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Chapter 10:: Clifford algebras

Abstract: NOTE: THESE NOTES, FROM 2009, MOSTLY TREAT CLIFFORD ALGEBRAS AS UNGRADED ALGEBRAS OVER $\mathbb{R}$ OR $\mathbb{C}$. A CONCEPTUALLY SUPERIOR VIEWPOINT IS TO TREAT THEM AS $\mathbb{Z}_2$-GRADED ALGEBRAS. SEE REFERENCES IN THE INTRODUCTION WHERE THIS SUPERIOR VIEWPOINT IS PRESENTED.

April 3, 2018

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1. Introduction

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Note added April 3, 2018: For the reader willing to invest some time first learning some $\mathbb{Z}_2$-graded- , or super- linear algebra a much better treatment of this material can be found in:

http://www.physics.rutgers.edu/~gmoore/695Fall2013/CHAPTER1-QUANTUMSYMMETRY-OCT5.pdf (Chapter 13)

http://www.physics.rutgers.edu/~gmoore/PiTP-LecturesA.pdf (Section 2.3)

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Clifford algebras and spinor representations of orthogonal groups naturally arise in theories with fermions. They also play an important role in supersymmetry and supergravity. They are used quite frequently in the various connections of physics to geometry. It also turns out that they are central to modern topology.

While the details that follow can get rather intricate, it is worth the investment it takes to learn them.

It is important to have a clear understanding about the meaning of a complex structure and a quaternionic structure on a real vector space, what realification means etc. See chapter two for these linear algebra basics.

2. Clifford algebras

Definition. Let V be a vector space over a field F and let Q be a quadratic form on V valued in F. The *Clifford algebra* $C\ell(Q)$ is the algebra over F generated by V and defined by the relations:

$$v_1 v_2 + v_2 v_1 = 2Q(v_1, v_2) \cdot 1 \tag{2.1}$$

where 1 is the unit, considered to be the multiplicative unit in the ground field F.

Remarks

• One can give a much more formal definition by taking the quotient of the tensor algebra $T(V)$ by the 2-sided ideal generated by $v_1 \otimes v_2 + v_2 \otimes v_1 - 2Q(v_1, v_2)1$. Unlike the tensor algebra the Clifford algebra is *not* $\mathbb{Z}$-graded, since two vectors can multiply to a scalar. Nevertheless it is $\mathbb{Z}_2$-graded, and this $\mathbb{Z}_2$-grading is important. We can define an algebra automorphism λ on $C\ell(Q)$ by taking $\lambda(v) = -v$ for $v \in V$ and extending this to be an algebra automorphism. The even and odd parts of the $\mathbb{Z}_2$ grading are the $\lambda = \pm 1$ subspaces. We will use this grading later in this chapter and in the next.

• Since the relation is quadratic the embedding $V \to C\ell(Q)$ has no kernel. To lighten the notation we will usually not distinguish between a vector $v \in V$ and its image in $C\ell(Q)$, trusting to context to resolve the ambiguity.

• If $Q = 0$ the resulting algebra is called a *Grassmann algebra* and is isomorphic to the exterior algebra on V. (See more on this below). Usually, when people speak of a "Clifford algebra" it is understood that Q is nondegenerate.

To write a vector space basis of $C\ell(Q)$ let $\{e^\mu | \mu = 1, \dots d\}$ be a basis of V. Then the e^μ generate the algebra. The quadratic form in this basis will be $Q^{\mu\nu}$. Since it is nondegenerate we can define an inverse $Q_{\mu\nu}$ so that $Q_{\mu\nu} Q^{\nu\lambda} = \delta_\mu^{\ \lambda}$. We will also use the notation $e_\mu = Q_{\mu\nu} e^\nu$.

For simplicity, let us choose a basis in which Q is diagonal [1]

When we multiply $e_{\mu_1} \cdots e_{\mu_p}$ we can use the relations to move vectors with the same index next to each other and then "annihilate them" (i.e., use the relation) to get scalars. Define

$$e^{\mu_1 \cdots \mu_p} := e^{\mu_1} \cdots e^{\mu_p} \tag{2.2}$$ eq:multiindx

when the indices are all different. Note that this expression is completely antisymmetric on $\mu_1, \dots, \mu_p$.

Clearly the $e^{\mu_1 \cdots \mu_p}$ for $\mu_1 < \mu_2 < \cdots < \mu_p$ are all linearly independent and moreover form a vector space basis over F for $C\ell(Q)$. In particular we find

$$\dim_F C\ell(Q) = 2^d = \sum_{p=0}^{d} \binom{d}{p} \tag{2.3}$$

[1] We can also speak of Clifford modules over a ring. In this case we might not be able to diagonalize Q. This might happen, for example, if we considered Clifford rings over $\mathbb{Z}$.

