by $A_\mu \to A_\mu + \partial_\mu \chi$. The orbit space is parametrized by the fieldstrength $F_{\mu\nu}$.

We cannot cover this topic in proper detail here, but just indicate some examples and definitions to give a taste of the subject.

Remarks

1. As an example, let us consider the case where $X = G$ is itself a group. Let $H \subset G$ be a subgroup and consider the right-action of H on the set X. The set of orbits is G/H. This is in accord with our notation for the set of left-cosets of a subgroup H in a larger group G. Note that the set of orbits still admits a left-action by G. If $K \subset G$ is another subgroup then the set of orbits of the left-action of K on G/H is known as a *double coset* and denoted $K\backslash G/H$.

2. Warning! The notation X/G is somewhat ambiguous, as a space X can admit more than one group action. For example, if $X = G$ and we use right-translation $R(g)$ then X/G is a single point. On the other hand, if $X = G$ and the G action is by conjugation:
$$\phi_g(h) := ghg^{-1} \tag{4.1}$$
then the space X/G is the set of conjugacy classes in G and always has more than one point, if G is nontrivial.

3. When G is a topological group acting on a topological space X we can make X/G into a topological space: We use the quotient topology under the equivalence relation of being G-related. Put differently, the topology on X/G is defined by requiring that $p : X \to X/G$ be a continuous map. That is, $\mathcal{U} \subset X/G$ is open if $p^{-1}(\mathcal{U})$ is open. Even when X and G are relatively "simple" the quotient can be quite subtle and complicated. For example, even if X and G are Hausdorff the quotient might not be Hausdorff. So, it is useful to have some criteria for when quotients are "well-behaved."

4. If G is a discrete group we defined the notion of a "properly discontinuous action" above. In this case we have a nice

   Theorem: Let G be a discrete group acting freely and properly discontinuously on a Hausdorff manifold X. Then the quotient X/G is a Hausdorff manifold.

   To prove this, just note that around each point $x \in X$ there is an open neighborhood U homeomorphic to $\mathbb{R}^n$ such that $g \cdot U \cap U = \emptyset$ for all $g \neq 1$. This U will serve as a local neighborhood. To check that it is Hausdorff show that if x_1, x_2 are on distinct orbits then we can then there are compact neighborhoods K_1 and K_2 of x_1, x_2 which are disjoint. For a more complete proof see Thurston, Proposition 3.5.7.

5. When G is a Lie group one useful criterion is the

   Theorem: [Quotient manifold theorem]: Let G be a Lie group acting smoothly, freely, and properly, on a smooth manifold X. Then X/G is a manifold of dimension $\mathrm{dim}X - \mathrm{dim}G$ and the projection map $p : X \to X/G$ is a submersion.

   For a proof see, J. Lee, *Introduction to smooth manifolds*, pp. 218-223.

♣SHOULD GIVE IDEA OF THE PROOF! ♣

4.1 Simple examples

Example 1 . $G = \mathbb{Z}$ acts properly discontinuously on $\mathbb{R}$ via $n \cdot x = x + n$. The orbits are in 1-1 correspondence with $[0,1]/\sim$ where $\sim$ identifies $0 \sim 1$. Note that therefore the set of orbits is in one-one correspondence with $\mathbb{R}/\mathbb{Z}$, and can be identified with points on the circle S^1.

Example 2. This generalizes: Let Λ be a two-dimensional lattice. It acts properly discontinuously on $\mathbb{R}^2$ by translation. Then

$$\mathbb{R}^2/\Lambda \tag{4.2}$$

is geometrically realized as a doughnut, or torus. Of course, nothing is special about two dimensions in this example, if Λ is an n-dimensional lattice in $\mathbb{R}^n$ then it acts on $\mathbb{R}^n$ by translations and the set of orbits is an n-dimensional torus.

Example 3: If we take $X = S^2$ and $G = SO(2)$ acting by rotations around some axis then the space of orbits is the closed interval.

Example 4: If $X = \mathbb{R}^{n+1} - \{0\}$ and $G = \mathbb{R}^*$ acting by scalar multiplication then $X/G \cong \mathbb{RP}^n$. Similarly if $X = \mathbb{C}^{n+1} - \{0\}$ and $G = \mathbb{C}^*$ acting by scalar multiplication then $X/G \cong \mathbb{CP}^n$. In these cases the quotient manifold theorem is not so useful because the groups involved are not compact so it is not easy to check the map is proper. However, if we think of $\mathbb{CP}^n$ as the moduli space of lines through the origin then we can put the standard Hermitian structure on $\mathbb{C}^{n+1}$ and observe that for every line there exists a basis vector v of norm one: $v^* \cdot v = 1$. But

$$\{v \in \mathbb{C}^{n+1} | v^* \cdot v = 1\} \cong S^{2n+1} \tag{4.3}$$

This ON vector is not unique. We can define an action of $U(1)$ on S^{2n+1} by $v \to e^{\mathrm{i}\theta} v$. Then $S^{2n+1}/U(1)$ is a manifold by the quotient manifold theorem, and is another presentation of $\mathbb{CP}^n$. Now that we see that a given space of orbits can be realized as a quotient space if different ways we should note a third presentation: $GL(n+1,\mathbb{C})$ acts on $\mathbb{CP}^n$ by linear action on the homogeneous coordinates. The action is clearly transitive. Therefore, by the stabilizer-orbit theorem we need only find the stabilizer subgroup of a convenient point, such as $[1 : 0 : \cdots : 0]$. The stabilizer subgroup of this point is the subgroup B of $GL(n+1,\mathbb{C})$ consisting of matrices of the form:

$$\begin{pmatrix} g_{11} & g_{12} \cdots g_{1,n+1} \\ 0_{n\times 1} & \tilde{g} \end{pmatrix} \tag{4.4}$$

where $\tilde{g} \in GL(n,\mathbb{C})$ and $g_{11} \in \mathbb{C}^*$. Then $\mathbb{CP}^n$ can be identified with the complex homogeneous space $GL(n+1,\mathbb{C})/B$.

Example 5: If we take $X = \mathbb{R}^2$ with $G = SO_0(1,1)$ acting via Lorentz transformations then we saw above that the space of orbits X/G is a union of a 5 special points together with four copies of $(0,\infty)$. In this case X/G is not a Hausdorff space. See Section **** below.

♣Really belongs below in the section on how things can go wrong. ♣

4.2 Fundamental domains

One way to try to get a picture of the space of orbits of a group action on a space X is to form a *fundamental domain.* In order to define this notion recall that given a G action on X we can define an equivalence relation, saying $x \sim y$ (“x is G-related to y” if there is a $g \in G$ with $y = g \cdot x$. In these terms a fundamental domain is a subset $\mathcal{F} \subset X$ of elements providing a complete set of representatives of the distinct equivalence classes for this relation. Then:

$$X = \amalg_{g \in G} g \cdot \mathcal{F}, \tag{4.5}$$

so the points in $\mathcal{F}$ are in one-one correspondence with the points of X/G. Put differently:

Definition: A *fundamental domain* for a group action of G on X is a subset $\mathcal{F} \subset X$ such that:

1. No two distinct points in $\mathcal{F}$ are G-related.
2. Every point of X is G-related to some point in $\mathcal{F}$.

Example 1 . *Finite covers of the circle.* Consider:

$$X = U(1) \cong \{z : |z| = 1\} \cong S^1 \tag{4.6}$$

and let $G \cong \mathbb{Z}/N\mathbb{Z}$ be the subgroup of N^{th} roots of 1. This acts on X by multiplication. A fundamental domain is the set of $z = e^{i\theta}$ with θ varying over a half-open interval of length $2\pi/N$. For example we could take $0 \le \theta < 2\pi/N$. Then X/G is another copy of S^1. The map $p : X \to X/G$ can be identified with the map $p(z) = z^N$.

Example 2 . If we consider $X/G = \mathbb{R}/\mathbb{Z} \cong S^1$ then a fundamental domain would be <u>any</u> set of points $\theta \le x < \theta + 1$, for any θ. As we see from the previous two examples, fundamental domains are not unique.

Example 3 . If we consider $\mathbb{R}^2/\mathbb{Z}^2$ where $\mathbb{Z}^2$ acts by translations by independent vectors e_1, e_2 then one choice of fundamental domain would be the set of points $\vec{x}_0 + t_1 e_1 + t_2 e_2$ where we take the union

$$\{(t_1, t_2) : 0 < t_1, t_2 < 1\} \amalg \{(0, t_2) : 0 < t_2 < 1\} \amalg \{(t_1, 0) : 0 < t_1 < 1\} \amalg \{(0, 0)\} \tag{4.7}$$

Note that $\vec{x}_0$ is arbitrary, and any choice gives a fundmental domain.

Example 4 . In general *unit cell* $\bar{\mathcal{F}} \subset \mathbb{R}^n$ of an embedded lattice $\Lambda \subset \mathbb{R}^n$ is the closure of a fundamental domain for $\mathbb{R}^n/\Lambda$. One natural choice is given by *choosing* a basis e_i for the lattice together with a vector $\vec{x}_0 \in \mathbb{R}^n$ and defining

$$\bar{\mathcal{F}} = \vec{x}_0 + \{\sum_i t_i e_i | 0 \le t_i \le 1\}. \tag{4.8}$$

Since $\mathbb{R}^n$ has a natural Euclidean metric the lattice Λ inherits an inner product. In the basis e_i it is given by the symmetric form $G_{ij} := e_i \cdot e_j$. Note that the volume of the unit cell is

$$\mathrm{vol}(\bar{\mathcal{F}}) = \sqrt{\det G_{ij}} \tag{4.9}$$ eq:unitcell

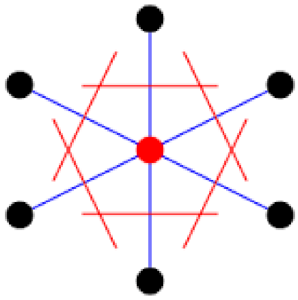

Figure 7: Constructing a Wigner-Seitz (or Voronoi) cell for the triangular lattice. The cells are regular hexagons. Figure from Wikipedia. fig:WignerSeit

Example 5 . Given an embedded lattice $\Lambda \subset \mathbb{R}^n$ we can use the metric to produce a canonical (i.e. basis-independent) set of fundamental domains, known as *Voronoi cells* in mathematics and as *Wigner-Seitz cells* in physics. Choose any lattice point $v \in \Lambda$ and take $\bar{\mathcal{F}}$ to be the set of all points in $\mathbb{R}^n$ which are closer to v than to any other point. (If the points are equidistant to another lattice point we include them in the closure $\bar{\mathcal{F}}$. Thus, for the regular triangular lattice a Wigner-Seitz cell would be a regular hexagon centered on a lattice point. See Figure 7. In reciprocal space, the Wigner-Seitz cell for the reciprocal lattice is known in solid state physics as the *Brillouin zone*. Note that there is a clear algorithm for constructing $\bar{\mathcal{F}}$: Starting with v we look at all other points $v' \in \Lambda$. We consider the hyperplane perpendicular to the line between v and v' and take the intersection of all the half-planes containing v. It is also worth remarking that the concept of Voronoi cell does not require a lattice and applies to any collection of points, indeed, any collection of subsets of $\mathbb{R}^n$.

Example 6 . *The modular group.* The modular group is $PSL(2,\mathbb{Z}) := SL(2,\mathbb{Z})/\{\pm 1\}$, where $SL(2,\mathbb{Z})$ is the subgroup of $SL(2,\mathbb{R})$ of matrices all of whose matrix elements are integers. Recall that this group acts effectively on the complex upper half-plane $\mathcal{H}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau := \frac{a\tau + b}{c\tau + d} \tag{4.10}$$

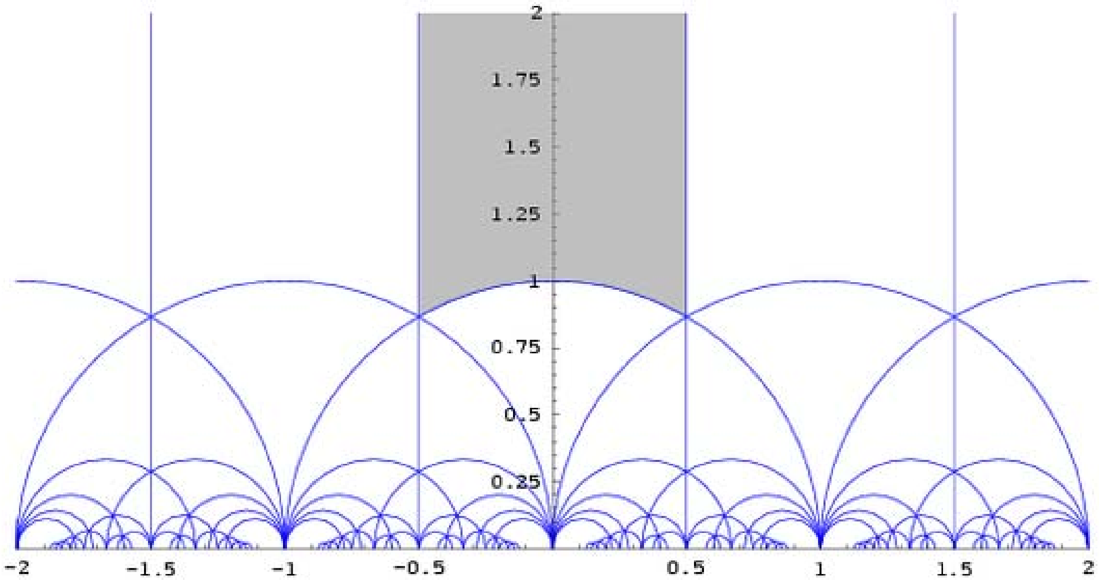

Figure 8: The keyhole region, a standard choice of fundamental domain for the action of $PSL(2,\mathbb{Z})$ on the complex upper half-plane. Figure from Wikipedia article on "Modular Group". fig:KeyholeReg

We will find a fundamental domain for this group action, and in the process prove that $SL(2,\mathbb{Z})$ is generated by the group elements S and T defined by:

$$S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.11}$$

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{4.12}$$

Denote their images in $PSL(2,\mathbb{Z})$ by $\bar{S}, \bar{T}$. [9]

Let:

$$\tilde{\mathcal{F}} := \{\tau \in \mathcal{H} || \tau| \geq 1 \quad \& \quad |\mathrm{Re}(\tau)| \leq \frac{1}{2}\} \tag{4.13}$$

This is almost, but not quite the canonical fundamental domain for the modular group. It is the famous *keyhole region* shown in Figure 8. Let $\bar{G}$ be the subgroup generated by $\bar{S}$

[9] We are here following a very nice argument by J.-P. Serre, *A Course in Arithmetic*, Springer GTM 7, pp. 78-79.

and $\bar{T}$. We claim that $\cup_{g\in\bar{G}} g\cdot\tilde{\mathcal{F}}$ is the entire half-plane. To prove this recall that, for any $g\in SL(2,\mathbb{Z})$,

$$\mathrm{Im}(g\cdot\tau)=\frac{\mathrm{Im}\tau}{|c\tau+d|^2} \tag{4.14}$$

Now, for any fixed $\tau\in\mathcal{H}$ the function $|c\tau+d|$ is bounded below on $SL(2,\mathbb{Z})$, and hence on G. Indeed, decomposing $\tau=x+iy$ into its real and imaginary parts

$$|c\tau+d|^2=(cx+d)^2+c^2y^2\geq\begin{cases} y^2 & c\neq 0\\ d^2\geq 1 & c=0\end{cases} \tag{4.15}$$

Therefore, for any fixed τ there will exist a group element $g\in\bar{G}$ such that $\mathrm{Im}(g\cdot\tau)$ takes a maximal value as a function of g. Note that multiplying g on the left by a power of $\bar{T}$ or $\bar{T}^{-1}$ does not change this property, so there is not a unique g which maximizes $\mathrm{Im}(g\cdot\tau)$. We can fix the ambiguity by requiring $|\mathrm{Re}(g\cdot\tau)|\leq\frac{1}{2}$. Choose such a group element g. We claim that for this transformation, $\tau'=g\cdot\tau\in\tilde{\mathcal{F}}$. We need only check that $|\tau'|\geq 1$. If not, then $|\tau'|<1$ but then $\mathrm{Im}(\bar{S}\cdot\tau')=\mathrm{Im}(\tau')/|\tau'|^2>\mathrm{Im}(\tau')$, contradicting the definition of g. In conclusion, every element of the upper half-plane can be brought to $\tilde{\mathcal{F}}$ by a suitable element of $\bar{G}$.

Now we need two Lemmas:

Lemma 1: If $g\in SL(2,\mathbb{Z})$ and τ have the property that both $\tau\in\tilde{\mathcal{F}}$ and $g\cdot\tau\in\tilde{\mathcal{F}}$ then

1. $|\mathrm{Re}(\tau)|=\frac{1}{2}$ and $g\cdot\tau=\tau\pm 1$, <u>or</u>
2. $|\tau|=1$

To prove this note that, WLOG, we may assume that $\mathrm{Im}(g\cdot\tau)\geq\mathrm{Im}\tau$. (If not replace $g\to g^{-1}$ and $\tau\to g^{-1}\tau$.) But this equation implies $1\geq|c\tau+d|$ which in turn implies:

$$\begin{aligned}1&\geq|c\tau+d|^2\\ &=(cx+d)^2+c^2y^2\\ &=c^2|\tau|^2+2cdx+d^2\\ &\geq c^2|\tau|^2-|cd|+d^2\\ &=c^2(|\tau|^2-\frac{1}{4})+(|d|-\frac{1}{2}|c|)^2\\ &\geq\frac{3}{4}c^2+(|d|-\frac{1}{2}|c|)^2\end{aligned} \tag{4.16}$$

eq:Inequals

From (4.16) we conclude:

1. $(c=0,d=\pm1)$ <u>or</u> $(c=\pm1,d=0,\pm1)$.
2. The inequalities are saturated iff $2cdx=-|cd|$ and $|\tau|=1$.

If $c=0$ and $d=\pm1$ then $g\cdot\tau=\tau\pm1$. In this case it is clear that $|\mathrm{Re}(\tau)|=\frac{1}{2}$. If $c=\pm1$, and $d=0,\pm1$ then the inequality is saturated, and hence $|\tau|=1$.

Lemma 2: If $\tau\in\tilde{\mathcal{F}}$ and $\bar{g}\cdot\tau=\tau$ with $\bar{g}\in PSL(2,\mathbb{Z})$ and $\bar{g}\neq 1$ then either

1. $\tau = \mathrm{i}$ and the stabilizer group is $\{1, \bar{S}\}$
2. $\tau = \omega = e^{2\pi \mathrm{i}/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$ and the stabilizer group is

$$\{1, \bar{S}\bar{T}, (\bar{S}\bar{T})^2\} \tag{4.17}$$

3. $\tau = -\omega^2 = e^{\pi \mathrm{i}/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$ and the stabilizer group is

$$\{1, \bar{T}\bar{S}, (\bar{T}\bar{S})^2\} \tag{4.18}$$

In particular, for all other points $\tau \in \tilde{\mathcal{F}}$, the stabilizer group is the trivial group.

Lemma 2 follows quickly from Lemma 1: If $g\tau = \tau$ then we must be in the case $c = \pm 1$. If $d = 0$ then $a - 1/\tau = \tau$ for some integer a. But we must also have $|\tau| = 1$ and hence $a = \tau + \bar{\tau}$. Since a is an integer we quickly find that $a = 0$ with $\tau = \mathrm{i}$, or $a = \pm 1$ with $\tau = \omega$ or $-\omega^2$. If $d = \pm 1$ then from the saturation condition $2cdx = -|cd|$ we get $x = -\frac{1}{2}d/|d|$ and hence $\tau = \omega$ or $= -\omega^2$.

Now we can finally prove:

Theorem: $SL(2,\mathbb{Z})$ is generated by S and T.