Note that using this basis we can define an isomorphism of $C\ell(Q)$ with $\Lambda^*(V)$ as *vector spaces*:

$$\frac{1}{p!}\omega_{\mu_1\dots\mu_p}e^{\mu_1\cdots\mu_p} \to \frac{1}{p!}\omega_{\mu_1\dots\mu_p}e^{\mu_1}\wedge\cdots\wedge e^{\mu_p} \tag{2.4}$$ eq:isomf

where $\omega_{\mu_1\dots\mu_p}$ is a totally antisymmetric tensor.

Even though $\Lambda^*(V)$ is an algebra, (2.4) does *not* define an algebra isomorphism since $(e^\mu)^2 \neq 0$ in the Clifford algebra (when Q is diagonal) while $e^\mu \wedge e^\mu = 0$ in $\Lambda^*(V)$. Indeed, $\Lambda^*(V)$ is isomorphic to the Grassmann algebra, as an algebra.

Remarks

• An important anti-automorphism, the *transpose* is defined as follows: $1^t = 1$ and $v^t = v$ for $v \in V$. Now we extend this to be an anti-automorphism so that $(\phi_1\phi_2)^t = \phi_2^t\phi_1^t$. In particular:

$$(e^{\mu_1}e^{\mu_2}\cdots e^{\mu_k})^t = e^{\mu_k}e^{\mu_{k-1}}\cdots e^{\mu_2}e^{\mu_1} \tag{2.5}$$

A little computation shows that

$$(e^{\mu_1\cdots\mu_k})^t = e^{\mu_k\cdots\mu_1} = (-1)^{\frac{1}{2}k(k-1)}e^{\mu_1\cdots\mu_k} \tag{2.6}$$

• The functions $f(k) = (-1)^{\frac{1}{2}k(k-1)}$ and $g(k) = (-1)^{\frac{1}{2}k(k+1)}$ appear frequently in the following. Note that $f(k)$ and $g(k)$ only depend on $k\mathrm{mod}4$, $f(k) = g(-k)$ and

$$(-1)^{\frac{1}{2}k(k-1)} = \begin{cases} +1 & k = 0, 1\mathrm{mod}4 \\ -1 & k = 2, 3\mathrm{mod}4 \end{cases} \tag{2.7}$$ eq:qnfen

$$(-1)^{\frac{1}{2}k(k+1)} = \begin{cases} +1 & k = 0, 3\mathrm{mod}4 \\ -1 & k = 1, 2\mathrm{mod}4 \end{cases} \tag{2.8}$$ eq:qnfena

3. The Clifford algebras over $\mathbb{R}$

In quantum mechanics we work over the complex numbers. Nevertheless, in theories of fermions it is often important to take into account reality constraints, and hence the properties of the Clifford algebras over $\mathbb{R}$ is quite relevant to physics, particularly when we discuss supersymmetric field theories and string theories in various spacetime dimensions. Moreover, in the physics-geometry interaction the beautiful structure of the Clifford algebras over $\mathbb{R}$ plays an important role. It is thus well worth the extra effort to understand the structures here.

By Sylvester's theorem, when working over $\mathbb{R}$ it suffices to consider

$$Q(e^\mu, e^\nu) = \eta^{\mu\nu} \tag{3.1}$$

where $\eta^{\mu\nu}$ is diagonal with ± 1 entries. Since there are different conventions in the literature, we will denote the Clifford algebra with

$$\begin{aligned} (e^1)^2 = \cdots = (e^{r_+})^2 &= +1 \\ (e^{r_++1})^2 = \cdots = (e^{r_++s_-})^2 &= -1 \end{aligned} \tag{3.2}$$

by $C\ell(r_+, s_-)$. The Clifford algebra is thus specified by a pair of nonnegative integers labelling the number of $+1$ and -1 eigenvalues. If the form is definite we simply denote $C\ell(r_+)$ or $C\ell(s_-)$. [2]

Exercise
Let p_μ be a vector on the pseudo-sphere

$$p_\mu p_\nu \eta^{\mu\nu} = R^2. \tag{3.3}$$

Show that $(p_\mu e^\mu)^2 = R^2 \cdot 1$.