Proof: Let τ_0 be in the interior of $\tilde{\mathcal{F}}$. Then choose any element $g \in SL(2,\mathbb{Z})$ with $\bar{g} \neq 1$. Then there is an element $g' \in \bar{G}$ so that $g'g \cdot \tau_0 \in \tilde{\mathcal{F}}$. Moreover, this element must be in the interior of $\tilde{\mathcal{F}}$ by Lemma 1 and and hence, by Lemma 2, $g'g = 1$ in $PSL(2,\mathbb{Z})$. Therefore $\bar{g} \in \bar{G}$, which means $\bar{G} = PSL(2,\mathbb{Z})$. Moreover, $S^2 = -1$, and hence S and T generate $SL(2,\mathbb{Z})$. ♠

1. The exact fundamental domain $\mathcal{F}$ must be chosen so that no two distinct points on the boundary are G-related. So, for example, we could choose the part of the boundary with $\mathrm{Re}(\tau) \geq 0$.
2. One can show that the relations on S and T are:

$$S^2 = -1 \qquad (ST)^3 = -1 \tag{4.19}$$

3. Given $g \in SL(2,\mathbb{Z})$ it is possible to write the word in S, T giving g by applying the Euclidean algorithm to (a, c) and interpreting the standard equations there in terms of matrices. See Chapter 1, Section 8.
4. Although the keyhole region is the standard fundamental domain there is no unique choice of fundamental domain. For example, one could equally well use any of the images shown in Figure 8 (and of course there are infinitely many such regions). Moreover, we could displace $\mathcal{F} \to \mathcal{F} + \epsilon$ and still produce a fundamental domain.

5. The action of $PSL(2,\mathbb{Z})$ is properly discontinuous on $\mathcal{H}$, but not quite free. If we consider finite-index subgroups that do not contain the stabilizer groups mentioned above then the action will be free and the quotient space will be a nice Riemann surface.

Exercise

a.) The *face-centered-cubic* lattice in $\mathbb{R}^3$ is the sublattice of $\mathbb{Z}^3$ of all points such that $\sum_i x_i = 0 \mathrm{mod} 2$. Construct the Wigner-Seitz cell for the fcc lattice.

b.) The *body-centered-cubic* lattice in $\mathbb{R}^3$ is the sublattice of $\mathbb{Z}^3$ such that the coordinates x_i are all even or all odd. Construct the Wigner-Seitz cell for the bcc lattice.

Exercise

a.) The group $\Gamma_0(2)$ is the subgroup of $SL(2,\mathbb{Z})$ of matrices with $c = 0 \mathrm{mod} 2$. Find a fundamental domain for $\bar{\Gamma}_0(2)$ acting on $\mathcal{H}$.

b.) The group $\Gamma(2)$ is the subgroup of $SL(2,\mathbb{Z})$ of matrices congruent to 1 modulo 2. Find a fundamental domain for $\bar{\Gamma}(2)$.

4.3 Algebras and double cosets

TO BE WRITTEN

4.4 Orbifolds

An interesting class of examples where the quotient space is "almost" a manifold are called "orbifolds" or "V-manifolds."

In a manifold, neighborhoods of points locally look like copies of $\mathbb{R}^n$.

In an orbifold neighborhoods of points locally look like copies of $\mathbb{R}^n/\Gamma$ where Γ is a finite group acting on $\mathbb{R}^n$. The finite group Γ can depend on the point in question. For example, for "most" points it might be trivial. But for some other points it might be nontrivial

Example 1: Let $\mathbb{Z}_2$ act on $\mathbb{R}^n$ by $\sigma \cdot \vec{x} = -\vec{x}$. Then in the quotient space $\mathbb{R}^n/\mathbb{Z}_2$ the neighborhood of every point is homeomorphic to $\mathbb{R}^n$ except for the origin. A neighborhood of $[\vec{0}]$ is a cone on $\mathbb{RP}^{n-1}$. Since $\mathbb{RP}^{n-1}$ is not homotopy equivalent to S^{n-1} for $n > 2$ this space cannot be homeomorphic to $\mathbb{R}^n$ for $n > 2$.

Example 2: Consider the n-dimensional torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$. Let $\Gamma = \mathbb{Z}_2$ act on T^n with the action induced from $\sigma \cdot \vec{x} = -\vec{x}$ on $\mathbb{R}^n$. There are now 2^n fixed points. At each fixed point the local neighborhood is homeomorphic to $\mathbb{R}^n/\mathbb{Z}_2$.

Example 3: Let $G \cong \mathbb{Z}/N\mathbb{Z}$ be the group of N^{th} roots of 1 acting on the complex plane by multiplication. The quotient $\mathbb{C}/\mathbb{Z}_N$ is an orbifold. The stabilizer group is trivial everywhere except the origin, where it is all of G. A neighborhood of 0 should be viewed as a cone with opening angle $2\pi/N$.

Example 4: Recall that we showed that we can identify $\mathcal{H} \cong SL(2,\mathbb{R})/SO(2)$. Now consider the double-coset

$$\Gamma\backslash SL(2,\mathbb{R})/SO(2) \tag{4.20}$$

where $\Gamma = SL(2,\mathbb{Z})$. We can identify this as the set of orbits of Γ on $\mathcal{H}$. From our analysis of the fundamental domain above we see that it is topologically a sphere with two orbifold singularities. The one at [i] has a neighborhood modeled on $\mathbb{C}/\mathbb{Z}_2$ and the one at $[\omega] = [-\bar{\omega}]$ has a neighborhood modeled on $\mathbb{C}/\mathbb{Z}_3$. In addition, there is one puncture (at $\tau = \mathrm{i}\infty$).

Exercise *Weighted projective spaces*

Choose positive integers $p_1, \cdots p_{n+1}$. Then the *weighted projective space* $W\mathbb{P}[p_1, \dots, p_{n+1}]$ is the space defined by $(\mathbb{C}^{n+1} - \{\vec{0}\})/\mathbb{C}^*$ with $\mathbb{C}^*$ action:

$$\lambda \cdot (z_1, \dots, z_{n+1}) := (\lambda^{p_1} z_1, \dots, \lambda^{p_{n+1}} z_{n+1}) \tag{4.21}$$

Show that the resulting quotient space is a well-defined orbifold. What are the orbifold singularities?

♣Do something with double cosets and Hecke algebras. ♣

4.5 Examples of quotients which are not manifolds

When we divide by a *noncompact group* like $\mathbb{C}^*$ or $GL(n,\mathbb{C})$ then, even when the action is free the quotient space M/G can be a "bad" (e.g. non-Hausdorff, or very singular and difficult-to-work-with) space.

Example 1 Consider the circle $S^1 \cong \mathbb{R}/\mathbb{Z}$. Consider translation $x \to x + \alpha$ where α is *irrational.* This generates an action of $\mathbb{Z}$ on S^1 (or $\mathbb{Z} \oplus \mathbb{Z}$ on $\mathbb{R}$) that is *not* properly discontinuous. The quotient space is *not* a manifold.

Example 2 . A similar example. Consider the two-dimensional torus, identified with $S^1 \times S^1 \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. Thus we can use coordinates (x, y) where x, y are defined modulo 1. Consider the group action by $\mathbb{R}$:

$$(x, y) \to (x + tv_1, y + tv_2) \tag{4.22}$$

where $t \in \mathbb{R}$, (v_1, v_2) is some vector. If the slope v_2/v_1 is a rational number the orbits are compact and the space of orbits is a nice space. (Exercise: What is this space?) But if the slope is irrational we again have a non-Hausdorff space since the orbits are all dense, so it is impossible to separate open sets around these orbits.

Example 3: Consider $X = \mathbb{C}^{n+1}$ and $G = \mathbb{C}^*$ acting as

$$\lambda \cdot \vec{z} = \lambda \cdot (z_1, \dots, z_{n+1}) := (\lambda z_1, \dots, \lambda z_{n+1}) \tag{4.23}$$

for $\lambda \in \mathbb{C}^*$. Then $X/\mathbb{C}^*$ is not Hausdorff. For consider the equivalence class $[\vec{0}]$. Let $\pi : X \to X/\mathbb{C}^*$ be the projection. An open neighborhood U of $[\vec{0}]$ is such that $\pi^{-1}(U)$ is an open neighborhood of $\vec{0}$. Now consider any other point $[\vec{w}]$ where $\vec{w} \neq 0$. There will always be a $\lambda \in \mathbb{C}^*$ so that $\lambda \vec{w} \in \pi^{-1}(U)$. Therefore, $[\vec{w}] \in U$ for *all* points $[\vec{w}] \in X/\mathbb{C}^*$! In particular one cannot separate $[\vec{0}]$ and $[\vec{w}]$ by open sets. Clearly, there is one bad actor here, the point $\vec{0}$. Indeed if we eliminate it then the $\mathbb{C}^*$ action on $\mathbb{C}^* - \{0\}$ produces a good manifold

$$\mathbb{C}P^n = (\mathbb{C}^* - \{0\})/\mathbb{C}^* \tag{4.24}$$

Example 4: For a very similar example let us consider a $\mathbb{C}^*$ action on $\mathbb{C}^2$ with two coordinates ϕ_1, ϕ_2 and action:

$$\begin{aligned} \phi_1 &\to \lambda \phi_1 \\ \phi_2 &\to \lambda^{-1} \phi_2 \end{aligned} \tag{4.25}$$

eq:scalen

Generic orbits are labelled by the "gauge invariant" quantity $\phi_1 \phi_2$. Note however that there are 3 special orbits corresponding to the value $\phi_1 \phi_2 = 0$:

$$\begin{aligned} \mathcal{O}_1 &= \{(\phi_1, 0) | \phi_1 \neq 0\} \\ \mathcal{O}_2 &= \{(0, \phi_2) | \phi_2 \neq 0\} \\ \mathcal{O}_3 &= \{(0, 0)\} \end{aligned} \tag{4.26}$$

Unlike projective space now the space

$$[\mathbb{C}^2 - \mathcal{O}_3]/\mathbb{C}^* \tag{4.27}$$

is NOT a Hausdorff space. Indeed the orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ project to two distinct points $[\mathcal{O}_1]$ and $[\mathcal{O}_2]$ and we claim these cannot be separated from each other by open sets in the quotient topology. See the exercise below.

We could just consider the space

$$[\mathbb{C}^2 - (\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3)]/\mathbb{C}^* \tag{4.28}$$

eq:easyquot

Now the quotient by $\mathbb{C}^*$ is Hausdorff and is a nice algebraic variety. It is just a copy of $\mathbb{C}^*$.

But we could compactify (4.28). One way to do this is to omit the axes $\phi_1 = 0$ and $\phi_2 = 0$, but include the origin, then the quotient space $\mathbb{C}^2 - (\mathcal{O}_1 \cup \mathcal{O}_2)$ is well-defined and in fact isomorphic to $\mathbb{C}$. But in fact, there are three ways of getting a good quotient by omitting any two of the three special orbits and then dividing by $\mathbb{C}^*$.

Remark: We did not use the complex numbers in an essential way in Example 4 so we can apply the very same ideas to the quotient of Minkowski space by the component of the identity of the Lorentz group. We can consider the real subspace $\mathbb{R}^2 \subset \mathbb{C}^2$ and interpret ϕ_1, ϕ_2 as light-cone coordinates. Then we consider the subgroup of $\mathbb{C}^*$ corresponding to $\mathbb{R}_+$. We studied the orbits in Section **** above. Using the same kind of reasoning as above we see that the quotient is not a Hausdorff space since the origin cannot be separated from the lightlike orbits. Even if we remove the origin the quotient is not Hausdorff since we cannot separate timelike and spacelike orbits from lightlike orbits. If we remove the origin and the lightlike orbits *then* the quotient space is a Hausdorff space. It is naturally thought of as four copies of $\mathbb{R}_+$.

Example 5: Another example similar to the above is the following. Let us consider the set of conjugacy classes of all matrices in $M \in M_n(\mathbb{C})$ under the action of $S \in GL(n, \mathbb{C})$:

$$M \mapsto SMS^{-1} \tag{4.29}$$

Let us consider the conjugacy classes of matrices:

$$\mathcal{C}(n) := M_n(\mathbb{C})/GL(n, \mathbb{C}) \tag{4.30}$$

we could consider this as the set of linear transformations $T : V \to V$ up to change of basis, where V is an n-dimensional complex vector space.

The key fact here is that every matrix can be brought to Jordan canonical form: (See Linear Algebra User's Manual, chapter **** for more details, application, and a proof.)

Take the characteristic polynomial $p(x) = \det(x\mathbf{1} - M)$ and factor it into its distinct complex roots $p(x) = \prod_{i=1}^{s}(x - \lambda_i)^{r_i}$ where $r_i > 0$. Then, first of all we can bring M to block diagonal form $M = \oplus_{i=1}^{s} M_i$, where each block M_i is itself a block of Jordan matrices of type i, that is, there are positive integers n_α^i, $1 \le \alpha \le k_i$ with $\sum_\alpha n_\alpha^i = k_i$ such that

$$M_i = \oplus_{\alpha=1}^{k_i} J_{\lambda_i}^{(n_\alpha^i)} \tag{4.31}$$

where $J_\lambda^{(1)}$ is the 1×1 matrix with entry λ and, for $n > 1$, $J_\lambda^{(n)} = \lambda \mathbf{1}_{n\times n} + \sum_{j=1}^{n-1} e_{j,j+1}$ cannot be brought to a simpler form, and certainly cannot be diagonalized.

If a matrix M has nontrivial Jordan form then the quotient space $M_n(\mathbb{C})/GL(n, \mathbb{C})$ is non-Hausdorff in the neighborhood of $[M]$. To see why it suffices to consider the case $n = 2$: The characteristic polynomial $p(x) := \det(x\mathbf{1} - M)$ is now order two. When there are two distinct roots M is diagonalizable. When two roots coincide, say $p(x) = (x - \lambda)^2$, then there are two possibilities: Either $M = \lambda \mathbf{1}_{2\times 2}$ for some λ, or, M is conjugate to Jordan form:

$$M = S \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} S^{-1} \tag{4.32}$$

The existence of matrices with nontrivial Jordan blocks leads to non-Hausdorff behavior of the quotient. To see this it suffices to consider conjugation by matrices of the form:

$$g = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \in GL(2, \mathbb{C}) \tag{4.33}$$

then

$$g \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} g^{-1} = \begin{pmatrix} \lambda & z \\ 0 & \lambda \end{pmatrix} \tag{4.34}$$

with $z = t_1/t_2$ so

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \sim \begin{pmatrix} \lambda & z \\ 0 & \lambda \end{pmatrix} \tag{4.35}$$

for any $z \in \mathbb{C}^*$.

Now, suppose $\bar{V} \subset \mathcal{C}(2)$ is an open set in the quotient topology and moreover

$$[\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}] \in \bar{V} \tag{4.36}$$

$p^{-1}(\bar{V})$ is an open set in $M_2(\mathbb{C})$ containing the matrix $\lambda \mathbf{1}_{2\times 2}$. But any such open set must also contain

$$\begin{pmatrix} \lambda & z \\ 0 & \lambda \end{pmatrix} \tag{4.37}$$

for some sufficiently small z. Therefore,

$$[\begin{pmatrix} \lambda & z \\ 0 & \lambda \end{pmatrix}] = [\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}] \tag{4.38}$$

for $z \neq 0$, is a distinct point from

$$[\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}] \tag{4.39}$$

and yet any neighborhood of $[\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}]$ contains $[\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}]$. So, these distinct points cannot be separated by open sets.

If, by hand, we consider the subset of diagonalizable matrices in $M_2(\mathbb{C})$ then the quotient is better. If $m \in GL(2,\mathbb{C})$ is of the form:

$$m = g \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} g^{-1} \qquad\qquad z_1 \neq z_2 \tag{4.40}$$

then $[m]$ is parametrized by the *unordered* pair $\{z_1, z_2\}$. This is just the configuration space $\mathbb{C}^2/\mathbb{Z}_2$ and is just an orbifold, with a $\mathbb{Z}_2$ orbifold singularity along the diagonal $\{z, z\}$.

Returning to the general case, if we consider $M_n^{\mathrm{ss}}(\mathbb{C}) \subset M_n(\mathbb{C})$ of diagonalizable matrices then the quotient $M_n^{\mathrm{ss}}(\mathbb{C})/GL(n,\mathbb{C})$ is much better behaved. It is an orbifold, with orbifold singularities corresponding to various symmetric groups along loci where eigenvalues coincide.

Remarks

1. Making nice quotients is part of the general subject of "geometric invariant theory." See, Fogarty, Kirwan, and Mumford, *Geometric Invariant Theory* for a sophisticated presentation of conditions in algebraic geometry for when quotients are "good."

2. The mathematics of Geometric Invariant theory finds several applications in supersymmetric field theory and string theory. First of all, it is important in forming various moduli spaces. (For example, we saw it was necessary in forming a good quotient space corresponding to the moduli space of lines.) Closely related to this, it is important in understanding the moduli spaces of vacua in SUSY field theory. ♣Explain more. Section in GMP2010 ch.5, section 6. ♣

3. The examples of quotients by ergodic actions such as the irrational rotation on a circle forms one of the primary examples in the study of *noncommutative geometry.* See, A. Connes, *Noncommutative Geometry* for much detailed discussion.

Exercise

Consider a small open neighborhood of $[\mathcal{O}_1]$:

$$\{[(1,z_2)] : |z_2| < \epsilon\} \tag{4.41}$$

and a small open neighborhood of $[\mathcal{O}_2]$:

$$\{[(z_1,1)] : |z_1| < \epsilon'\} \tag{4.42}$$

a.) Show that when $n > 0$ in (4.25) these sets will intersect each other no matter how small we take ϵ, ϵ'

Therefore we cannot separate $[\mathcal{O}_1]$ from $[\mathcal{O}_2]$ and the quotient space is not Hausdorff.

4.6 When is the quotient of a manifold by an equivalence relation another manifold?

A natural question which arises from these examples is, more generally, when the quotient of a manifold by a general equivalence relation is another manifold.

An equivalence relation $\sim$ on any set X has a *graph* which is the subset $R \subset X \times X$ defined by

$$R = \{(x,y)|x \sim y\} \tag{4.43}$$

Conversely, one could define an equivalence relation from subsets of $X \times X$ satisfying certain properties.

We have already discussed how, if X is a topological space and $\sim$ is an equivalence relation then we can define a topological space with a continuous projection $p : X \to X/\sim$. A question which frequently arises is this: If M is a manifold, when is $M/\sim$ a manifold?

One criterion for answering this question is the following theorem: [10]

Theorem: If M is a smooth manifold and $\sim$ is an equivalence relation then the following are equivalent:

[10] For a proof see Theorem 8.3 of the notes "Differential Geometry," by R.L. Fernandes, in http://www.math.illinois.edu/~ruiloja/Math519/notes.pdf.

1. $M/\sim$ is a smooth manifold and $p : M \to M/\sim$ is a submersion.

2. The graph $R \subset M \times M$ is a proper smooth submanifold and the projection $p_1 : M \times M$ onto the first factor, when restricted to R, is a proper submersion.

♣SHOULD ILLUSTRATE HOW THIS THEOREM IS USEFUL. ♣

5. Isometry groups

♣AT THIS POINT PROBABLY SHOULD SPLIT OFF SEPARATE CHAPTERS ON ISOMETRY GROUPS AND REGULAR POLYTOPES AND ON SYMMETRIC FUNCTIONS. ♣

One natural way in which group actions appear in physics is via the set of transformations preserving some notion of distance. For example, we could consider transformations preserving lengths and angles. To fix ideas, let us consider $\mathbb{R}^3$ with Euclidean metric

$$\parallel \vec{x} \parallel^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 \tag{5.1}$$

Definition: An *isometry* of the Euclidean metric is a map $T : \mathbb{R}^3 \to \mathbb{R}^3$ which is norm-preserving:

$$\parallel T(\vec{x}) - T(\vec{y}) \parallel = \parallel \vec{x} - \vec{y} \parallel \tag{5.2}$$

The group of all isometries is denoted $E(3)$. More generally, the group of isometries of Euclidean $\mathbb{R}^n$ is denoted $E(n)$.

Two examples:

1.) **Translations**

$$T_{\vec{a}} : \vec{x} \mapsto \vec{x} + \vec{a} \tag{5.3}$$

Now

$$T_{\vec{a}} \circ T_{\vec{b}} = T_{\vec{a}+\vec{b}} \tag{5.4}$$

so the $T_{\vec{a}}$'s form an abelian subgroup $\mathcal{T}$ of $E(3)$. We have $\mathcal{T} \cong \mathbb{R}^3$.

2.) **Rotations**

For $R \in O(3)$ we let $R \cdot \vec{x}$ be the vector with components $(R\vec{x})^i = R^i{}_j x^j$.

One can show [11] that the *full* isometry group of $\mathbb{R}^3$ is the semidirect product of $\mathcal{T}$ with $O(3)$:

$$E(3) \cong \mathcal{T} \ltimes O(3) \tag{5.5}$$

To see that this is a semidirect product let

$$(\vec{a}, R) \cdot \vec{x} = R\vec{x} + \vec{a}. \tag{5.6}$$

Then

$$(a_1, R_1)(a_2, R_2) = (a_1 + R_1 a_2, R_1 R_2) \tag{5.7}$$

[11] This is surprisingly hard. See P. Yale, *Geometry and Symmetry*, Holden-Day, San Francisco, California, 1968, for a proof.

clearly defines a twisted multiplication law. Put differently, every element of $E(3)$ is a product of a translation and a rotation and the subgroup of translations is a normal subgroup. Indeed, $O(3)$ acts as a group of automorphisms on $\mathcal{T}$ where $O(3) \to Aut(\mathcal{T})$ is defined by

$$\alpha_R : T_{\vec{a}} \to T_{R\vec{a}} \tag{5.8}$$

This follows from

$$RT_{\vec{a}}R^{-1} = T_{R\vec{a}} \tag{5.9}$$

We can define a homomorphism det : $E(3) \to \mathbb{Z}_2$, by taking $\mathrm{det}R$. The kernel of this homomorphism are the *proper transformations* and the complement are the *improper transformations.*

It is useful to have simple pictures of the different group elements in $E(3)$. We begin with the following theorem:

Theorem:

a.) Every rotation in $SO(3)$ is a rotation about an axis.

b.) More generally, every rotation $R \in SO(n)$ in an odd number of dimensions fixes some line.

Proof: Note that for $R \in SO(n)$, $R - I = R(I - R^{tr})$ and $\mathrm{det}R = 1$ so

$$\mathrm{det}(R - I) = \mathrm{det}(I - R^{tr}) = \mathrm{det}(I - R) = (-1)^n \mathrm{det}(R - I) \tag{5.10}$$

so $\mathrm{det}(R - I) = 0$ for n odd. ♠

Note: In part (b) we are not asserting that R is a rotation about an axis.

Exercise *Geometrical interpretation of the elements and conjugacy classes of* $E(3)$

Denote by $C_{\vec{k}}(\theta)$ a rotation by θ about an axis through a vector $\vec{k}$.

a.) If $0 < \theta < 2\pi$ and $\vec{a}$ is perpendicular to $\vec{k}$ show that $(\vec{a}, C_{\vec{k}}(\theta))$ has an infinite number of fixed points, giving a straight line in $\mathbb{R}^3$ parallel to the axis through $\vec{k}$. Indeed $(\vec{a}, C_{\vec{k}}(\theta))$ is a rotation by θ around this axis.

b.) Show that every proper Euclidean transformation is a *screw displacement.* This is a rotation about some axis followed by a translation along that axis.

c.) Let $\sigma_{\vec{k}}$ be the reflection in a plane through the origin perpendicular to $\vec{k}$. Show that

$$\sigma_{\vec{k}}\vec{x} = \vec{x} - \frac{2(\vec{x} \cdot \vec{k})}{(\vec{k} \cdot \vec{k})}\vec{k} \tag{5.11}$$

d.) Show that any improper rotation in $O(3)$ can be written as $S_{\vec{k}}(\theta) = \sigma_{\vec{k}} C_{\vec{k}}(\theta)$.

e.) Show that $(\vec{a}, \sigma_{\vec{k}})$ is the reflection in a plane parallel to the plane through the origin and perpendicular to $\vec{k}$, and translation by a vector in that plane. This is called a *glide reflection.*

f.) Show that $(\vec{a}, S_{\vec{k}}(\theta))$ for $0 < \theta < 2\pi$ is a rotation-inversion about some point.

g.) The distinct conjugacy classes in $E(3)$ are:

1. Translations: $\{T_{\vec{a}} | \parallel \vec{a} \parallel = r\}$, $r > 0$. Infinite conjugacy class.
2. Screw displacements: $\{(\vec{a}, C_{\vec{k}}(\theta)), 0 < \theta < \pi$. Parametrized by TS^2.
3. Glide reflections: $\{(\vec{a}, \sigma_{\vec{k}})\}$.
4. Rotation-inversions $(\vec{a}, S_{\vec{k}}(\theta))$ for $0 < \theta < 2\pi$. Parametrized by $\mathbb{R}^3 \times (SO(3) - \{1\})$.

A good reference for this material is
Miller, Symmetry Groups and Their Applications

6. Symmetries of regular objects

One very natural occurance of groups in physics, and elsewhere, is as symmetries of subsets of Euclidean space.

Suppose $\mathbb{R}^n$ has the Euclidean metric. Consider a subset $X \subset \mathbb{R}^n$. Then a *symmetry group of* X is a subgroup $G \subset E(n)$ such that G maps X to itself. The group of all isometries preserving X is the complete symmetry group of X.

Example : If X is the sphere then the complete symmetry group is $O(n)$.

In this section and the next we will examine the symmetry groups of various regular polygons and solids in $\mathbb{R}^2$ and $\mathbb{R}^3$. These are subgroups of $E(3)$. Now, classification of all subgroups would be very hard. For example, already in $SO(2)$ we could consider the group generated by several rotations by incommensurate irrational multiples of 2π.

It is good to restrict attention to *discrete groups*. These are groups such that if $B(r, \vec{x})$ is the sphere of radius r about $\vec{x}$ then for any $\vec{y}$, there are a finite number of points in the G-orbit of $\vec{y}$ in $B(r, \vec{x})$.

If X is a bounded region, then a symmetry group G of X must have some fixed point $\vec{y}_{cm}$ in $\mathbb{R}^3$. For a discrete group if $\vec{y} \in X$ the G-orbit of $\vec{y}$ must be finite (because X is bounded). Let us denote those points by $\{\vec{y}_1, \dots, \vec{y}_n\}$. Then we can form the center of mass

$$\vec{y}_{cm} = \frac{1}{n} \sum_i \vec{y}_i \tag{6.1}$$

and it is easily seen to be a fixed point of G. By the classification of conjugacy classes above we see that G must be a subgroup of rotations or rotation-inversions about $\vec{y}_{cm}$, and is hence conjugate to a finite subgroup of $O(2)$ or $O(3)$.

subsec:sdisc

The finite subgroups of $SO(2)$ are just the cyclic groups of order n, denoted $C_n \subset SO(2)$. [12]

C_n is by definition the group of rotations about the origin $\vec{0}$ in $\mathbb{R}^2$ by $\theta = \frac{2\pi k}{n}$ $\quad k = 0, 1, \cdots, n-1$ in the counterclockwise direction:

$$C_n := \{R(\frac{2\pi k}{n}) = \begin{pmatrix} \cos\frac{2\pi k}{n} & -\sin\frac{2\pi k}{n} \\ \sin\frac{2\pi k}{n} & \cos\frac{2\pi k}{n} \end{pmatrix}, k = 0, \dots, n-1\} \tag{6.2}$$

[12] There are two common notations for subgroups of $E(3)$, the Schoenfliess notation and the international notation. We will be using the Schoenfliess notation.

when n is understood we denote the rotations by R_k.

Exercise
Show that $R_{k_1} R_{k_2} = R_{k_1+k_2 \mathrm{mod} n}$ and deduce the isomorphism: $\mathbb{Z}/n\mathbb{Z} \cong C_n$.

If we now consider $O(2)$ there is a new geometrical operation corresponding to parity. Suppose we take P to be reflection in the x-axis:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.3}$$

Note that $P^2 = 1$ and

$$PR(\theta)P = R(-\theta) \tag{6.4}$$ eq:reflc

so the group generated by P and C_n is nonabelian for $n \geq 3$.

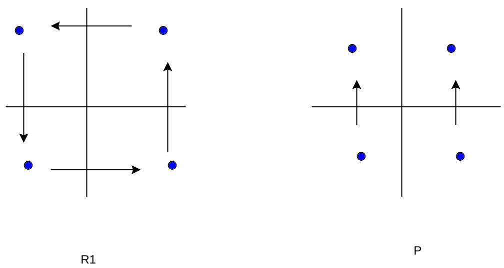

Figure 9: Operations of C_4 and P on 4 points. fig:prtbdiag

For $n = 4$ the action of R_1, P are shown in 9.

Moreover, a reflection in any other axis through the origin can be computed by rotating the axis to the x-axis, say by an angle ψ, then reflecting, and then rotating back. In equations this gives:

$$P_\psi = R(-\psi)PR(\psi) = PR(2\psi) = R(-2\psi)P \tag{6.5}$$

also satisfies

$$P_\psi^2 = 1 \qquad\qquad P_\psi R(\theta) P_\psi = R(-\theta) \tag{6.6}$$

Thus the groups generated by P_ψ and C_n, call them $D_{n,\psi}$ are all isomorphic as abstract groups, although they are different discrete subgroups of $O(2)$. Indeed, the $D_{n,\psi}$ are conjugate subgroups within $O(2)$. The common abstract group is known as the *dihedral group* and denoted as D_n.

Put more precisely D_n *is the group generated by* x, y *subject to the relations:*

$$\begin{aligned} x^2 &= 1 \\ y^n &= 1 \\ xyx^{-1} &= y^{-1} \end{aligned} \tag{6.7}$$ eq:rels

There is an isomorphism to the discrete subgroups of $O(2)$ given by $y \to R_1$ and $x \to P_\psi$.

Exercise *Generators and relations for the dihedral group D_n*

a.) Show that the relation $xyx^{-1} = y^{-1}$ can equivalently be written as

$$(xy)^2 = 1 \qquad \text{or} \qquad xy = y^{-1}x \qquad \text{or} \qquad xyx = y^{-1} \tag{6.8}$$

b.) Show that every element of D_n can be written as y^j or xy^j, and hence conclude that the order of D_n is $2n$.

c.) Show that there are n elements of order 2.

d.) The relations (6.7) show that D_n is nonabelian for $n \geq 3$. C_n is a normal subgroup. Show that

$$D_n/C_n \cong \mathbb{Z}_2 \tag{6.9}$$

e.) Indeed, define a homomorphism $\psi : D_n \to \mathbb{Z}_2$ whose kernel is C_n and deduce that

$$1 \to C_n \to D_n \to \mathbb{Z}_2 \to 1 \tag{6.10}$$

f.) Show that $\mathbb{Z}_2$ acts as a group of automorphisms on C_n and

$$D_n \cong C_n \ltimes \mathbb{Z}_2 \tag{6.11}$$

Exercise

Show that $D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

Exercise

Define a homomorphism $D_n \times \mathbb{Z}_2 \to D_{2n}$ as follows. Let

$$D_n = \langle x, z | x^2 = 1, z^n = 1, xzx = z^{-1} \rangle \tag{6.12}$$

$$D_{2n} = \langle x, y | x^2 = 1, y^{2n} = 1, xyx = y^{-1} \rangle \tag{6.13}$$

and let $\mathbb{Z}_2$ be generated by σ. Then define ϕ by its action on generators:

$$\begin{aligned} z &\to y^2 \\ x &\to x \\ \sigma &\to y^n \end{aligned} \tag{6.14}$$

a.) Show that ϕ is a homomorphism.

b.) Show that ϕ is an *isomorphism* for n odd, but is not an isomorphism for n even.

c.) Show that $\pi : D_{2n} \to D_n$ defined by $\pi : y \to z$ and $\pi : x \to x$ defines a homomorphism. Show that the kernel is isomorphic to $\mathbb{Z}_2$. Show that a section can be defined by

$$s : z \to y^{n+1} \qquad s : x \to x \tag{6.15}$$

and that this is a homomorphism for n odd, but not for n even.

d.) Show that D_{2n} is a central extension of D_n by $\mathbb{Z}_2$:

$$1 \to \mathbb{Z}_2 \to D_{2n} \to D_n \to 1 \tag{6.16}$$

And it is a split central extension for n odd.

6.1 Symmetries of polygons in the plane

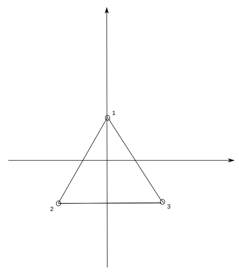

Figure 10: The symmetry group of the equilateral triangle is isomorphic to the symmetric group S_3. `fig:triangle`

Consider the symmetries of the equilateral triangle.

There are clearly 3 axes of reflection symmetry. If we label the vertices of the triangle $1, 2, 3$ they correspond to the three transpositions (12), (13), (23).

There are also 3-fold rotational symmetries about the center of mass of the triangle. These generate the subgroup C_3, and correspond to the cyclic transformations $1, (123), (132)$.

Thus, the group of symmetries is the full symmetric group S_3. On the other hand, it is also clearly the dihedral group D_3. Indeed, $S_3 \cong D_3$. The explicit isomorphism is:

$$\begin{aligned} 1 &\to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ (23) &\to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ (123) &\to \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \end{aligned} \tag{6.17}$$

Note that this is a matrix representation of S_3.

Example .: The symmetries of the square.

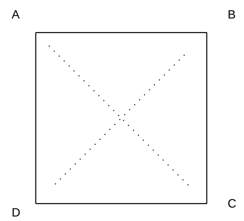

Figure 11: A square with labeled vertices. `fig:symsq`

Label the vertices A, B, C, D and consider the rotations and reflections in the plane preserving the square. Each such symmetry defines a permutation of the set $\{A, B, C, D\}$, and hence an element of S_4. Thus, the symmetries of the square form a subgroup of S_4.

Exercise

Label transformations as permutations of 1,2,3,4. Show there are 8 elements. Work out the multiplication table. Identify the symmetry group of the square as the dihedral group D_4.

Note that the full symmetry group is only a subgroup of S_4:

$$\{1, (1234)^j, (13), (24), (12)(34), (14)(23)\} \tag{6.18}$$

Exercise

D_4 can also be understood as a central extension of $\mathbb{Z}_2 \times \mathbb{Z}_2$ by $\mathbb{Z}_2$:

$$1 \to \mathbb{Z}_2 \to D_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1 \tag{6.19}$$

Interpret this central extension geometrically.

Hint: $x \to \sigma_1, y \to \sigma_1\sigma_2$.

Figure 12: A regular n-gon. `fig:ngoni`

Finally, let us consider the regular n-gon in the plane, centered at the origin. It is clear that C_n is a subgroup, and there are reflection symmetries, so

The complete symmetry group of the regular n-gon in the plane is D_n.

Figure 13: Order two symmetry axis of the n-gon for n odd. fig:ngonii

Figure 14: Order two symmetry axis of the n-gon for n even. fig:ngoniii

Remarks *Conjugacy classes of D_n.* It is interesting to work out the conjugacy classes in D_n. First, let us think of it geometrically.

Now let us consider the elements of order 2, i.e. reflections in some axis.

1. If n is odd, each symmetry axis passes through a vertex and a face as in 13. There are n elements of order two, corresponding to PR_k, and they are all conjugate.

2. If n is even there are $n/2$ axes through antipodal vertices and $n/2$ axes through antipodal edge mid-points as in 14. Again we find n elements of order 2, but they separate into two sets of distinct conjugacy classes.

Now let us consider the question algebraically:

Note that $(xy^k)y^j(xy^k)^{-1} = y^{n-j}$. Geometrically, we are reflecting a rotation through an axis. Also note that $x(xy^j)x = xy^{n-j}$, while $y^k(xy^j)y^{-k} = xy^{j-2k}$. Now we must distinguish between n odd and n even.

For n odd we have

$$\begin{aligned} C(1) &= \{1\} \\ C(x) &= \{x, xy, \dots, xy^{n-1}\} \\ C(y^j) &= \{y^j, y^{n-j}\} \qquad j = 1, \dots, (n-1)/2 \end{aligned} \tag{6.20}$$

eq:dncci

This gives a total of $(n+3)/2$ distinct conjugacy classes.

For n even we have

$$\begin{aligned} C(1) &= \{1\} \\ C(x) &= \{x, xy^2, \dots, xy^{n-2}\} \\ C(xy) &= \{xy, xy^3, \dots, xy^{n-1}\} \\ C(y^j) &= \{y^j, y^{n-j}\} \qquad j = 1, \dots, \frac{n}{2} - 1 \\ C(y^{n/2}) &= \{y^{n/2}\} \end{aligned} \tag{6.21}$$

eq:dncciev

This gives a total of $3 + n/2$ conjugacy classes. Note that there are two distinct conjugacy classes of order two elements. These are the two kinds of reflection axes.

Figure 15: A regular n-pyramid. fig:regpyr

6.2 Symmetry groups of some regular solids in $\mathbb{R}^3$

1. Consider the regular n-pyramid. Its base is a regular n-gon, and the distance to the vertex is not equal to the length of a side of the n-gon. A symmetry group is plainly C_n, rotations by multiples of $2\pi/n$ about the axis of the pyramid. To this we can add reflections in the symmetry planes to produce a subgroup of $O(3)$ known as C_{nv} in the Schoenfliess system to get the complete symmetry group of the n-pyramid. This group is isomorphic to D_n.

Figure 16: A regular n-prism. The isomorphic to D_n. The complete symmetry group is $D_n \times \mathbb{Z}_2$. I_3 is not a symmetry for n odd. fig:regprism

2. Consider the regular n-prism. Its base is a regular n-gon and the height is not equal to the length of a side of the n-gon.

The symmetries are generated by rotations about the axis by multiples of $2\pi/n$, reflections in the planes defined by the axes of symmetry of the n-gon times the z-direction, and reflection in the xy plane. For the latter we choose coordinates so that the midpoint of the axis of symmetry is at the origin, and the polygons are parallel to the xy plane.

Note that we have therefore the group $D_n \times \mathbb{Z}_2$ as the complete symmetry group of the prism. The D_n symmetries act in the 2×2 block leaving the z-coordinate fixed,

$$\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \qquad g \in D_n \subset O(2) \tag{6.22}$$ eq:prsmsymm

while the $\mathbb{Z}_2$ can be taken to be

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{6.23}$$

eq:prsii

The group of *proper* symmetries is in fact D_n, but it is not the "same" D_n subgroup as in (6.22). Rather we take matrices of the form (6.22) for $g \in C_n$, but for g of the form xy^j we multiply by (6.23). Thus we have two nonconjugate copies of D_n in $O(3)$ acting as symmetries.

Note that in the case when n is even, $g = -1_{2\times 2} \in C_n$ and hence the reflection I_3 in all three coordinates is a symmetry.

For n odd, I_3 is *not* a symmetry. The complete group of proper symmetries is D_n. We can generate it by rotations about the axis, together with a product of reflections in the symmetry planes times a reflection in the xy plane. The complete symmetry group is then obtained by adding the reflection in the xy plane. Thus, the symmetry group is again $D_n \times \mathbb{Z}_2$. Notice that this group is isomorphic to D_{2n} (which is how it is presented in tables).

The complete symmetry group of the n-prism – the prismatic symmetry group - is denoted D_{nh} in the Schoenfliess system

6.3 The symmetry group of a baseball

Figure 17: Sketch of a baseball.

fig:baseball

As a final example, let us consider the symmetry group of a baseball, drawn in 17. Note that we have symmetry elements acting as transformations in $O(3)$:

$$R = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{6.24}$$

and

$$P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.25}$$

Check that this gives group isomorphic to D_4 since it can be presented using generators and relations:

$$\langle R, P : R^4 = 1, P^2 = 1, PRP^{-1} = R^{-1} \rangle \tag{6.26}$$

Note that this is a little tricky: R is an improper transformation but it is of order 4. This is the group D_{2d} in the Schoenfliess notation.

As we will see later, it is a direct sum of two irreducible representations of D_4.