Exercise *The Clifford volume element*

a.) There is a basic anti-automorphism, the transpose automorphism that acts as $(\phi_1\phi_2)^t = \phi_2^t\phi_1^t$. Show that:

$$e^{\mu_1\cdots\mu_k} = (-1)^{\frac{1}{2}k(k-1)} e^{\mu_k\cdots\mu_1} \tag{3.4}$$

b.) Show that the volume element in $C\ell(r_+, s_-)$

$$\omega = e^1 e^2 \cdots e^d \tag{3.5}$$

$d = r_+ + s_-$, satisfies

$$\omega^2 = (-1)^{\frac{1}{2}(s_- - r_+)(s_- - r_+ + 1)} = \begin{cases} +1 & for (s_- - r_+) = 0, 3 \mathrm{mod} 4 \\ -1 & for (s_- - r_+) = 1, 2 \mathrm{mod} 4 \end{cases} \tag{3.6}$$

[Answer: The easiest way to compute is to write

$$\omega \cdot \omega = (-1)^{\frac{1}{2}d(d-1)} \omega\omega^t = (-1)^{\frac{1}{2}d(d-1)+s} = (-1)^{\frac{1}{2}(s-r)(s-r+1)}] \tag{3.7}$$

c.) Show that under a change of basis $e^\mu \to f^\mu = \sum g^{\nu\mu} e^\nu$ where $g \in O(\eta)$ we have $\omega' = \det g \omega$, so that ω indeed transforms as the volume element.

d.) $\omega e^\mu = (-1)^{d+1} e^\mu \omega$. Thus ω is central for d odd and is not central for d even.

Note:

1. $d_T = s_- - r_+$ generalizes the number of dimensions *transverse* to the light cone in Lorentzian geometry.

2. $\omega^2 = 1$ and ω is central only for $d_T = s - r = 3 \mathrm{mod} 4$.

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[2] I find it impossible to remember whether the first or second entry should be the number of positive or negative signs. Therefore, I will explicitly put a subscript $\pm$ to indicate the signature. Lawson-Michelsohn take $C\ell(r, s) = C\ell(r_-, s_+)$.

3.1 The real Clifford algebras in low dimension

In 0 dimensions $C\ell(0) = \mathbb{R}$.

3.1.1 $C\ell(1_-)$

To see this note that the general element in $C\ell(1_-)$ is $a+be^1$. But we have a faithful matrix representation $e^1 \to \sqrt{-1}$. (Note that in fact we have *two* inequivalent representations, depending on the sign of the squareroot). Thus $a + be^1 \to a + ib$ defines an isomorphism to $\mathbb{C}$, regarded as an algebra over $\mathbb{R}$. Thus

$$C\ell(1_-) \cong \mathbb{C} \tag{3.8}$$

There is only one real irreducible representation up to equivalence: $V \cong \mathbb{R}^2$ and

$$\rho(a + be^1) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \tag{3.9}$$

Note that

$$\rho'(a + be^1) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{3.10}$$

defines an equivalent representation.

Over $\mathbb{C}$ these matrices can be diagonalized, leading to *two* inequivalent complex representations of $\mathbb{C}\ell(1)$. Applying the rule for the complexification of a real vector space with complex structure we have $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \overline{\mathbb{C}}$ so the inequivalent representations are:

$$\rho_+(a + be^1) = a + bi \tag{3.11}$$

eq:inqa

and

$$\rho_-(a + be^1) = a - bi \tag{3.12}$$

eq:inqb

3.1.2 $C\ell(1_+)$

The multiplication law of elements $a + be^1$ is simply

$$(a + be^1) \cdot (a' + b'e^1) = (aa' + bb') + (ab' + a'b)e^1 \tag{3.13}$$

In this dimension we can introduce central projection operators

$$P_\pm = \frac{1}{2}(1 \pm e^1) \tag{3.14}$$

Then $P_+C\ell(1_+)$ and $P_-C\ell(1_+)$ are subalgebras and we can write a direct sum of algebras:

$$C\ell(1_+) = P_+C\ell(1_+) \oplus P_-C\ell(1_+) \tag{3.15}$$

In this case each of the subalgebras is one-dimensional:

$$a + be^1 = (a + b)(\frac{1 + e^1}{2}) + (a - b)(\frac{1 - e^1}{2}) \tag{3.16}$$

We have

$$C\ell(1_+) = \mathbb{R} \oplus \mathbb{R} \tag{3.17}$$

This algebra is also known as the “double numbers.”

There are two inequivalent real representations:

$$\rho_+(a + be^1) = a + b \tag{3.18}$$

$$\rho_-(a + be^1) = a - b \tag{3.19}$$

These representations are not faithful. We have a faithful matrix rep:

$$a + be^1 \to \begin{pmatrix} a & b \\ b & a \end{pmatrix} \tag{3.20}$$

eq:doublnumone

Of course the representation (3.20) is in fact reducible. it is equivalent to matrices of the form

$$\begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix} \tag{3.21}$$

eq:doublenumii

However, one needs both diagonal entries to get a faithful representation. Later we will talk about $\mathbb{Z}_2$ graded representations. The minimal $\mathbb{Z}_2$ graded representation is 2-dimensional and given by (3.20).