7. The symmetries of the platonic solids

A *regular polyhedron* is a solid whose boundaries are regular polygons such that at each vertex, the nearest neighbor vertices form a regular polygon. There are 5 regular polyhedrons in three dimensions, much loved and studied since the time of the ancient Greek mathematicians. These are known as the platonic solids.

The 5 platonic solids are the tetrahedron, cube, octahedron, icosahedron, and dodecahedron.

The number of vertices V, edges E, and faces F is given by:

	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
F	4	6	8	12	20
E	6	12	12	30	30
V	4	8	6	20	12

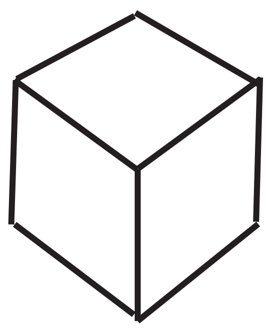

Figure 18: A cube viewed down an axis through antipodal vertices has 3-fold symmetry. fig:cubeone

Notice that $V - E + F = 2$, in each case, in accord with Euler's theorem. The boundary surfaces of these polyhedrons are all topologically spheres.

A convex solid has a dual given by the convex hull of the centers of its faces. Duality exchanges vertices and faces and preserves the number of edges. The tetrahedron is self-dual. The octahedron and cube are dual. The icosahedron and dodecahedron are dual. Dual solids have the same symmetry group.

When investigating the symmetries of the platonic solids it is helpful to keep in mind the theorem mentioned above that every rotation in $SO(3)$ is a rotation about some axis.

7.1 The cube ("hexahedron") and octahedron

In order to investigate the symmetries of the solids (and to construct them) it is easiest to begin with the cube 18.

If we draw the cube projected on the plane through an axis running through antipodal vertices then we get 18. The isotropy group of P is a discrete subgroup of $O(2)$ which is plainly seen to be D_3, since the figure is preserved by reflection and rotations of order 3.

Similarly, we can rotate by π through axes running through antipodal edge midpoints.

Finally we have 4-fold symmetry axes running through parallel faces.

Let us now account for the rotational symmetries of the cube:

There are

1. $V/2 = 4$ distinct axes of 3-fold symmetry yielding 8 distinct elements not equal to the identity.
2. $E/2 = 6$ axes through antipodal edge-midpoints of 2-fold symmetry yielding 6 distinct elements not equal to the identity.
3. $F/2 = 3$ axes through face centers of 4-fold symmetry yielding 9 distinct elements not equal to the identity.

Together with the identity we get a group of order

$$2 \times 4 + 1 \times 6 + 3 \times 3 + 1 = 24 \tag{7.1}$$

called the *octahedral group*, and denoted O. Since parity is a symmetry the full symmetry group of the cube is the group of order 48:

$$O_h = O \cup PO \tag{7.2}$$

eq:cubesym

Let us check this using orbits. The symmetry group of the cube acts transitively on the vertices. The isotropy group of a vertex is the group $D_3 \cong S_3$. There are 8 vertices so

$$|G|/|H| = 8 \tag{7.3}$$

so $|G| = 6 \times 8 = 48$, as expected.

Let us now understand the structure of the group O. We can label the vertices 1 to 8 so O_h is clearly a subgroup of S_8, but it is a rather small subgroup, consisting of only 48 out of $8! = 40320$ permutations. Notice for example that any two antipodal points

Figure 19: A cube with labeled vertices at $(\pm 1, \pm 1, \pm 1)$. fig:cubetwo

must remain antipodal, which would hardly be the case for a general permutation. Indeed, let us label pairs of antipodal points of the cube as A, B, C, D (or equivalently, the 3-fold symmetry axes as AA, BB, CC, DD). Any permutation of the axes can be implemented by some proper symmetry of the cube and hence we can identify the octahedral group as a subgroup of the symmetric group on 4 letters:

$$O \cong S_4 \tag{7.4}$$

As a check note that they both have order 24.

Since parity commutes with these rotations, the complete symmetry group of the octahedron, or cube, is

$$O_h \cong S_4 \times \mathbb{Z}_2 \tag{7.5}$$

and is a direct product.

Another point of view can be useful. Since the cube may be realized as the convex hull of the set of points

$$(\epsilon_1, \epsilon_2, \epsilon_3) \tag{7.6}$$

with $\epsilon_i = \pm 1$, as shown in 19, the group of orthogonal transformations preserving the cube is given by

$$(x_1, x_2, x_3) \to (\epsilon_1 x_{\sigma(1)}, \epsilon_2 x_{\sigma(2)}, \epsilon_3 x_{\sigma(3)}) \tag{7.7}$$

This exhibits the group as a semidirect product of S_3 with $(\mathbb{Z}_2)^3$.

Exercise

Identify each of the conjugacy classes of S_4 with symmetry operations on the cube.

Answer:

1. Rotations around 2-fold axes through edge midpoints; (ab)
2. Rotations by $\pi/2$ around a face: $(abcd)$
3. Rotation by π around a face: $(ac)(bd)$
4. Rotations on 3-fold axes: (abc).

Exercise *O as the symmetries of the octahedron*

a.) Show that the dual octahedron is the convex hull of the 6 vertices $(\epsilon_1, 0, 0), (0, \epsilon_2, 0), (0, 0, \epsilon_3)$.

b.) Show that there are

$V/2 = 3$ axes of 4-fold symmetry

$E/2 = 6$ axes of 2-fold symmetry

$F/2 = 4$ axes of 3-fold symmetry

c.) Label the 4 pairs of parallel faces of the octahedron A, B, C, D and identify the permutation associated with the different group elements.

7.2 Tetrahedron

We can obtain the tetrahedral symmetry group by embedding the tetrahedron in the cube. This can be done in two distinct ways:

Figure 20: Two embeddings of the tetrahedron in the cube.

fig:embtet

One embedding has vertices (x_1, x_2, x_3) whose product is -1 and the other has vertices whose product is $+1$.

The complete symmetry group of the tetrahedron is thus the subgroup of O_h preserving these vertices. It follows immediately that

$$|T_d| = 24 \tag{7.8}$$

Let us consider the subgroup T generated by rotations. The group T of order 12 is known as the *tetrahedral group*. To account for the elements of T we have

1. $V = F = 4$ axes through vertices, or equivalently, face-centers, with 3-fold symmetry, yielding 8 distinct elements not equal to the identity.

2. $E/2 = 3$ axes through edge-midpoints of 2-fold symmetry yielding 3 distinct elements not equal to the identity.

Together with the identity these give 12 elements:

$$1 + 4 \times 2 + 3 \times 1 = 12 \tag{7.9}$$

Now, recall that $O \cong S_4$, so T is a subgroup of the symmetric group S_4 of order 12. By considering the map between the symmetries of the cube and elements of S_4 (see above exercise) we see that

1. Rotations through an axis of the tetrahedron passing through a vertex and an opposite face are rotations through a diagonal of the cube, and hence correspond to the conjugacy class of the shape (abc).

2. Rotations through edge-midpoints are rotations of 180 degrees through an axis of the cube passing through opposite faces. These correspond to the conjugacy class of type $(ac)(bd)$. Thus, we have the *even* permutations:

$$T \cong A_4 \tag{7.10}$$

Now, reflection in a plane that bisects the tetrahedron acts to permute just two of the vertices, so this gives the permutations of type (ab). Hence, the complete symmetry group of the tetrahedron is S_4. It is denoted T_d in the Schoenfliess system.

Note that the tetrahedron is the only one of the platonic solids which does not have inversion symmetry, so that T_d is not a direct product $T \times \mathbb{Z}_2$, a fact nicely encoded in the fact that $S_4 \neq A_4 \times \mathbb{Z}_2$.

Exercise

Show that although $O \cong S_4$ and $T_d \cong S_4$, these are not "the same" S_4. That is, show that T_d is not a subgroup of O, but only of O_h. Describe the embedding of T_d in O_h.

Thus, T_d and O are nonconjugate subgroups of $O(3)$, but both are isomorphic to S_4. They provide inequivalent 3-dimensional representations of S_4.

7.3 The icosahedron

The icosahedron is the regular solid with 20 triangular sides.

We can construct the icosahedron as follows.

Label alternate sides of the octahedron black and white and orient the edges so that black is on the left as we move along the arrow.

Divide the edges in the ratio given by the golden section. [13] Thus, the edge from $(0,1,0)$ to $(0,0,1)$ is $(0,a,b)$ with $a+b=1$ and we choose the point with

$$\frac{a}{b} = \frac{\sqrt{5}-1}{2} \tag{7.11}$$

The twelve points

$$(0,\pm a,\pm b), \qquad (\pm b,0,\pm a), \qquad (\pm a,\pm b,0) \tag{7.12}$$

form the vertices of the icosahedron.

[13] Recall: φ, the golden section, or golden ratio, is the ratio a/b defined by $\frac{a+b}{a} = \frac{a}{b}$. Thus $\varphi = \frac{1+\sqrt{5}}{2}$. Note that $\varphi^{-1} = \frac{\sqrt{5}-1}{2}$

Let us consider the rotational symmetries.

1. There are $V/2 = 6$ axes through antipodal vertices with 5-fold symmetry yielding 24 elements.

2. There are $E/2 = 15$ axes through antipodal edge-midpoints with 2-fold symmetry yielding 15 elements

3. There are $F/2 = 10$ axes through face centers with 3-fold symmetry yielding 20 elements.

Together with the identity these form the icosahedral group Y of

$$1 + 6 \times 4 + 15 \times 1 + 10 \times 2 = 60 \tag{7.13}$$

elements. It is a subgroup of $SO(3)$.

If we consider the complete symmetry group of the icosahedron then we need only include inversion $I_3 : \vec{x} \to -\vec{x}$ to get the complete symmetry group of the icosahedron:

$$Y_h = Y \cup PY \cong Y \times \mathbb{Z}_2 \tag{7.14}$$

of order 120.

Again we can check this with orbits: G is transitive on the vertices. The isotropy group of a vertex is isomorphic to a discrete subgroup of $O(2)$ isomorphic to D_5. There are 12 vertices so $|Y_h| = 120$, and we have accounted for all elements.

Figure 21: There are 5 triplets of dyad axes. Each triplet of axes is labelled AA,BB,CC,DD,EE. fig:tetrafive

Now, to describe the group structure of Y we focus on the notion of a dyad axis. This is an axis through antipodal edge midpoints. Using a model you can convince yourself that one can form an orthogonal coordinate system using 3 sets of dyad axes. This partitions up the $30/2 = 15$ dyad axes into 5 sets of 3 mutually orthogonal dyad axes. Let us call these 5 coordinate systems $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5$. Every symmetry in Y permutes the S_i and no symmetry in Y fixes all the $\mathcal{S}_i$, so Y is isomorphic to a subgroup of S_5. The only 60 element subgroup of S_5 is A_5, and hence we conclude

$$Y \cong A_5 \tag{7.15}$$

Remark Consider the $6 = 12/2$ *pentad axes* through antipodal vertices marked $\infty, 0, 1, 2, 3, 4$ in the figure. Interpreting these points as elements of the projective plane $\mathbb{PF}_5$ the symmetries of the icosahedron map to the action of $PSL(2, \mathbb{F}_5)$ on this plane by Mobius transformations.

Exercise
Is Y transitive on the vertices?

Exercise
Show that a $2\pi/5$ rotation about an axis passing through antipodal vertices maps to a cycle $(abcde)$ in A_5.

Exercise *The dodecahedron*
a.) Show that the Dodecahedron is the convex hull of the set of points

$$(0, \pm\varphi^{-1}, \pm\varphi), (\pm\varphi^{-1}, \pm\varphi, 0), (\pm\varphi, 0, \pm\varphi^{-1}), (\pm 1, \pm 1, \pm 1) \tag{7.16}$$

b.) Describe Y and Y_h as the group of symmmetries of the dodecahedron.

7.4 No more regular polyhedra

Suppose a regular polyhedron of p-gons have q faces meeting at vertices.

At each vertex of a regular p-gon the tangent vector makes a turn by angle $2\pi/p$. Therefore the internal angles of the p-gon is

$$(1 - \frac{2}{p})\pi \tag{7.17}$$

Now, in our regular polyhedron there will be q faces meeting at a vertex, so the sum of the angles around the vertex must be less than 2π. Therefore

$$\frac{1}{p} + \frac{1}{q} > \frac{1}{2} \tag{7.18}$$

There are only 5 solutions to this Diophantine inequality with $p, q > 2$, i.e. $p, q \geq 3$. Since $\frac{1}{6} + \frac{1}{3} = \frac{1}{2}$ we must have $p, q \leq 5$. It is now easy to enumerate the cases and find that out of the 9 possible pairs we must have:

$$(p, q) = (3,3), (3,4), (4,3), (3,5), (5,3) \tag{7.19}$$

7.5 Remarks on the platonic solids

The platonic solids and their symmetries arise in many fields of human endeavor, as well as in Nature. We just give some indications here, with references to the literature.

7.5.1 Mathematics

1. For a description of the subgroups and conjugacy classes of the icosahedral group see Kramer and Haase, "Group Theory of Icosahedral Quasicrystals."

   For a nice description of the icosahedral group and its representations see

2. J. Conway, *Monsters and Moonshine*, Math. Intell. **2**(1980) 165. Has a nice description of many different ways of looking at the icosahedral group.

   More about the relation to finite fields and some amusing applications are in

3. B. Kostant, "Structure of the truncated icosahedron (such as fullerene or viral coatings) and a 60-element conjugacy class in $PSL(2, 11)$," Proc. Natl. Acad. Sci. USA, Vol. 91, pp. 11714-11717, 1994.

4. The exceptional outer automorphism of S_6 is related to the icasahedron, as described in Joyner, sec. 8.6.

There is a rich literature on convex polytopes and their symmetry. See
1.P. du Val, *Homographies, Quaternions, and Rotations*
2. K. Lamotke, *Regular Solids and Isolated Singularities*
3. H.S.M. Coxeter, *Regular Polytopes*
4. H.S.M. Coxeter, *Regular Complex Polytopes*

7.5.2 History of Physics

In Plato's cosmology the four elements earth, air, fire and water were associated with four of the 5 platonic solids, and the fifth, the dodecahedron, corresponded to the heavens.

In Kepler's time there were only 6 known planets Mercury, Venus, Earth, Mars, Jupiter, and Saturn. Note that considering Earth to be a planet requires a heliocentric ordering of the solar system. By inscribing and circumscribing spheres on the 5 platonic solids Kepler wanted to derive the ratios of the radii of the orbits. The approximate success of this idea led him to support the Copernican theory. While this idea was obviously misguided, the broader principle - that there is an underlying beautiful mathematical basis of the laws of nature is still with us today. Moreover, Kepler's passionate pursuit of this idea led to his careful analysis of the orbits of planets, and hence the discovery that they are actually ellipsoidal, and not circular, and to the third law of planetary motion.

7.5.3 Molecular physics

Many molecules in nature have the symmetries we have discussed. For example, benzene C_6H_6 has dihedral symmetry, while CH_4, CCl_4 have tetrahedral symmetry. Uranium hexaflouride UF_6 has octahectral symmetry. These symmetries have implications for the spectrum of vibrations of these molecules.

See:
5. S. Sternberg, *Group Theory and Physics*

6. R.M. Hanson, *Molecular Origami.* This contains cutout models of many compounds in nature.

Recently there has been much excitement over the discovery of "buckyballs," or buckminsterfullerene. The molecule C_{60} is arranged in an icosahedron.

Some references:

8. Dresselhaus, et. al. *Science of Fullerenes and Carbon Nanotubes*

9. Kroto, Heath, et. al. Nature **318**(1985) 354

10. Fowler and Manolopoulos, *An Atlas of Fullerenes*

11. Battye and Sutcliffe, "Solitonic fullerene structures in light atomic nuclei," hep-th/0012215.

7.5.4 Condensed Matter Physics

As we will see below, crystals can have the symmetries T_d and O_h, but it turns out they cannot have icosahedral symmetry. Nevertheless, there is an interesting notion of icosahedral order.

A good reference is the book by David Nelson, "Defects and Geometry in Condensed Matter Physics." View online at

http://books.google.com

7.5.5 Mathematical Physics

1. There are some remarkable occurances of such structures in solitons in nonlinear sigma models in 3+1 dimensions. See, for example: Atiyah and Sutcliffe, arXiv:math-ph/0303071

2. The symmetries of the platonic solids also play and amusing role in the classification of unitary CFT's with $c < 1$: See Friedan, Qiu, and Shenker; Ginsparg; Dijkgraaf, Verlinde, Verlinde.

3. These symmetries also arise in algebraic geometry - classifying degenerations of $K3$ surfaces and simple singularities and have played an important role in string theory.

7.5.6 Biology

The regular polyhedra also occur in biology. For example Radiolaria species have shells in the form of regular polyhedra, e.g. *Circogonia icosahedra*, and *Circorrhegma dodecahedra.*

See the wonderful book:

Ernst Haeckel, *Kunstformen der Nature*: Look at the first radiolarian.

Available on-line at

http://caliban.mpiz-koeln.mpg.de/ stueber/haeckel/kunstformen/natur.html

http://www.flickr.com/photos/origomi/sets/72157601323433758/

A little surfing on the internet will teach you that the majority of viruses, including HIV, have protein coatings in the form of regular polyhedra, especially the icosahedron. [14]

See also:

https://simonsfoundation.org/multimedia/symmetric-structures/

[14] A good place to start is the Wikipedia article "Regular polyhedron."

7.5.7 Human culture: Architecture, art, music and sports

Regular solids have inspired architecture from the pyramids to the geodesic domes of Buckminster Fuller.

According to Weyl, in his book *Symmetry*, Princeton University Press, 1952, 4-fold symmetry in architecture is very common, while 5-fold symmetry is very rare, with the Pentagon in Washington D.C. providing a notable example. (As Weyl wryly notes, it thus provides an easy target for bombers, as indeed proved to be the case on 9/11/2001.) He goes on to say

"Leonardo da Vinci engaged in systematicaly determining the possible symmetries of a central building and how to attach chapels and niches without destroying the symmetry of the nucleus. In abstract modern terminology, his result is essentially [the classification of discrete subgroups of $O(2)$*]."*

The platonic solids also played a role in musical theory, according to Plato following on Pythagoras.

The platonic solids have appeared in artwork throughout the ages. Sometimes in subtle form. In the attached reprint of Albrecht Dürer's *Melancholia* you see a truncation of a platonic solid.

And let us not forget sports. Icosahedral dice were used in the Ptolemaic dynasty of ancient Egypt. If we truncate the icosahedron we get a soccar ball.

References:

1. Paul Calter, http://www.dartmouth.edu/ matc/math5.geometry/unit6/unit6.html#Plato

7.6 Regular polytopes in higher dimensions

An n-dimensional polytope is a generalization of the notion of polygon and polyhedron. It is a region of $\mathbb{R}^n$ bounded by $(n-1)$-dimensional polytopes.

A regular n-dimensional polytope can be defined recursively by requiring that its boundary is a union of regular $(n-1)$-dimensional polytopes, and its vertex-figure is a regular $(n-1)$-dimensional polytope. The vertex-figure is the $(n-1)$-dimensional polytope obtained by intersecting an $(n-1)$-dimensional hyperplane near an extremal vertex.

Ludwig Schläfli discovered that there are 6 regular 4-polytopes (or "polychorons") in 4-dimensions.

It turns out that in $n \geq 5$ dimensions there are only 3 regular polytopes:

1. The n-simplex, which is self-dual.
2. The n-dimensional cube.
3. The dual of the n-dimensional cube.

References:

1. Coxeter's book.
2. Wikipedia articles on Regular Polytopes.

8. Classification of the Discrete subgroups of $SO(3)$ and $O(3)$

Now we will prove that the symmetry groups above exhaust the discrete subgroups of

$SO(3)$. Of course, when classifying the discrete subgroups we understand two subgroups to be equivalent if they are conjugate.

Theorem The discrete subgroups of $SO(3)$ are

1. C_n, $n \geq 1$
2. D_n, $n \geq 1$
3. Symmetries of the tetrahedron, T
4. Symmetries of the cube = symmetries of the octahedron, O
5. Symmetries of the dodecahedron = symmetries of the icosahedron, Y

Note the similarity to the list of finite simple groups.

Figure 22: An element g of a finite subgroup $G \subset SO(3)$ fixes some point, and permutes other points in a G-orbit. fig:platonic

Proof: Any discrete subgroup of $SO(3)$ preserves the sphere, and any nontrivial element in the group is a rotation about an axis and therefore fixes two points on the sphere. Let X be the set of points on the sphere fixed by some nontrivial element of the finite subgroup $G \subset SO(3)$.

We begin by counting the elements of the set

$$\{(x,g) \in X \times G | g \neq 1 \qquad \textit{and} \qquad gx = x\} \tag{8.1}$$ eq:countset

We count in two different ways. First, we project $(x,g) \to g$. The fiber of this map consists of two points because every element $g \neq 1$ fixes precisely two points. Denoting $|G| = n$, it is clear that this number is

$$2(n-1) \tag{8.2}$$

On the other hand, we can proceed by counting the number of group elements fixing each x. The points x will form orbits O_i, $i = 1, \ldots, r$ under G. Suppose $|O_i| = p_i$ and the isotropy group of a point in O_i is of order n_i. (Recall that the isotropy group of two points on the same orbit are conjugate subgroups, so this is well-defined.) Note that

$$p_i = n/n_i \tag{8.3}$$ eq:orbtord

by the stabilizer-orbit theorem. Thus (8.1) has

$$\sum_{i=1}^{r}(n_i-1)p_i = n\sum_{i=1}^{r}(1-\frac{1}{n_i}) \tag{8.4}$$ eq:otherfs

points, and therefore we obtain

$$2(1-\frac{1}{n}) = \sum_{i=1}^{r}(1-\frac{1}{n_i}) = r - \sum\frac{1}{n_i} \tag{8.5}$$ eq:relst

Now we analyze this Diophantine equation.

First, note that $r=2$ or $r=3$. To see this we first prove that it is impossible to have $n_i=1$. Indeed, if $n_i=1$ then for that x no $g\neq 1$ fixes x, but this is impossible by the definition of X. Therefore, all $n_i>1$ and hence $\sum\frac{1}{n_i}<r/2$ so the RHS of (8.5) is $\geq r-\frac{1}{2}r=r/2$, but the LHS is less than 2. Thus, $r/2<2$ and hence $r=1,2,3$. On the other hand, we cannot have $r=1$: Note that we can assume $n>1$, and hence the LHS is ≥ 1, but comparing to the RHS we see that this implies $r>1$.

Now we examine the two cases $r=2$ and $r=3$:

$r=2$: We rewrite the equation as

$$0 = n_1(n-n_2)+n_2(n-n_1) \tag{8.6}$$

Now $n_i\leq n$ so both terms are nonnegative. Since $n_1>1$ and $n_2>1$ it must be that $n_1=n_2=n$. Thus, X consists of two antipodal points. Each point is an orbit fixed by the entire group. These are the cyclic groups C_n acting by rotation around some axis.

Now consider $r=3$. Then (8.5) simplifies to

$$\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3} = 1+\frac{2}{n} \tag{8.7}$$ eq:reslii

WLOG we assume $n_1\leq n_2\leq n_3$. Then the complete list of solutions is:

	n	(n_1,n_2,n_3)	p_1	p_2	p_3
D_k	$2k$	$(2,2,k)$	k	k	2
T	12	$(2,3,3)$	6	4	4
O	24	$(2,3,4)$	12	8	6
I	60	$(2,3,5)$	30	20	12

This is easily proved as follows: We must have $n_1=2$, for if $n_1>2$ then the LHS of (8.7) is ≤ 1. Similarly, we cannot have $n_2\geq 4$, since again the LHS (8.7) is ≤ 1. Therefore,

$n_2 = 2, 3$. If $n_2 = 2$ the equation becomes $n = 2n_3$. This is the first row. We have two orbits of length k and one orbit of length 2. Embedding the dihedral group D_k into $SO(3)$ by multiplying parity transformations of D_k with reflection in the x, y plane clearly gives such a solution.

If $n_2 = 3$ then

$$\frac{1}{n_3} = \frac{1}{6}(1 + \frac{12}{n}) \tag{8.8}$$

and hence $1/n_3 > 1/6$, so $n_3 < 6$. Now we simply plug in $n_3 = 3, 4, 5$.

Now we should show that each solution indeed corresponds to a discrete subgroup of $SO(3)$. But this follows from our discussion of the regular solids. ♠

It is interesting to work out the orbits on the sphere corresponding to the different symmetry groups of the regular solids.

Consider C_n embedded as rotations by $2\pi j/n$ around some axis. The set X indeed consists of two points, where the axis intersects the sphere, and the isotropy group at each point clearly has order n. This is the solution $r = 2$.

Now, for D_n we must embed it in $SO(3)$. We do this, as in the prismatic symmetry group by embedding

$$y^j \to \begin{pmatrix} R(2\pi j/n) & 0 \\ 0 & 1 \end{pmatrix} \tag{8.9}$$

$$x \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{8.10}$$

For the elements y^j the north and south poles are fixed points with isotropy group of order n. Note that x maps one to another, so they form an orbit of length two. The fixed points of xy^j necessarily lie in the $z = 0$ plane. We can find them by considering the circle to circumscribe the regular n-gon. Then consider the reflection axes of the n-gon. Each such vertex intersects the sphere in 2 points. The intersections of these axes with the sphere describe two regular n-gons. These are the other two orbits.

Next note that the last three entries nicely reproduce the number of faces, edges, and vertices of the platonic solids.

Consider, for example, the tetrahedral group T. If we embed a tetrahedron in the sphere, then the vertices form one orbit of length four. The inverted tetrahedron forms another orbit of length four. Indeed, the four 3-fold symmetry axes pierce the sphere in 2 sets of 4 points. These are the two orbits with $p_2 = p_3 = 4$. The 3 2-fold symmetry axes through edge midpoints pierce the sphere in 6 points. These form the orbit of $p_1 = 6$.

The other cases are similar. (See Lamotke's book for details.)

In conclusion we list without proof the generators and relations of the discrete subgroups of $SO(3)$: [15]

$$C_n = \langle \gamma | = \gamma^n = 1 \rangle \tag{8.11}$$

[15] See, e.g. Lamotke's book above.

$$D_n = \langle \beta, \gamma | \beta^2 = \gamma^n = (\beta\gamma)^2 = 1 \rangle \tag{8.12}$$

$$T = \langle \beta, \gamma | \beta^3 = \gamma^3 = (\beta\gamma)^2 = 1 \rangle \tag{8.13}$$

$$O = \langle \beta, \gamma | \beta^3 = \gamma^4 = (\beta\gamma)^2 = 1 \rangle \tag{8.14}$$

$$Y = \langle \beta, \gamma | \beta^3 = \gamma^5 = (\beta\gamma)^2 = 1 \rangle \tag{8.15}$$

Exercise
Verify the nontrivial relation on β, γ in each case above.

8.1 Finite subgroups of $O(3)$

Now we consider $O(3) = SO(3) \times \mathbb{Z}_2$.

We can reduce the classification of discrete subgroups of $O(3)$ to those of $SO(3)$ using the following construction.

Consider the determinant homomorphism $\det : O(3) \to \mathbb{Z}_2$. Suppose $G \subset O(3)$ is a discrete subgroup. If $\det : G \to \mathbb{Z}_2$ is trivial, then $G \subset SO(3)$, and we have already classified these. On the other hand, if $\det : G \to \mathbb{Z}_2$ is a nontrivial homomorphism (i.e. there are parity-reversing elements in G) then there is a kernel $K \subset G$ of order 2. It is a normal subgroup, and

$$G = K \amalg K g_0 \tag{8.16}$$

for some $g_0 \in G$. The elements of Kg_0 are rotation-inversions. We have an exact sequence

$$1 \to K \to G \to \mathbb{Z}_2 \tag{8.17}$$

eq:GtoK

If $I_3 : \vec{x} \to -\vec{x}$ is in G then we can simply take $g_0 = I_3$ and

$$G = K \amalg K I_3 \tag{8.18}$$

Note that I_3 is in the center of $O(3)$ and hence $G \cong K \times \mathbb{Z}_2$. The sequence (8.17) splits.

Now, it can happen that $I_3 \notin G$. For the most trivial example, consider a solid which is completely irregular except for a reflection plane of symmetry. (For example, the exterior of an idealized human being.) In this case $K = 1$ and $g_0 \neq I_3$. For more nontrivial examples consider the n-prism with n odd or the complete symmetry group of the tetrahedron, T_d.

If $I_3 \notin G$ then we can construct

$$G^+ := K \amalg K g_0 I_3. \tag{8.19}$$

Note that $K \cap Kg_0I_3 = \emptyset$ for otherwise there would be $k, k' \in K$ with $k = k'g_0P$ which implies $k'' = g_0I_3$ which implies $I_3 \in G$, since $g_0 \in G$, contrary to hypothesis. Moreover, we claim that G^+ is in fact a group. To verify this note that if $k_1, k_2 \in K$ then

$$k_1g_0I_3k_2g_0I_3 = k_1g_0k_2g_0 \tag{8.20}$$

but $k_1g_0k_2g_0 \in G$, and yet it is a proper rotation, so it must be in K.

Moreover note that this calculation also shows that, as abstract groups, G^+ is isomorphic to G via the map

$$\begin{aligned} \psi: \qquad & g \to g & \qquad g \in K \\ & g \to I_3g & \qquad g \notin K \end{aligned} \tag{8.21}$$

Warning: G^+ is a group of proper rotations, $G^+ \subset SO(3)$, but is distinct from G as a subgroup of $O(3)$. That is, there are inequivalent embeddings of G into $O(3)$.

Example: Consider the group denoted S_{2n} in the Schoenfliess classification. It is generated by

$$g_0 = \begin{pmatrix} R(\pi/n) & 0 \\ 0 & -1 \end{pmatrix} \tag{8.22}$$

Note that it is clearly isomorphic to C_{2n}. However, for n even $I_3 \notin S_{2n}$. Note that $K \cong C_n$ is generated by g_0^2. Moreover,

$$I_3g_0 = \begin{pmatrix} R(\frac{n+1}{2n}2\pi) & 0 \\ 0 & +1 \end{pmatrix} \tag{8.23}$$

generates $G^+ \cong C_{2n}$.

Now, using this construction we can construct the finite subgroups of $O(3)$. In addition to the finite subgroups of $SO(3)$ we have already listed, those with rotation-inversions are:

1. $C_n \amalg I_3C_n$
2. $D_n \amalg I_3D_n$
3. $T_h \cong T \amalg I_3T$
4. 4. $O_h \cong O \amalg I_3O$
5. $Y_h \cong Y \amalg I_3Y$
6. $G^+ = C_{2n}, K = C_n$
7. $G^+ = D_n, K = C_n$
8. $G^+ = D_{2n}, K = D_n$
9. $G^+ = O, K = T$

To prove this we look for homomorphisms G^+ *onto* $\mathbb{Z}_2$ where G^+ runs over the discrete subgroups of $SO(3)$. Any element mapping to the nontrivial generator must have order two. C_n only has an order two element for n even, this gives $G^+ = C_{2n}$. For D_n we can use the reflection, or, if n is even a rotation by π. This gives the next two examples. Now note that $T \cong A_4$ and $Y \cong A_5$ are simple groups, so cannot have nontrivial homomorphic images. This leaves $G^+ = O$, which has only one possibility since S^4 has only one index two subgroup. If we think of O as the group which takes $(x_1, x_2, x_3) \to (\epsilon_1 x_{\sigma(1)}, \epsilon_2 x_{\sigma(2)}, \epsilon_3 x_{\sigma(3)})$ then the homomorphism is $g \to \epsilon_1\epsilon_2\epsilon_3$.

The following is a table of the finite subgroups of $O(3)$ which are not in $SO(3)$, together with examples of solids for which they are the complete symmetry group.

G^+	K	Schoenfliess	Solid
$C_n \amalg I_3C_n$	C_n	S_{2n}, n odd	alternating $2n$-prism
$C_n \amalg I_3C_n$	C_n	C_{nh}, n even	shaved n-prism
$D_n \amalg I_3D_n$	D_n	D_{nh}, n even	n-prism
$D_n \amalg I_3D_n$	D_n	D_{nd}, n odd	twisted n-prism
$T \amalg I_3T$	T	T_h	decorated cube
$O \amalg I_3O$	O	O_h	cube, octahedron
$Y \amalg I_3Y$	Y	Y_h	dodecahedron, icosahedron
C_{2n}	C_n	S_{2n}, n even	alternating $2n$-prism
C_{2n}	C_n	C_{nh}, n odd	shaved n-prism
D_n	C_n	C_{nv}, $n \geq 2$	n-pyramid
D_{2n}	D_n	D_{nh}, n odd	n-prism
D_{2n}	D_n	D_{nd}, n even	twisted n-prism
O	T	T_d	tetrahedron

Note that several of the entries under G^+ are isomorphic to each other. However, the groups with different names in the Schoenfless classification are nonconjugate as subgroups of $O(3)$, despite being abstractly isomorphic as groups.

The twisted n-prism is obtained by gluing together two n-prisms rotated by π/n around their axis of symmetry. If we shave off the edges of an n-prism it receives and orientation and we get a shaved n-prism. If on a $2n$-prism we shave every other edge, we get an alternating n-prism. For pictures see pp. 96-97 of the book by Yale.

8.2 Finite subgroups of $SU(2)$ and $SL(2,\mathbb{C})$

In the next chapter we will study the rich relation between $SO(3), SU(2)$, and $SL(2,\mathbb{C})$. The finite subgroups are closely related (by a 2 : 1 homomorphism) to the above. They have generators:

1. $\mathcal{C}_n$

$$\begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \qquad \omega = e^{\pi i/n} \tag{8.24}$$

2. $\mathcal{D}_{n+2}$: "binary dihedral"

$$\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} \qquad \beta = e^{\pi i/n}, \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{8.25}$$

3. $\mathcal{T}$, "binary tetrahedral," is generated by $\mathcal{D}_4$ together with:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon^{-1} & \epsilon^{-1} \\ \epsilon^5 & \epsilon \end{pmatrix} \qquad \epsilon = e^{2\pi i/8} \tag{8.26}$$

4. $\mathcal{O}$, is generated by $\mathcal{T}$ together with:

$$\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \qquad \epsilon = e^{2\pi i/8} \tag{8.27}$$

5. $\mathcal{I}$, is generated by:

$$-\begin{pmatrix} \eta^3 & 0 \\ 0 & \eta^2 \end{pmatrix}, \frac{1}{\eta^2 - \eta^3} \begin{pmatrix} \eta + \eta^4 & 1 \\ 1 & -\eta - \eta^4 \end{pmatrix} \qquad \eta = e^{2\pi i/5} \tag{8.28}$$

9. Symmetries of lattices

9.1 The crystallographic restriction theorem

The *crystallographic restriction theorem* is an important restriction on the possible subgroups of $SO(2)$ or $SO(3)$ which can be automorphisms of 2 and 3-dimensional lattices, respectively.

Let $g \in SO(2)$ be a symmetry of a two-dimensional lattice. This generates a discrete subgroup of $SO(2)$, which must be $\cong C_n$ corresponding to rotations by $k \cdot \frac{2\pi}{n}$.

In an *integral basis* for the lattice the action of $g \in SO(2)$ must be an *integral matrix*:

$$g : \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix} \tag{9.1}$$

with $a, b, c, d \in \mathbb{Z}$.

On the other hand, in a *Cartesian basis* the transformation g must be represented by a rotation:

$$R(g) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{9.2}$$ eq:rotgee

On the other hand, the trace is independent of basis, so (9.2) must have an integral trace. Therefore:

$$2\cos\theta \in \mathbb{Z} \tag{9.3}$$ eq:lattcond

This is only possible if $\cos\theta = 0, \pm\frac{1}{2}, \pm 1$.

$\cos\theta$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$-\pi < \theta \leq \pi$	π	$\pm\frac{2\pi}{3}$	$\pm\frac{\pi}{2}$	$\pm\frac{\pi}{3}$	0

In other words, $\theta = \pm\frac{2\pi}{n}$ with

$$n = 1, 2, 3, 4, \quad \text{or} \quad 6. \tag{9.4}$$ eq:twodee

A very similar argument applies to three-dimensional lattices. Once again we choose a lattice basis e_i. Then a point-group symmetry must take

$$g \cdot e_i = \sum_j C_{ji} e_j \tag{9.5}$$

where C_{ji} is a matrix of integers. On the other hand, g is conjugate to

$$\pm \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9.6}$$

so

$$\mathrm{Tr}(g) = \pm(1 + 2\cos\phi) \in \mathbb{Z} \tag{9.7}$$

Once again we conclude that $2\cos\phi \in \mathbb{Z}$, and hence we again have the list (9.4).

9.2 Three-dimensional lattices

The symmetries of 3-dimensional lattices is an important subject in condensed matter physics. We summarize here the bare essentials:

Definition A subgroup of $E(3)$ mapping a three-dimensional lattice to itself and fixing a point of the lattice is called a *crystallographic point group*.

Because of the crystallographic restriction theorem there are only 32 possible crystallographic point groups. It is useful to separate those which contain I_3 from those which do not:

1. C_n, $n = 1, 2, 3, 4, 6$

2. D_n, $n = 2, 3, 4, 6$

3. T, O, *but not* Y!

4. C_{nh}, $n = 1, 3$

5. C_{nv}, $n = 2, 3, 4, 6$

6. D_{3h}

7. D_{2d}

8. T_d

9. S_n, $n = 2, 4, 6$

10. C_{nh}, $n = 2, 4, 6$

11. D_{nh}, $n = 2, 4, 6$

12. D_{nd} $n = 3$

13. T_h, O_h

Only the groups in 9-13, consisting of 11 groups, contain I_3.

Definition A *crystallographic space group* is a discrete subgroup $G \subset E(3)$ such that

$$L = G \cap \mathcal{T} \tag{9.8}$$

is a three-dimensional lattice group, i.e., isomorphic to $\mathbb{Z}^3$. Physically these classify lattice systems with a basis and are therefore symmetries of (idealized) crystals.

The question of how to classify crystallographic space groups is an intricate one. One's first guess is that we should classify crystallographic subgroups $G \subset E(3)$ up to conjugacy.

However, this leaves infinitely many parameters such as the lengths of basis vectors and the angles between them.

The standard way to classify space groups is via conjugacy within the group of all *affine transformations* of $\mathbb{R}^3$,

$$\vec{x} \to S\vec{x} + \vec{a} \tag{9.9}$$

eq:affinetmn

where $S \in GL(3, \mathbb{R})$.

The result of Fedorov, Schoenflies and Barrow from the 1890's is that:

1. There are 219 distinct isomorphism classes of space groups,

2. If we restrict the classification to conjugacy within the group of *proper* transformations, with $\det S > 0$ then there are 230 distinct classes of space groups. There are 11 conjugacy classes which are related by improper isometries, and hence crystallographers recognize 230 distinct space groups. In these 11 pairs it turns out that the conjugation taking one to the other changes the chirality of a screw displacement, but the chirality of the screw displacement does affect the physical properties of the crystal.

The lattice subgroup $L \subset G$ is a normal subgroup, and if an element $g \in G$ is $g = (\vec{a}, R)$ then conjugation by g acts as a group of automorphisms of L:

$$T_{\vec{a}} \in L \rightarrow g T_{\vec{a}} g^{-1} = T_{R \cdot \vec{a}} \tag{9.10}$$

Thus, $P = G/L$ acts a discrete subgroup of $O(3)$ on L.

Now, we have classified the discrete subgroups of $O(3)$, and we can use this information.

Definition The largest crystallographic point group at a point $\vec{x} \in L$ is called the *holohedry* of L at $\vec{x}$. Two lattices are in the same *crystal system* if their holohedries are conjugate subgroups of $E(3)$.

Lattices clearly have I_3 as a symmetry. Using this one can show that I_3 is always a member of the holohedry F. This shows that many of the crystallagraphic point groups cannot be holohedries. Using some detailed arguments (See, e.g. Miller's book or Sternberg, Appendix A) one can then show that of the 11 point groups containing I_3, only 7 can be holohedries.