Finally, over $\mathbb{C}$, i.e. for $\mathbb{C}\ell_1$ we could also have taken $(e^1)^2 = +1$ and

$$\rho_+(a + be^1) = a + b \tag{3.22}$$

eq:inqc

and

$$\rho_-(a + be^1) = a - b \tag{3.23}$$

eq:inqd

with $a, b \in \mathbb{C}$ reflecting the complexification of the two representations of $C\ell(1_+)$. Thus

$$\mathbb{C}\ell(1) \cong \mathbb{C} \oplus \mathbb{C} \tag{3.24}$$

eq:cclxone

3.1.3 Two dimensions

Let us introduce the very useful notation

$$\mathbb{K}(n) := Mat_{n\times n}(\mathbb{K}) \tag{3.25}$$

eq:kayenn

where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. This is an algebra over $\mathbb{R}$ of real dimension $n^2, 2n^2, 4n^2$, respectively.

In two dimensions we have

$$\begin{aligned} C\ell(2_+) &= \mathbb{R}(2) \\ C\ell(1_+, 1_-) &= \mathbb{R}(2) \\ C\ell(2_-) &= \mathbb{H} \end{aligned} \tag{3.26}$$

eq:tdclff

To see this consider first $C\ell(2_+)$: Give a faithful matrix rep:

$$e^1 \to \sigma_1 \qquad e^2 \to \sigma_3 \tag{3.27}$$

then

$$e^1 e^2 = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.28}$$

Now we can write an arbitrary 2×2 real matrix as a linear combination of $1, \sigma_1, \sigma_3, -i\sigma_2$:

$$\rho(a + be^1 + ce^2 + de^1e^2) = \begin{pmatrix} a+c & b-d \\ b+d & a-c \end{pmatrix} \tag{3.29}$$

Next for $C\ell(2-)$. Map to the imaginary unit quaternions:

$$\begin{aligned} e^1 &\to \mathbf{i} \\ e^2 &\to \mathbf{j} \\ e^1e^2 &\to \mathbf{k} \end{aligned} \tag{3.30}$$

Finally $C\ell(1,1)$. Again we can provide a faithful matrix rep:

$$e^1 \to \sigma_1 \qquad e^2 \to i\sigma_2 \tag{3.31}$$

thus

$$\rho(a + be^1 + ce^2 + de^1e^2) = \begin{pmatrix} a+d & b+c \\ b-c & a-d \end{pmatrix} \tag{3.32}$$

and the algebra is that of $\mathbb{R}(2)$. ♠

Remarks

• When we complexify there is no distinction between the signatures. Any of the above three algebras can be used to show that

$$\mathbb{C}\ell(2) \cong \mathbb{C}(2) \tag{3.33}$$

eq:ccxclff

• The representation matrices are always denoted as Γ matrices in the physics literature. Thus, for example, what we are saying above is that in $1+1$ dimensions we could choose an irreducible real representation

$$\Gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.34}$$

Exercise

We have now obtained two algebra structures on the vector space $\mathbb{R}^4$: $\mathbb{R}(2)$ and $\mathbb{H}$. Are they isomorphic? (Hint: Is $\mathbb{R}(2)$ a division algebra?)

3.2 Tensor products of Clifford algebras and periodicity

We will now examine how the Clifford algebras in different dimensions are related to each other. This will enable us to express the Clifford algebra in terms of matrix algebras for all (r_+, s_-). These relations are also very useful in physics in dimensional reduction.

There are two kinds of tensor products one could define, the graded and ungraded tensor product. For now we will focus on the *ungraded* tensor product. [3] The tensor product is then the standard tensor product of vector spaces. We define a Clifford multiplication on the tensor product by the rule:

$$(\phi_1 \otimes \psi_1) \cdot (\phi_2 \otimes \psi_2) := \phi_1 \cdot \phi_2 \otimes \psi_1 \cdot \psi_2 \tag{3.35}$$

This is the standard tensor product on two algebras. In section *** below we will discuss the *graded* tensor product which differs by some important sign conventions.