There are only 7 crystal systems with holohedries S_2, C_{2h}, D_{3d}, O_h and D_{nh} for $n = 2, 4, 6$. These satisfy the subgroup relations

$$\begin{array}{ccccccccc} S_2 & \subset & C_{2h} & \subset & D_{2h} & \subset & D_{4h} & \subset & O_h \\ & & \cap & & \cap & & & & \\ & & D_{3d} & \subset & D_{6h} & & & & \end{array} \tag{9.11}$$

Crystal System	Holohedry
triclinic	S_2
monoclinic	C_{2h}
orthorhombic	D_{2h}
tetragonal	D_{4h}
cubic	O_h
trigonal	D_{3d}
hexagonal	D_{6h}

If we now list the possible pairs (P, L) where P is a point group which is the maximal symmetry of L. The equivalence relation is that there exists an $S \in GL(3, \mathbb{R})$ with $\det S > 0$ such that $L' = SL$ and $P' = SPS^{-1}$. It turns out that the 7 crystal systems decompose into 14 "lattice types," known as the "Bravais lattices." [16]

For example, the generic lattice only has $P = S_2$ as its symmetry group. These are called the *triclinic* lattices.

We will not go through the detailed arguments here. See the references for that. Once we have the possible pairs (P, L) we can try to classify the possible crystallographic groups by studying the possible extensions:

$$1 \to L \to G \to P \to 1 \tag{9.12}$$

eq:extnesion

Not all of the extensions are split. The groups for which the sequence splits are called *symmorphic.*

Remarks:

1. The analogous question for 2-dimensional lattices leads to the 17 *wallpaper groups* which are discrete subgroups of $E(2)$ containing two independent translations. See the Wikipedia article http://en.wikipedia.org/wiki/Wallpaper-group for an account.

2. Hilbert's 18th problem asked if there are finitely many crystallographic space groups in higher dimensions. Bieberbach's theorem answered this in the affirmative.

References:
1. *Symmetry in Condensed Matter Physics*, L. Michel
2. Hahn, Theo, ed. (2002), International Tables for Crystallography, Volume A: Space Group Symmetry (5th ed.), Springer-Verlag
3. S. Sternberg, Group Theory and Physics
4. Miller, Symmetry Groups and Their Applications.
5. Wikipedia article on Space Groups

9.3 Quasicrystals

In particular, note that 5-fold symmetry of 2- and 3-dimensional lattices is forbidden. This is why the discovery of quasicrystals was such a shock! Certain materials have x-ray diffraction patterns with 5-fold symmetry.

For a reference on a group-theoretical approach to some quasicrystals see P. Kramer and R.W. Haase, "Group Theory of Icosahedral Quasicrystals," in *Introduction to the Mathematics of Quasicrystals* ed. by M. V. Jarić.

One way to produce almost periodic structures with symmetries unavailable for lattices is to *project* from a higher-dimensional lattice with forbidden symmetry.

[16] In 1835 Frankenheim published a paper listing 15 crystal types. In 1848, Bravais pointed out that two of them are isomorphic.

Example. Choose $\vec{v} = (v_1, 0; v_2, 0) \in \mathbb{R}^{2,2}$ and a rotation

$$\theta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix} \tag{9.13}$$

eq:thetr

This can be diagonalized to $\theta = Diag\{\xi, \xi^{11}, \xi^5, \xi^7\}$ with $\xi = e^{2\pi i/12}$. If

$$\begin{aligned} \langle \vec{v}, \vec{v} \rangle &= v_1^2 - v_2^2 = 2 \\ \langle \vec{v}, \theta \vec{v} \rangle &= \frac{\sqrt{3}}{2}(v_1^2 + v_2^2) = 2 \end{aligned} \tag{9.14}$$

then $\vec{v}, \theta\vec{v}, \theta^2\vec{v}, \theta^3\vec{v}$ generate a lattice in $\mathbb{R}^{2,2}$ with $\mathbb{Z}_{12}$ symmetry. Projecting to the positive definite 2-dimensional planes gives an aperiodic array with $\mathbb{Z}_{12}$ symmetry.

10. Tesselations by Triangles

11. Diversion: The Rubik's cube group

Reference: Joyner, *Adventures in Group Theory*

The Rubik's cube group G is generated by 90 degree rotations of the face slices. Note that rotations of the middle slice is equivalent to rotations of the two parallel face slices together with a rotation in space. Now notice that any rotation by a face slice leaves the central cube fixed. Thus, if the puzzle-solver is holding the cube we can describe the faces by U (top) D (bottom) F (facing the holder), B, (facing away from the holder), L, held by the left-hand and R held by the right-hand.

If one is looking down on a face the basic move is a 90 degree rotation in the *clockwise* direction. This defines six generators of the Rubik's cube group U, D, F, B, L, R.

These permute the $54 - 6 = 48$ remaining facets which are not centers of faces. Thus, the Rubik's cube group $\mathcal{R}$ is a subgroup of S_{48}. Now, the order of S_{48} is $48! \cong 1.24 \times 10^{61}$, and there are only about $\pi \times 10^7$ seconds in a year, so a little discretion is called for in solving the cube.

In fact, as with the symmetry groups of the platonic solids, we can cut down the group of potential transformations somewhat. Note that $\mathcal{R}$ separately permutes the 8 corner pieces (accounting for $8 \times 3 = 24$ facets) and the 12 edge pieces (accounting for $12 \times 2 = 24$ facets). In addition the facets have an orientation, the edge pieces have two possible orientations (since the painted faces always point outwards) and the corner pieces have three possible orientations. Therefore, the group $\mathcal{R}$ is a certain subgroup of

$$H = (C_3^{|V|} \ltimes S_V) \times (C_2^{|E|} \ltimes S_E) \tag{11.1}$$

Here V is the set of 8 vertex cubes and E is the set of 12 edge cubes. The factors C_3 and C_2 take care of the orientations of these vertex and edge cubes. $S_V \cong S_8$ is the group permuting the vertex cubes while S_E permutes the edge cubes. In the semidirect product $C_3^{|V|} \ltimes S_V$, S_V permutes the 8 factors of C_3.

The group H here is known as the *illegal Rubik's cube group* in Joyner's book. It is the group of permutations of the facets obtained by allowing one to take apart the cube and reassemble it, but one must keep the painted faces outward.

It is quite clear that $\mathcal{R}$ is a proper subgroup of H. Indeed, consider the basic move on a face. It only permutes the edge and corner pieces of that face which is being rotated. There is a relation between the permutation of the edge and face pieces: they are both cyclic permutations of the kind (1234). In particular, if $r \in S_V$ and $s \in S_E$ then

$$\epsilon(r) = \epsilon(s). \tag{11.2}$$

As for the orientation, choose a fixed orientation of the cube using some ordering of the central faces. For example, white side up, red side forward. Now in the solved position give each cube position a standard orientation, and each cube piece its intrinsic orientation. Comparing these two, for edge pieces we get an element of C_2 and for corner pieces and element of C_3. Any configuration of the Rubik's cube defines an element $\vec{v} \in C_3^8$ and $\vec{w} \in C_2^{12}$. Note that the basic 90-degree rotation move preserves

$$\sum w_i = 0 \bmod 2 \qquad \sum v_i = 0 \bmod 3. \tag{11.3}$$

The nontrivial theorem in Joyner's book states that these are the only relations:

Theorem Denote an element of H by $(\vec{v}, r, \vec{w}, s)$ with $\vec{v} \in C_3^8, \vec{w} \in C_2^{12}, r \in S_8, s \in S_{12}$. Then the Rubik's cube group is the subgroup consisting of elements such that

1. $\epsilon(r) = \epsilon(s)$
2. $\sum v_i = 0 \bmod 3$
3. $\sum w_i = 0 \bmod 2$

From this one can deduce that the order of $\mathcal{R}$ is

$$\begin{aligned} \frac{1}{2} \times \frac{1}{3} \times \frac{1}{2} \times 3^8 \times 8! \times 2^{12} \times 12! &= 2^{27} \cdot 3^{14} \cdot 5^3 \cdot 7^2 \cdot 11 \\ &= 43252003274489856000 \\ &\cong 4.325 \times 10^{19} \end{aligned} \tag{11.4}$$

Exercise
Consider a 2×2 Rubik's cube. What is the Rubik's cube group?

---

12. Group actions on function spaces

Suppose X, Y are two sets. We can then consider the space of maps

$$\mathcal{F}_{X,Y} := \{f : X \to Y\} \tag{12.1}$$

Now, if X or Y admits a G-action for a group G then it follows that the space of maps $\mathcal{F}(X, Y)$ also admits a G-action.

Figure 23: If G acts on the target Y then we simply define $g \cdot f$ on functions $X \to Y$ by post-composing with the G action. fig:gactfunone

If $\varphi : G \times Y \to Y$ is a G action then we define a G-action $\Phi : G \times \mathcal{F} \to \mathcal{F}$ (abbreviating $\mathcal{F} = \mathcal{F}_{X,Y}$) by

$$\Phi(g, f)(x) := \varphi(g, f(x)) \tag{12.2}$$

Figure 24: If G acts on the target X then we simply define $g \cdot f$ on functions $X \to Y$ by pre-composing with the G action. fig:gactfuntwo

as in 23. One checks that if φ is a left(right)-action then Φ is also a left(right)-action.

Similarly, if X admits a G-action $\varphi : G \times X \to X$ then $\mathcal{F}$ admits a G action $\Phi : G \times \mathcal{F} \to \mathcal{F}$ defined by

$$\Phi(g, f)(x) := f(\varphi(g, x)) \tag{12.3}$$

as in 24. Now one checks that if φ is a left, resp. right-action on X the Φ is a right, resp. left-action on $\mathcal{F}$.

If we want to restrict attention to left-actions then we can say that

he function space $\mathcal{F}_{X,Y}$ also has a left-G-action defined by $f \mapsto g \cdot f$ where the new function $g \cdot f$ is defined by the values: $(g$

Exercise

a.) Show that

$$g_1 \cdot (g_2 \cdot f) = (g_1 g_2) \cdot f \tag{12.4}$$

and hence defines a left group action.

b.) Show that

$$(f \cdot g)(x) := f(g \cdot x) \tag{12.5}$$

defines a *right*-action of G on functions.

Example 1. $SO(2, \mathbb{R})$ acts on $\mathbb{R}^2$

$$R_\phi^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \tag{12.6}$$

By our basic principle, $G = SO(2, \mathbb{R})$ also acts on the space of all functions

$$\mathcal{F} = \{f : \mathbb{R}^2 \to \mathbb{R}\} \tag{12.7}$$

$f \mapsto (R_\phi f)$ where the new function $(R_\phi f)$ is defined by the values $(R_\phi f)(x_1, x_2) \equiv f(x_1', x_2')$ and The association $f \to R_\phi f$ is an action of the group on $\mathcal{F}$.

The G-orbits in the space of functions are collections of functions which can be rotated into one another.

Exercise
a.) What is the image of the function $(x_1)^3$ under R_ϕ ?

Some orbits of $\mathcal{F}_{X,Y}$ are distinguished. These are the G-orbits consisting of one point, or, in plain English, the *invariant functions* In example 1 above these are simply the radially symmetric functions.

$$f(x^1, x^2) = f(r) \tag{12.8}$$

Remarks

1. The above principle is frequently used in quantum mechanics. The wavefunction $\psi : X \to \mathbb{C}$ where X is some configuration space of a dynamical system. If X admits a symmetry group then that group acts on the space of wavefunctions.

13. The simple singularities in two dimensions

There is a purely algebraic approach to geometry: Characterize a variety by the kinds of polynomial functions that make sense on it.

For $\mathbb{C}^2/\Gamma$, where Γ is a discrete subgroup of $SL(2,\mathbb{C})$ these varieties are very beautiful and important both in math and in string theory.

Example Consider $\mathbb{C}^2$ with the action of $\mathbb{Z}_k$:

$$(z_1, z_2) \to (\omega z_1, \omega^{-1} z_2) \tag{13.1}$$

where ω is a primitive k^{th} root of 1. This acts on the ring of polynomials $\mathbb{C}[z_1, z_2]$. *What is the ring of invariant polynomials?*

There are three obvious invariant polynomials:

$$X = z_1^k \qquad Y = z_2^k \qquad Z = z_1 z_2 \tag{13.2}$$

Any polynomial in X, Y, Z is therefore an invariant polynomial. Note that X, Y, Z are not independent, but satisfy

$$XY = Z^k \tag{13.3}$$ eq:akrel

A nontrivial fact is that the ring of invariant polynomials is *generated* by X, Y, Z, subject to the single relation (13.3), that is:

$$\mathbb{C}[z_1, z_2]^G = \mathbb{C}[X, Y, Z]/(XY - Z^k). \tag{13.4}$$

This describes the polynomial functions on the orbifold $\mathbb{C}^2/\mathbb{Z}_k$.

Example Now consider the binary dihedral group $\mathcal{D}_{n+2}$ of order $4n$. This acts on $\mathbb{C}^2$:

$$
\begin{aligned}
(z_1, z_2) &\to (\beta z_1, \beta^{-1} z_2) \\
(z_1, z_2) &\to (+z_2, -z_1)
\end{aligned}
\tag{13.5}
$$

eq:bindih

where $\beta = e^{i\pi/n}$. Again, let us ask for the ring of invariant polynomials. Some obvious invariants are:

$$
Z = (z_1 z_2)^2 \qquad X = z_1^{2n} + z_2^{2n} \qquad Y = (z_1 z_2)(z_1^{2n} - z_2^{2n}) \tag{13.6}
$$

Now note that

$$
X^2 = z_1^{4n} + 2 z_1^{2n} z_2^{2n} + z_2^{4n} \tag{13.7}
$$

$$
Y^2 = (z_1 z_2)^2 (z_1^{4n} - 2 z_1^{2n} z_2^{2n} + z_2^{4n}) \tag{13.8}
$$

so, again, these three have a single independent relation

$$
ZX^2 - Y^2 = 4Z^{n+1} \tag{13.9}
$$

eq:deerel

Again, it is true, but not obvious that all invariant polynomials are generated by X, Y, Z subject to the relation (13.9).

Remarks:

1. Similar, but more difficult arguments give the invariant polynomials $\mathbb{C}[z_1, z_2]^G$ for all the finite subgroups G of $SL(2,\mathbb{C})$. The ring is $\mathbb{C}[X,Y,Z]/I$ where I is the ideal generated by:

$\mathcal{C}_n$	$XY = Z^n$
$\mathcal{D}_{n+2}$	$ZX^2 + Y^2 = Z^{n+1}$
$\mathcal{T}$	$X^2 + Y^3 + Z^4 = 0$
$\mathcal{O}$	$X^2 + Y^3 + YZ^3 = 0$
$\mathcal{I}$	$X^2 + Y^3 + Z^5 = 0$

2. The groups $\mathcal{C}_n, \mathcal{D}_{n+2}, \mathcal{T}, \mathcal{O}, \mathcal{I}$ will turn out to be closely related to the Lie algebras $A_n, D_{n+2}, E_6, E_7, E_8$. The geometry of $\mathbb{C}^2/\Gamma$ has amazing relations to these Lie algebras. This is part of something called the McKay correspondence. We'll give a very rough idea of what happens here. The equation $XY - Z^n = 0$ is an equation for a

complex "surface" in $\mathbb{C}^3$. (That is the solutions space has 2 complex dimensions, or 4 real dimensions.) Note that it is a cone: If (X_0, Y_0, Z_0) is on the surface then so is $(\lambda^n X_0, \lambda^n Y_0, \lambda^2 Z_0)$. So the surface is singular. We can "resolve the singularity" by considering a deformation

$$XY - Z^n = \epsilon \tag{13.10}$$

This equation turns out to have n "independent" 2-dimensional spheres. (It might help to think about the real case, where a deformation of a cone $X^2 - Y^2 - Z^k = 0$ in $\mathbb{R}^3$ by putting ϵ on the RHS gives a smooth space, but the resolution has induced a noncontractible circle, hence nontrivial topology.) These spheres intersect in points within the surface. The pattern of intersection turns out to be the Dynkin diagram of the corresponding Lie group. That's just the beginning...

**** REFERENCES .... *****

Exercise

Describe the ring of invariant polynomials under

$$(z_1, z_2) \to (\omega z_1, \omega^k z_2) \tag{13.11}$$

where $\omega = e^{2\pi i/n}$.

Warning: This is hard.

---

14. Symmetric functions

14.1 Structure of the ring of symmetric functions

One important example of group actions on functions spaces is the theory of symmetric functions.

The permutation group S_n acts on $\mathbb{R}^n$:

$$\sigma \cdot (x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)}) \tag{14.1}$$

Exercise

a.) Show that an orbit is uniquely labelled by a point in the subspace $x^1 \geq x^2 \geq \cdots \geq x^n$. That is, this subspace is a fundamental domain for the action of the symmetric group on $\mathbb{R}^n$.

b.) Compute the isotropy groups which arise.

By our basic principle S_n acts on functions on $\mathbb{R}^n$:

$$(\sigma^{-1}\psi)(x_1,\dots,x_n) = \psi(x_{\sigma(1)},\dots,x_{\sigma(n)}) \tag{14.2}$$ eq:funcs

The invariant functions are the *symmetric functions.*

One way these come up in physics is

1. As wavefunctions of identical bosonic particles. We will discuss this more below.
2. As characters of representations of Lie groups.

We now summarize a few of the main mathematical facts about symmetric functions. A standard reference is:

I.G. MacDonald, *Symmetric Functions and Hall Polynomials* Oxford 1995.

This book has the proofs of the theorems we quote.

Let us consider the ring of polynomials with integer coefficients [17]

$$R = \mathbb{Z}[x_1,\dots,x_n] \tag{14.3}$$

We would like to understand the structure of the ring of invariants

$$\Lambda_n := R^{S_n}. \tag{14.4}$$

What kind of symmetric polynomials can we make? For any polynomial f the sum

$$\sum_{\sigma\in S_n} \sigma\cdot f \tag{14.5}$$

is clearly a symmetric polynomial. Thus, the most obvious thing to do is to apply this construction to monomials. If $\alpha = (\alpha_1,\dots,\alpha_n)\in\mathbb{Z}_+^n$ is a sequence of nonnegative integers, then let

$$x^\alpha := x_1^{\alpha_1}\cdots x_n^{\alpha_n} \tag{14.6}$$

and consider the sum over distinct permutations of α. There will be a unique permutation of α so that it takes the form of a sequence of nonnegative integers $\lambda = (\lambda_1,\dots,\lambda_n)$ such that

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \tag{14.7}$$

We can label the resulting function m_λ.

Clearly, the m_λ form an integral basis for the symmetric polynomials. However, the symmetric polynomials form a (graded) ring, and to understand this ring it is better to use other functions.

In order to describe these functions let us recall the notion of a *partition* λ. This is a sequence of integers

$$\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \cdots \tag{14.8}$$ eq:seg

such that $\lambda_i = 0$ for i larger than some integer. For a partition we can define $|\lambda| := \sum \lambda_i$. The partitions of n we met earlier in chapter 1 are the partitions with $|\lambda| = n$.

The functions m_λ are labeled by partitions.

[17] In fact, we can replace $\mathbb{Z}$ by any ring A.

Figure 25: Young diagrams fig:young

It is traditional to associate a *Young diagram* to partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$. These are diagrams of boxes with λ_i boxes in the i^{th} row. They are very useful in representation theory. Some examples are shown in 25. There are of course different conventions used by different authors. These conventions are related by the group D_4.

By reflecting in the diagonal we have a map of partitions $\lambda \to \lambda'$ the conjugate partition. For example the conjugate of (1^s) is $(s, 0, 0, \dots)$, or if $\lambda = (3, 1, 0, \dots)$ then $\lambda' = (2, 1, 1, 0, \dots)$. Formally, we may define

$$\lambda'_i = |\{\lambda_j | \lambda_j \geq i\}| \tag{14.9}$$

Figure 26: Ordering on Young diagrams fig:orderyoung

One can put various orderings on partitions. The only one we need is the partial ordering which says that $\lambda \geq \mu$ if

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i \qquad \forall i \tag{14.10}$$

If in addition, for some i there is strict inequality we write $\lambda > \mu$. Note that this is a partial order. One cannot say that (31^3) is greater or less than (2^3).

Now to get a generating set for the ring of symmetric polynomials we consider the *Elementary symmetric functions* defined by

$$E(t) := \sum_{r=0}^{n} e_r t^r := \prod_{i=1}^{n} (1 + x_i t) \tag{14.11}$$ eq:esym

So, for example:

$$
\begin{aligned}
e_0 &= 1 \\
e_1 &= x_1 + \cdots + x_n \\
e_2 &= x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n \\
e_3 &= x_1x_2x_3 + \cdots \\
e_r &= \sum_{1\le i_1 < i_2 < \cdots < i_r \le n} x_{i_1}\cdots x_{i_r}
\end{aligned}
\tag{14.12}
$$

eq:eet

The first fundamental theorem on symmetric functions is: **Theorem 1**. The ring of

invariant polynomials, $\mathbb{Z}[x_1,\dots,x_n]^{S_n}$, is itself a polynomial ring and

$$
\mathbb{Z}[x_1,\dots,x_n]^{S_n} = \mathbb{Z}[e_1,\dots,e_n] \tag{14.13}
$$

eq:rings

That is, *any* symmetric polynomial in n variables with integer coefficients can be uniquely written as a polynomial, with integer coefficients, in the functions $e_r(x)$.

Example:

$$
\sum_{i=1}^{n} x_i^2 = e_1^2 - 2e_2 \tag{14.14}
$$

Sketch of Proof: We have seen that the m_λ define an integral basis for the ring. For a partition $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots)$ define

$$
e_\lambda := e_{\lambda_1}e_{\lambda_2}\cdots \tag{14.15}
$$

Then the key identity to establish is

$$
e_{\lambda'} = m_\lambda + \sum_{\mu<\lambda} a_{\lambda\mu}m_\mu \tag{14.16}
$$

eq:keyrel

where $a_{\lambda\mu}$ are nonnegative integers.

To check this let us say $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r > 0 \ge 0\cdots)$ and $\lambda' = (\lambda_1' \ge \lambda_2' \ge \cdots \ge \lambda_s' > 0 \ge 0\cdots)$. Note that $r = \lambda_1'$ and $s = \lambda_1$. Similarly, the number of rows in λ' with at least two elements is λ_2, and so forth.

Now imagine evaluating the symmetric polynomial $e_{\lambda'}(x)$ in the regime were $x_1 \gg x_2 \gg \cdots \gg x_n$. Then the leading term in e_λ is

$$
(x_1x_2\cdots x_{\lambda_1'} + \cdots)(x_1x_2\cdots x_{\lambda_2'} + \cdots)\cdots(x_1x_2\cdots x_{\lambda_r'} + \cdots) = x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_n^{\lambda_n} + \cdots \tag{14.17}
$$

To see this note that the leading power of x_1 is clears x_1^s. But $s = \lambda_1$. This accounts for the power of x_1. Next, the leading power of x_2 is the number of rows in λ' with at least two elements. But this is just λ_2. And so forth.

Moreover the other monomials one obtains from expanding out e_λ are subleading, which indeed means that the other monomials we obtain have leading term μ with $\lambda > \mu$. Thus we have (14.16).

Since this is an upper triangular transformation any symmetric function can be uniquely expressed as a polynomial in the e_λ. Thus they are algebraically independent. ♠

Another natural set of symmetric functions are the *complete symmetric functions.* These are defined by

$$H(t) := \prod_{i=1}^{n} \frac{1}{1-x_i t} := \sum_{k=0}^{\infty} h_k t^k \tag{14.18}$$ eq:complete

That is, h_k is the sum, with coefficient 1, of all monomials of degree k in the x_i:

$$\begin{aligned} h_0 &= 1 \\ h_1 &= x_1 + \cdots + x_n \\ h_2 &= x_1^2 + \cdots + x_n^2 + x_1x_2 + \cdots + x_{n-1}x_n \\ \cdots\cdots & \\ h_r &= \sum_{1\le i_1 \le i_2 \le \cdots \le i_r \le n} x_{i_1}\cdots x_{i_r} \end{aligned} \tag{14.19}$$ eq:hfuncs

Note that

$$H(-t) = 1/E(t) \tag{14.20}$$ eq:htoee

and hence we can express the h_k in terms of the e_k by an upper triangular integral matrix with 1 on the diagonal:

$$\begin{aligned} h_0 &= e_0 \\ h_1 &= e_1 \\ h_2 &= -e_2 + e_1^2 \\ h_3 &= e_3 - 2e_1e_2 + e_1^3 \\ \cdots\cdots & \end{aligned} \tag{14.21}$$

Therefore we can also say:

$$\mathbb{Z}[x_1, \dots, x_n]^{S_n} = \mathbb{Z}[h_1, \dots, h_n] \tag{14.22}$$ eq:ringsii

Conversely, using $E(t) = 1/H(-t)$ we can express the e_i as polynomials in the h_i with integral coefficients.

Another obvious way to make symmetric functions is to define the *Power sum functions*:

$$s_r(x) := \sum_{i=1}^{n} x_i^r \tag{14.23}$$

These can be related to the other symmetric functions by considering the generating function

$$P(t) := \sum_{r=1}^{\infty} s_r(x) t^{r-1} = \sum_{i=1}^{n} \frac{x_i}{1-x_i t} \tag{14.24}$$

Note that

$$P(t) = H'(t)/H(t) \tag{14.25}$$

or, equivalently

$$P(-t) = E'(t)/E(t) \tag{14.26}$$

This implies *Newton's identities*:

$$ne_n = \sum_{r=1}^{n} (-1)^{r-1} s_r e_{n-r} \tag{14.27}$$ eq:newtons

$$nh_n = \sum_{r=1}^{n} s_r h_{n-r} \tag{14.28}$$ eq:newtonsp

The first few give:

$$\begin{aligned}
e_1 &= s_1 \\
e_2 &= \frac{1}{2}(s_1^2 - s_2) \\
e_3 &= \frac{1}{3!}s_1^3 - \frac{1}{2!}s_1 s_2 + \frac{1}{3}s_3 \\
e_4 &= \frac{1}{4!}s_1^4 - \frac{1}{4}s_2 s_1^2 + \frac{1}{8}s_2^2 + \frac{1}{3}s_1 s_3 - \frac{1}{4}s_4 \\
e_5 &= \frac{1}{5!}(s_1^5 - 10 s_1^3 s_2 + 15 s_1 s_2^2 + 20 s_1^2 s_3 - 20 s_2 s_3 - 30 s_1 s_4 + 24 s_5)
\end{aligned} \tag{14.29}$$ eq:newiden

(See the exercise below for an efficient way to generate these polynomials.)

Thus we have proved

Theorem 2. *Any* symmetric polynomial in n variables (with rational coefficients) can be uniquely written as a polynomial in the functions $s_r(x)$. That is:

$$\mathbb{Q}[x_1, \dots, x_n]^{S_n} = \mathbb{Q}[s_1, \dots, s_n] \tag{14.30}$$

Remarks

1. Note that the relations between the h_i, e_i, s_i are *universal*. That is, they do not depend on the number of variables n of which h_i, e_i, s_i are functions. Thus we can, in a sense, take $n \to \infty$ and consider these as functions of infinitely many variables so long as $x_j = 0$ for j sufficiently large. The mathematically precise way to do this involves "injective limits."

2. There are many uses of these formulae. One common use is in discussions of class functions of matrices. Recall that by the Jordan canonical form theorem the conjugacy class of a matrix $A \sim SAS^{-1}$ has a representative given by Jordan canonical form. It is easy to evaluate traces and determinants in this form.

3. Symmetric functions appear very naturally in topology. The elementary symmetric functions e_i are related to Chern classes c_i of vector bundles, while the power functions are related to the Chern characters: $\mathrm{ch}_k \sim \frac{1}{k!}s_k$. Thus, the above identities relate Chern classes and characters of vector bundles.

Exercise
Show that

$$\exp\left(\sum_{r=1}^{\infty} s_r \frac{t^r}{r}\right) = \sum_{k=0}^{\infty} h_k t^k \tag{14.31}$$

$$\exp\left(-\sum_{r=1}^{\infty} s_r \frac{(-t)^r}{r}\right) = \sum_{k=0}^{\infty} e_k t^k \tag{14.32}$$

and thereby derive (14.29).

Exercise
Show that

$$\begin{aligned}
s_1 &= e_1 \\
s_2 &= -2e_2 + e_1^2 \\
s_3 &= 3e_3 - 3e_2e_1 + e_1^3 \\
s_4 &= -4e_4 + 4e_3e_1 + 2e_2^2 - 4e_2e_1^2 + e_1^4 \\
s_5 &= 5(e_5 - e_4e_1 - e_2e_3 + e_3e_1^2 + e_2^2e_1 - e_2e_1^3) + e_1^5 \\
s_n &= (-1)^{n+1}ne_n + \cdots + e_1^n
\end{aligned} \tag{14.33}$$

eq:chceeii

Exercise
a.) Write the e_i as polynomials in the h_i.
b.) Write the h_i as polynomials in the s_i.

Exercise
Show that any 2×2 matrix satisfies

$$\det A = \frac{1}{2}(\mathrm{Tr}A)^2 - \frac{1}{2}\mathrm{Tr}A^2 \tag{14.34}$$

Show that *any* 3×3 matrix A satisfies:

$$\det A = \frac{1}{3}\mathrm{Tr}A^3 - \frac{1}{2}(\mathrm{Tr}A)(\mathrm{Tr}A^2) + \frac{1}{6}(\mathrm{Tr}A)^3 \tag{14.35}$$

Show that any 4×4 matrix satisfies

$$\det A = \frac{1}{4!}(\mathrm{Tr}A)^4 - \frac{1}{4}(\mathrm{Tr}A^2)(\mathrm{Tr}A)^2 + \frac{1}{8}(\mathrm{Tr}A^2)^2 + \frac{1}{3}(\mathrm{Tr}A)(\mathrm{Tr}A^3) - \frac{1}{4}\mathrm{Tr}A^4 \tag{14.36}$$

And in general the expression for e_n as a polynomial in the s_i gives a relation between the determinant of an $n \times n$ matrix and its traces.

Exercise

Show that the roots $\theta_1, \dots, \theta_n$ of a monic polynomial

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 \tag{14.37}$$

are related to the coefficients via the elementary symmetric functions

$$a_{n-j} = (-1)^j e_j(\theta_1, \dots, \theta_n) \tag{14.38}$$

♣ EXERCISE ON EXPRESSING THE COEFFICIENTS IN THE CHARACTERISTIC POLYNOMIAL IN TERMS OF ELEMENTARY SYMMETRIC FUNCTIONS ♣

Exercise *The discriminant of a polynomial*

The discriminant of a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_0$ with $a_i \in \mathbb{C}$ and $a_n \neq 0$ is defined as

$$a_n^{2n-2} \prod_{i<j} (\theta_i - \theta_j)^2 \tag{14.39}$$

where θ_i are the complex roots.

a.) Noting the relation of a_i to the elementary symmetric functions in the roots, express the discriminant as a polynomial in the coefficients a, b, c, d for the cubic polynomial $ax^3 + bx^2 + cx + d$:

$$b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd \tag{14.40}$$

For $x^3 + px + q$ this simplifies to

$$D = -4p^3 - 27q^2 \tag{14.41}$$

b.) Suppose a, b, c, d are real. Show that $D > 0$ implies there are three real roots and $D < 0$ implies there is one real and two complex roots. If $D = 0$ two roots coincide.

c.) By considering how the functions a_{n-j} scale if we scale the roots $\theta_j \to t\theta_j$ show that the monomial $a_0^{p_0} \cdots a_n^{p_n}$ can only appear in the discriminant if

$$n(n-1) = np_0 + (n-1)p_1 + \cdots + p_{n-1} \tag{14.42}$$

In fact, there is an explicit formula for the discriminant in terms of the resultant of $f(x)$ and $f'(x)$. In general the resultant of two polynomials $f(x) = a_n x^n + \cdots + a_0$ and $g(x) = b_m x^m + \cdots + b_0$ is defined as the determinant of the $(n+m) \times (n+m)$ Sylvester matrix

♣Clarify the structure of this matrix ♣

$$R(f,g) := \det \begin{pmatrix} a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 & 0 & 0 & \cdots & 0 \\ 0 & a_n & a_{n-1} & \cdots & a_2 & a_1 & a_0 & 0 & \cdots & 0 \\ 0 & 0 & a_n & \cdots & a_3 & a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ b_m & b_{m-1} & b_{m-2} & \cdots & 1 \cdot b_0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & b_m & b_{m-1} & b_{m-2} & \cdots & b_0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & b_m & b_{m-1} & b_{m-2} & \cdots & b_0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & b_m & b_{m-1} & b_{m-2} & \cdots & \cdots & b_0 \end{pmatrix} \tag{14.43}$$

The discriminant is proportional to $R(f, f')$.

14.2 Free fermions on a circle and Schur functions

Figure 27: Fermi sea and particle/hole excitations. fig:Fermisea

Consider a system of N free fermions on a circle $z = e^{i\theta}$, $\theta \sim \theta + 2\pi$, with Hamiltonian $H = -\sum_i \frac{d^2}{d\theta_i^2}$. The one-particle wavefunctions are

$$\psi_n(z) = z^n \tag{14.44}$$

eq:onpart

with energy n^2. Consider a state of N fermions occupying levels n_i. By Fermi statistics these levels must all be distinct and we can assume

$$n_N > n_{N-1} > \cdots > n_1 \tag{14.45}$$

The N-fermion wavefunction is the "Slater determinant"

$$\Psi_{\vec{n}}(z_1, \dots, z_N) = \det_{1 \le i,j \le N} z_i^{n_j} \tag{14.46}$$ eq:slater

and has energy $\sum_i n_i^2$. States can be visualized as in 27.

For simplicity let us assume N is odd. Then there is a unique groundstate obtained by filling up the states $-n_F \le n \le n_F$ with

$$n_F = \frac{N-1}{2} \tag{14.47}$$ eq:fermil

(If N is even there are two groundstates.) This defines the Fermi sea. The ground state wavefunction is

$$\Psi_{gnd}(z_1, \dots, z_N) = \det_{1 \le i,j \le N} z_i^{j-1-n_F} = (z_1 \dots z_N)^{-n_F} \Delta_0(z) \tag{14.48}$$ eq:grnd

The ratio of the wavefunction to groundstate wavefunction

$$\frac{\Psi_{\vec{n}}}{\Psi_{gnd}} = \frac{\det_{1 \le i,j \le N} z_i^{n_j}}{\det_{1 \le i,j \le N} z_i^{j-1-n_F}} \tag{14.49}$$ eq:srat

is a totally symmetric function of the z_i. If $n_1 \ge -n_F$ then it is a symmetric polynomial known as a *Schur function.*

The Schur functions form a linear basis for $\mathbb{Z}[x_1, \dots, x_N]^{S_N}$ where basis elements are associated with partitions. To define them mathematically let us return to a monomial $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ with the $\alpha_i \ge 0$ and now try to skew-symmetrize:

$$\sum_{\sigma \in S_N} \epsilon(\sigma) \sigma \cdot x^\alpha \tag{14.50}$$

Note that this function is totally antisymmetric, and moreover vanishes unless the α_i are all distinct. WLOG assume that

$$\alpha_N > \alpha_{N-1} > \cdots > \alpha_1 \ge 0. \tag{14.51}$$

The smallest such α is

$$N-1 > N-2 > \cdots > 1 > 0 \ge 0 \tag{14.52}$$

defining a vector $\delta := (N-1, N-2, \dots, 1, 0)$ i.e. $\delta_j = j-1$, $1 \le j \le N$. Using δ we define ν by

$$\alpha = \nu + \delta \tag{14.53}$$

Note that $\nu_j = \alpha_j - \delta_j \ge 0$ and moreover

$$\nu_j + j - 1 > \nu_{j-1} + j - 2 \qquad \Rightarrow \qquad \nu_j \ge \nu_{j-1} \tag{14.54}$$

Thus, ν is a partition. It is more convenient to use ν than α so we define:

$$\Delta_\nu(x) := \sum_{\sigma \in S_N} \epsilon(\sigma) \sigma \cdot x^{\nu+\delta} \tag{14.55}$$

Note that this can also be written as a determinant:

$$\Delta_\nu(x) := \det(x_i^{\nu_j+N-j})$$
$$= \det \begin{pmatrix} x_1^{\nu_1} & x_1^{\nu_2+1} & \cdots & x_1^{\nu_N+N-1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ x_N^{\nu_1} & x_N^{\nu_2+1} & \cdots & x_N^{\nu_N+N-1} \end{pmatrix} \tag{14.56}$$

eq:deltsun

Note that $\Delta_0(x)$ is the famous Vandermonde determinant:

$$\Delta_0(x) := \prod_{i>j}(x_i - x_j) \tag{14.57}$$

Exercise
Show that

$$\Delta_\nu(\sigma \cdot x) = \epsilon(\sigma)\Delta_\nu(x) \tag{14.58}$$

where $\epsilon(\sigma)$ is the sign homomorphism.

Now we define the *Schur function* to be the ratio:

$$\Phi_\nu(x) := \frac{\Delta_\nu(x)}{\Delta_0(x)} \tag{14.59}$$

eq:schurf

This is a totally symmetric function. In fact, it is a polynomial in the x_i. To see this consider it as a meromorphic function of a complex variable x_1. Note that it is in fact an entire function since the potential poles at $x_1 = x_i$ are cancelled by zeroes of the numerator. Next note that the growth at infinity is obviously x_1^m for an integer m.

Example. Put $N = 2$. Then $\nu_2 \geq \nu_1 \geq 0$,

$$\Delta_\nu(x) = (x_1x_2)^{\nu_1}(x_2^{\nu_2-\nu_1+1} - x_1^{\nu_2-\nu_1+1}) \tag{14.60}$$

eq:deltatwo

$$\Phi_\nu(x_1, x_2) = (x_1x_2)^{\nu_1}\frac{x_2^{\nu_2-\nu_1+1} - x_1^{\nu_2-\nu_1+1}}{x_2 - x_1} = (x_1x_2)^{\nu_2}h_{\nu_2-\nu_1}(x_1, x_2) \tag{14.61}$$

eq:phitwo

Returning to the free fermion example, we have

$$\frac{\Psi_{\vec{n}}}{\Psi_{\vec{0}}} = \frac{\Delta_\nu(z)}{\Delta_0(z)} \tag{14.62}$$

eq:fration

for $n_j + n_F = \nu_j + j - 1$, that is

$$\nu_j = n_j - (j - 1) - n_F \tag{14.63}$$

eq:snet

Remarks

1. The second main theorem of symmetric polynomials is:

Theorem. The Schur functions $\Phi_\nu(x)$ form a linear integral basis for $\mathbb{Z}[x_1, \dots, x_N]^{S_N}$.

That is, any symmetric polynomial with integral coefficients can be written as a linear combination, with integral coefficients, of the $\Phi_\nu(x)$. Note that we are *not* forming polynomials in the Φ_ν.

2. The module A_N of skew-symmetric polynomials in $x_1, \dots, x_N$ is isomorphic to Λ_N via multiplication by Δ_0. Therefore the Δ_ν form a $\mathbb{Z}$-basis for A_N. This close relation between completely symmetric and antisymmetric functions comes up in the theory of matrix models – integrals over space of $N \times N$ matrices. It also suggests a relation between bosons and fermions, at least in $1 + 1$ dimensions. That indeed proves to be the case – there is a nontrivial isomorphism known as *bosonization* which is an isomorphism of quantum field theories of bosons and fermions in 1+1 dimensions.

3. There is an elegant expression for the Schur functions Φ_ν as a determinant of a matrix whose entries involve the h_k. The expression is

$$\Phi_\nu = \det(h_{\nu_i - i + j}) \tag{14.64}$$

`eq:phiach`

where we take an $N \times N$ matrix for N variables and it is understood that $h_0 = 1$ and $h_i = 0$ for $i < 0$. Equivalently

$$\Phi_\nu = \det(e_{\nu'_i - i + j}) \tag{14.65}$$

`eq:phiachp`

For a proof see Macdonald, p.41.

Example Consider the partition $(\nu_1, \nu_2, 0, 0, \dots)$. Then applying 14.64 we get

$$\begin{aligned}\Phi_\nu &= \begin{pmatrix} h_{\nu_1} & h_{\nu_1+1} & h_{\nu_1+2} & h_{\nu_1+3} & h_{\nu_1+4} & \cdots \\ h_{\nu_2-1} & h_{\nu_2} & h_{\nu_2+1} & h_{\nu_2+2} & h_{\nu_2+3} & \cdots \\ 0 & 0 & 1 & h_1 & h_2 & \cdots \\ 0 & 0 & 0 & 1 & h_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \\ &= h_{\nu_1} h_{\nu_2} - h_{\nu_1+1} h_{\nu_2-1}\end{aligned} \tag{14.66}$$

`eq:phitwoa`

With a little algebra one can confirm that this is indeed (14.61). Note, however, that both sides of this equality make sense for $N > 2$.

4. Define the $\mathbb{Z} \times \mathbb{N}$ matrix Ξ whose $(p, q)^{th}$ entry is h_{q-p} where $h_0 = 1$ and $h_k = 0$ for $k < 0$. Here $-\infty < p < \infty$ labels the rows, while $q = 1, 2, \dots$ labels the columns. Thus the matrix looks like 11.

Figure 28: A matrix for defining Schur functions. fig:schrmtx

Define the integers $s_k = k - \nu_k$. These eventually become $s_k = k$. Then Φ_ν is the determinant of the matrix formed from Ξ by keeping only the rows labeled by s_k. This leads to an upper triangular matrix is the determinant of the finite matrix above.

5. The definition of Φ_ν only makes sense as a function of N such that $\nu_i = 0$ for $i > N$. However, the relations between Φ_ν and the h_i are universal.

6. We will return to Schur functions after we have defined a few terms in representation theory.

Exercise

Verify that (14.66) and (14.61) are equal for $N = 2$.

Exercise

a.) Let $\nu = (n, 0, 0, 0, \dots)$. Show that $\Phi_\nu = h_n$.

b.) Let $\nu = (1^n, 0, 0, \dots)$. Then $\Phi_\nu = e_n$.

Exercise

Using Cauchy's determinant identity:

$$\det\left(\frac{1}{1 - x_i y_j}\right) = \frac{\Delta_0(x_i)\Delta_0(y_j)}{\prod_{i,j}(1 - x_i y_j)} \tag{14.67}$$

Show that

$$\exp\left[\sum_k \frac{1}{k} s_k(x) s_k(y)\right] = \sum_\nu \Phi_\nu(x)\Phi_\nu(y) \tag{14.68}$$

Hint: Multiply both sides by $\Delta_0(x)\Delta_0(y)$. For the answer see Hammermesh, p. 195

Exercise *Laughlin's wavefunctions*

Another natural place where symmetric wavefunctions appear is in the quantum Hall effect, where the 1-body wavefunctions in the lowest Landau level are

$$\psi_n(z) = \frac{1}{\sqrt{\pi n!}} z^n e^{-\frac{1}{2}|z|^2} \tag{14.69}$$

Laughlin introduced a fascinating set of approximate eigenfunctions of 2d *interacting* electrons in a magnetic field:

$$\Psi = \prod_{i<j} (z_i - z_j)^{2n+1} e^{-\frac{1}{2}\sum |z_i|^2} \tag{14.70}$$

Express these in terms of Schur functions.

Many other interesting trial wavefunctions in the FQHE can be generated using theorems about symmetric functions.

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FOLLOWING MATERIAL SHOULD BE MOVED TO CHAPTER ON REPRESENTATIONS. IT WOULD MAKE MORE SENSE TO TALK ABOUT REPS OF THE SYMMETRIC GROUP FIRST.

14.2.1 Schur functions, characters, and Schur-Weyl duality

Let V be the N-dimensional representation of $U(n)$. Consider the tensor spaces $V^{\otimes N}$. These are both representations of the symmetric group S_N as well as representations of $U(n)$. Then a beautiful result, *Schur-Weyl duality* states that

The relation between the Φ_ν and the power functions $s_k(x)$ is very nice, and important for representation theory. Let $(\ell) = (\ell_1, \ell_2, \dots, \ell_N)$ denote a tuple of nonnegative integers. Then let

$$s_{(\ell)} := s_1^{\ell_1} s_2^{\ell_2} \dots s_N^{\ell_N} \tag{14.71}$$

Then we have the important result of Frobenius:

$$\boxed{s_{(\ell)} = \sum_{\{\nu\}} \chi_\nu(\ell) \Phi_\nu} \tag{14.72}$$

eq:frobenius

We will prove this later as a consequence of Schur-Weyl duality. It will turn out that the coefficients $\chi_\nu(\ell)$ in this expansion are the characters of the representations of the symmetric group S_N in the conjugacy class labelled by $(1)^{\ell_1}(2)^{\ell_2}\cdots$. We can restrict to $\sum j\ell_j = N$ and $\sum \nu_j = N$ here.

3. Later we will view the $\Phi_\nu(x)$ as characters of representations of $U(N)$. Briefly, by Schur-Weyl duality we can label the representations of $U(N)$ by Young diagrams. Then for a given representation, its character is a function only of the conjugacy class of the matrix. Restricting to the maximal torus $Diag\{x_1, \dots, x_N\}$ the character is the Schur polynomial in the x_i. (This is the Weyl character formula.)

Remark: If we think of Φ_ν as a character of a representation of $U(N)$. Then the Hamiltonian H is closely related to the quadratic Casimir of that representation. See Cordes et. al. eq. (4.3)

Exercise *Weyl dimension formula*
Show that

$$\Phi_\nu(1, 1, \dots, 1) = \prod_{1 \le i < j \le N} \frac{(\nu_i - \nu_j + j - i)}{(j - i)} \tag{14.73}$$

eq:weyldim

We will see later that this is a special case of the Weyl dimension formula.
Hint: Put $x_j = e^{jt}$ and let $t \to 0$.

14.3 Bosons and Fermions in 1+1 dimensions

14.3.1 Bosonization

Finally, we want to describe a truly remarkable phenomenon, that of bosonization in 1+1 dimensions.

Certain quantum field theories of bosons are equivalent to quantum field theories of fermions in 1+1 dimensions! Early versions of this idea go back to Jordan [18] The subject became important in the 1970's. Two important references are

S. Coleman, Phys. Rev. **D11**(1975)2088

S. Mandelstam, Phys. Rev. **D11** (1975)3026

The technique has broad generalizations, and plays an important role in string theory.

To get some rough idea of how this might be so, let us consider the *loop group* $S^1 \to U(1)$. For winding number zero, loop group elements can be written as:

$$z \to g(z) = \exp[i \sum_{n=-\infty}^{+\infty} j_n z^n] \tag{14.74}$$

with $j_n^* = j_{-n}$. This group acts on the one-body wavefunction by

$$z^n \to g(z) z^n \tag{14.75}$$

where $g(z)$ is in the loop group of $U(1)$.

Under such a change of one-body wavefunctions the Slater determinant changes by:

$$\begin{aligned}
\Delta_0(z) &\to \det g(z_i) z_i^{j-1} \\
&= \det \begin{pmatrix} g(z_1) & g(z_1)z_1 & g(z_1)z_1^2 & \cdots \\ g(z_1) & g(z_2)z_2 & g(z_2)z_2^2 & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix} \\
&= \prod g(z_i) \Delta_0(z) \\
&= \exp[i \sum j_n s_n(z)] \Delta_0(z)
\end{aligned} \tag{14.76}$$

eq:coherentone

Note that there are two Fermi levels, and if N is large they are "far apart" meaning that operators such as $\sum_i z_i^n$ for small n will not mix states near the two respective levels. Let us therefore imagine taking N large and focussing attention on one of the Fermi levels. Therefore we adjust our energy level so that the groundstate wavefunctions are $1, z^{-1}, z^{-2}, \dots$ and we imagine that we have taken $N \to \infty$. Moreover, let us extend the $LU(1)$ action to $L\mathbb{C}^*$. Then we can separately consider the action of

$$g_-(z) = \exp[i \sum_{-\infty}^{0} \varphi_n z^n] \tag{14.77}$$

[18] P. Jordan, Z. Phys. **93**(1935)464.

and

$$g_+(z) = \exp[i\sum_1^\infty \varphi_n z^n] \tag{14.78}$$

on the groundstate Ψ_0. Note if we think of the groundstate wavefunction as a Slater determinant then acting with $g_-(z)$ takes one column to a linear combination of lower columns and hence does not change the wavefunction. On the other hand, by (14.76) acting with $g_+(z)$ has a nontrivial action and generates all possible symmetric functions of the z_i.

In this - rather heuristic - sense, the action of the loop group "generates the whole Hilbert space of fermionic states." Moreover, by mapping antisymmetric wavefunctions to symmetric wavefunctions we now view the Hilbert space as the space of polynomials in the s_n. [19]

Now we observe the following. The ring of symmetric functions (extending scalars to $\mathbb{C}$) is the polynomial ring $\mathbb{C}[s_1, s_2, \dots]$. We now make this space into a *Hilbert space*. We introduce the inner product on polynomials in s_i by

$$\langle f(s)|g(s)\rangle := \int \prod_{k=1}^\infty \frac{ds_k \wedge d\bar{s}_k}{2\pi i k} \overline{f(s)} g(s) e^{-\sum_{k=1}^\infty \frac{1}{k}|s_k|^2} \tag{14.79}$$

eq:innerpf

This is the coherent state representation of an infinite system of harmonic oscillators:

$$[a_k, a_j] = k\delta_{k+j,0} \qquad\qquad -\infty < j,k < \infty \tag{14.80}$$

with $a_{-j} = a_j^\dagger$. These are realized as follows: $a_k^\dagger$ is multiplication by s_k and a_k is $k\frac{\partial}{\partial s_k}$. The state

$$(a_1^\dagger)^{\ell_1}(a_2^\dagger)^{\ell_2}\cdots(a_N^\dagger)^{\ell_N}|0\rangle \tag{14.81}$$

corresponds to the symmetric function $s_{(\ell)}$.

Now one can form the quantum field:

$$\phi := i\sum_{k\neq 0}\frac{1}{k}a_k z^{-k} \tag{14.82}$$

Then the coherent states

$$|\phi(z)\rangle = \exp i\oint \phi(z) j(z)|0\rangle \tag{14.83}$$

give an action of the loopgroup on the bosonic Fock vacuum acting as

$$\exp[i\sum_{n=1}^\infty j_n a_n^\dagger]|0\rangle \tag{14.84}$$

eq:coherenttwo

thus producing a nontrivial isomorphism between a bosonic and fermionic Fock space.

[19] This statement is rather loose. See the Pressley-Segal book for a mathematically precise treatment. We are discussing an irreducible representation of a centrally extended group.

To make the isomorphism between bosonic and fermionic Fock spaces more explicit we introduce a second-quantized formalism. B^+_{-n} creates a fermionic state with wavefunction z^n, B_n annihilates it, so we introduce:

$$\begin{aligned}\Psi(\theta) &= \sum_{n\in\mathbb{Z}} B_n e^{in\theta}\\ \Psi^\dagger(\theta) &= \sum_{n\in\mathbb{Z}} B^+_{-n} e^{-in\theta}.\end{aligned} \tag{14.85}$$

eq:fermflds

The filled Fermi sea satisfies the constraints:

$$\begin{aligned}B_n|0\rangle &= 0, \qquad |n| > n_F\\ B^+_{-n}|0\rangle &= 0, \qquad |n| \le n_F.\end{aligned} \tag{14.86}$$

eq:fifer

When we have decoupled systems it is appropriate to define two independent sets of Fermi fields:

$$\begin{aligned}\Psi(\theta) &= e^{i(n_F+\frac{1}{2})\theta} b(\theta) + e^{-i(n_F+\frac{1}{2})\theta}\bar{b}(\theta)\\ \Psi^\dagger(\theta) &= e^{-i(n_F+\frac{1}{2})\theta} c(\theta) + e^{i(n_F+\frac{1}{2})\theta}\bar{c}(\theta).\end{aligned} \tag{14.87}$$

eq:bcsys

We introduce complex coordinates $z = e^{i\theta}$, and define the mode expansions:

$$\begin{aligned}b(z) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} b_n z^n\\ c(z) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} c_n z^n\\ \bar{b}(z) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} \bar{b}_n \bar{z}^n\\ \bar{c}(z) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} \bar{c}_n \bar{z}^n\end{aligned} \tag{14.88}$$

eq:bcsysi

$b_n, c_n, \bar{b}_n, \bar{c}_n$ are only unambiguously defined for $\mid n \mid << N$. That is, we focus on operators that do not mix excitations around the two Fermi levels. The peculiar half-integral moding is chosen to agree with standard conventions in CFT. In terms of the original nonrelativistic modes we have:

$$\begin{aligned}c_n &= B^+_{-n_F-\epsilon+n}\\ b_n &= B_{n_F+\epsilon+n}\\ \bar{c}_n &= B^+_{n_F+\epsilon-n}\\ \bar{b}_n &= B_{-n_F-\epsilon-n}\end{aligned} \tag{14.89}$$

eq:mapmodes

where $\epsilon = \frac{1}{2}$, so that

$$\{b_n, c_m\} = \delta_{n+m,0} \qquad \{\bar{b}_n, \bar{c}_m\} = \delta_{n+m,0} \tag{14.90}$$ eq:anticomms

and all other anticommutators equal zero.

We may now reinterpret the fields $b, c, ...$ Defining $z = e^{i\theta+\tau}$ we see that these may be extended to fields in two-dimensions, and that they are (anti-) chiral, that is, they satisfy the two-dimensional Euclidean Dirac equation. Upon continuation to Minkowski space we have $z \to e^{i(\theta+t)}$ and $\bar{z} \to e^{i(\theta-t)}$. We are thus discussing two relativistic massless Fermi fields in $1+1$ dimensions. The b, c fields generate a Fermionic Fock space built on the product of vacua $\mid 0\rangle_{bc} \otimes \mid \bar{0}\rangle_{\bar{b}\bar{c}}$ where $b_n \mid 0\rangle_{bc} = c_n \mid 0\rangle_{bc} = 0$ for $n > 0$. Explicitly, the b, c fields by themselves generate the Fock space:

$$\mathcal{H}_{bc} = \text{Span}\left\{\prod b_{n_i} \prod c_{m_i} \mid 0\rangle_{bc}\right\} \qquad . \tag{14.91}$$ eq:cftstspc

The space $\mathcal{H}_{bc}$ has a natural grading according to the eigenvalue of $\sum_{n\in\mathbb{Z}} : b_n c_{-n} = \oint bc$ (called "bc-number"):

$$\mathcal{H}_{bc} = \oplus_{p\in\mathbb{Z}} \mathcal{H}_{bc}^{(p)} \tag{14.92}$$ eq:grdspce

and the states obtained by moving a state from the Fermi sea to an excited level correspond to the subspace of zero bc number:

$$\mathcal{H}_{\text{chiral}} := \mathcal{H}_{bc}^{(0)} \tag{14.93}$$ eq:chrlspc

NEED TO GIVE HERE THE TRANSLATION BETWEEN a state corresponding to a Young Diagram and an explicit fermionic Fock state.

The bc system defines a conformal field theory. Indeed we may introduce

$$L_n = \sum_{m=-\infty}^{\infty} (n/2 + m) c_{-m} b_{m+n} \tag{14.94}$$ eq:frmvrop

and compute that the L_n satisfy the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0} \tag{14.95}$$ eq:virasoroalg

with $c = 1$.

Now let us consider the second quantized operators corresponding to z^n. Before taking the limit these are given by:

$$\Upsilon_n = \sum z_i^n \qquad \longleftrightarrow \qquad \int d\theta \Psi^\dagger(\theta)\, e^{in\theta} \Psi(\theta). \tag{14.96}$$ eq:upsop

Now let us take the large N limit and focus on terms that do not mix the two Hilbert spaces of excitations around the two Fermi levels. Cross terms between barred and unbarred fields involve operators that mix the two Fermi levels. Since we are only interested in the case of the decoupled Fermi level excitations we may replace:

$$\begin{aligned}\Upsilon_n &\to \oint dz\ z^{-1-n} c\ b(z) + \oint d\bar{z}\ \bar{z}^{-1+n} \bar{c}\ \bar{b}(\bar{z}) \\ &= \sum_m c_{n-m} b_m + \bar{c}_{m-n} \bar{b}_{-m} \\ &= \alpha_n + \bar{\alpha}_{-n}\end{aligned} \tag{14.97}$$

eq:upsii

where we have introduced a field $bc = i\partial_z \phi(z)$ which has expansion

$$\begin{aligned}\partial_z \phi(z) &= i \sum_{m\in\mathbb{Z}} \alpha_m z^{m-1} \\ [\alpha_m, \alpha_n] = [\bar{\alpha}_m, \bar{\alpha}_n] &= m\delta_{m+n,0} \\ [\alpha_m, \bar{\alpha}_n] &= 0.\end{aligned} \tag{14.98}$$

eq:bosalg

In terms of α_n, we may introduce a representation of the Virasoro algebra:

$$L_n = \frac{1}{2} \sum \alpha_{n-m} \alpha_m \tag{14.99}$$

eq:virop

which satisfy (14.95), again with $c = 1$. Using the α we can define a vacuum $\alpha_n \mid 0\rangle = 0$ for $n \geq 0$ and a statespace

$$\mathcal{H}_\alpha = \mathrm{Span}\left\{\mid \vec{k}\rangle \equiv \prod (\alpha_{-j})^{k_j} \mid 0\rangle\right\}. \tag{14.100}$$

eq:achalph

Bosonization states that there is a natural isomorphism:

$$\mathcal{H}_\alpha \cong \mathcal{H}_{bc}^{(0)} \tag{14.101}$$

eq:bsnztin

We will not prove this but it can be made very plausible as follows. The Hilbert space may be graded by L_0 eigenvalue. The first few levels are:

$$\begin{array}{lll} L_0 = 1 & \{b_{-\frac{1}{2}} c_{-\frac{1}{2}} \mid 0\rangle\} & \{\alpha_{-1} \mid 0\rangle\} \\ L_0 = 2 & \{b_{-\frac{1}{2}} c_{-\frac{3}{2}} \mid 0\rangle, b_{-\frac{3}{2}} c_{-\frac{1}{2}} \mid 0\rangle\} & \{\alpha_{-2} \mid 0\rangle, (\alpha_{-1})^2 \mid 0\rangle\} \end{array} \tag{14.102}$$

eq:firstlev

At level $L_0 = n$, the fermion states are labeled by Young diagrams Y with n boxes. At level $L_0 = n$, the Bose basis elements are labeled by partitions of n. We will label a partition of n by a vector $\vec{k} = (k_1, k_2, \dots)$ which has almost all entries zero, such that $\sum_j jk_j = n$. Bosonization states that the two bases are linearly related:

$$\mid Y\rangle = \sum_{\vec{k}\in \mathrm{Partitions}(n)} \langle \vec{k} \mid Y\rangle \mid \vec{k}\rangle \tag{14.103}$$

eq:linrel

This last relation can be understood as the relation (14.72).
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1. Need to explain this some more.

2. Look up papers of the Kyoto school, Jimbo et. al. for perhaps helpful ways of presenting this material.

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For much more about this see the references below.

1. Pressley and Segal, *Loop Groups*, Chapter 10.

2. M. Stone, "Schur functions, chiral bosons, and the quantum-Hall-effect edge states," Phys. Rev. **B42** 1990)8399 (I have followed this treatment.)

3. S. Cordes, G. Moore, and S. Ramgoolam, "Lectures on 2D Yang-Mills Theory, Equivariant Cohomology and Topological Field Theories," Nucl. Phys. B (Proc. Suppl 41) (1995) 184, section 4. Also available at hep-th/9411210.

4. M. Douglas, "Conformal field theory techniques for large N group theory," hep-th/9303159.