Lemma:
- $C\ell(r_+, s_-) \otimes C\ell(2_+) \cong C\ell((s+2)_+, r_-)$
- $C\ell(r_+, s_-) \otimes C\ell(1_+, 1_-) \cong C\ell((r+1)_+, (s+1)_-)$
- $C\ell(r_+, s_-) \otimes C\ell(2_-) \cong C\ell(s_+, (r+2)_-)$

Proofs:

• Let e^μ be generators of $C\ell(r_+, s_-)$, f^α, $\alpha = 1, 2$ be generators of $C\ell(2+)$. Note that the obvious set of generators $e^\mu \otimes 1$ and $1 \otimes f_\alpha$, do not satisfy the relations of the Clifford algebra, because they do not anticommute. On the other hand if we take

$$\tilde{e}_\mu := e_\mu \otimes f_{12} \qquad \tilde{e}_{d+\alpha} := 1 \otimes f_\alpha \tag{3.36}$$

where $f_{12} = f_1 f_2$, then $\tilde{e}_M$, $M = 1 \dots, d+2$ satisfy the Clifford algebra relations and also generate the tensor product. Now note that $(f_{12})^2 = -1$ and hence:

$$(e_\mu \otimes f_{12})^2 = -(e_\mu)^2 \tag{3.37}$$

(no sum on μ). The same proof works for item 3 above.

• Once again we can take generators as above, now we need only notice that in signature $(1_+, 1_-)$ we have $(f_{12})^2 = +1$ and hence:

$$(e_\mu \otimes f_{12})^2 = +(e_\mu)^2 \tag{3.38}$$

(no sum on μ). ♠

Remarks These isomorphisms, and the consequences below are very useful because they relate Clifford algebras and spinors in different dimensions. Notice in particular, item 2, which relates the Clifford algebra in a spacetime to that on the transverse space to the lightcone.

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[3] In section *** below we discuss the graded tensor product.

Exercise
Show that $C\ell((s+1)_+, r_-) \cong C\ell((r+1)_+, s_-)$.

Exercise
Show $C\ell(r_+, s_-) \cong \mathbb{R}(2^r)$ when $r = s$. This is always a matrix algebra over the reals. Further understanding of why this is so comes from the model for Clifford algebras in terms of contraction and wedge product of differential forms (and free fermions) described below.

In general, two algebras related by $A \cong B \otimes \mathrm{Mat}_n(\mathbb{R})$ for some n are said to be *Morita equivalent*.

3.2.1 Special isomorphisms

For $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ let $K(n)$ denote the algebra over $\mathbb{R}$ of all $n \times n$ matrices with entries in K. We have the following special isomorphisms of matrix algebras over $\mathbb{R}, \mathbb{C}, \mathbb{H}$:

$$\mathbb{R}(n) \otimes_{\mathbb{R}} \mathbb{R}(m) \cong \mathbb{R}(nm) \tag{3.39}$$ eq:specisoma

$$\mathbb{R}(n) \otimes_{\mathbb{R}} K \cong K(n) \tag{3.40}$$ eq:specisomb

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C} \tag{3.41}$$ eq:specisomc

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{C}(2) \tag{3.42}$$ eq:specisomd

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{R}(4) \tag{3.43}$$ eq:specisome

To prove (3.41) note that the tensor product is generated (*over* $\mathbb{R}$) by $1 \otimes 1$, $1 \otimes i$, $i \otimes 1$, $i \otimes i$. Now we have an explicit isomorphism

$$\begin{aligned}
(1,0) &\to \frac{1}{2}(1 \otimes 1 + i \otimes i) \\
(i,0) &\to \frac{1}{2}(i \otimes 1 - 1 \otimes i) \\
(0,1) &\to \frac{1}{2}(1 \otimes 1 - i \otimes i) \\
(0,i) &\to \frac{1}{2}(i \otimes 1 + 1 \otimes i)
\end{aligned} \tag{3.44}$$

This should be compared with the isomorphism

$$\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C} \tag{3.45}$$

The isomorphism (3.42) follows from the familiar representation of quaternions in terms of complex 2×2 matrices that gives us the quaternions as $x_\mu \tau_\mu$. If we now take x_μ to be complex then we get all 2×2 complex matrices.

For isomorphism (3.43) we identify $\mathbb{H} \cong \mathbb{R}^4$ and note that to an $q_1 \otimes q_2 \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}$ we can associate a linear map $\mathbb{R}^4 \to \mathbb{R}^4$ given by

$$x \to q_1 x \bar{q}_2 \tag{3.46}$$

Extending by linearity this defines an algebra homomorphism $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \to \mathrm{End}(\mathbb{R}^4) = \mathbb{R}(4)$. We claim the map is an isomorphism. To see this let us try to compute the kernel. This would be an element $\sum_{\mu\nu} a_{\mu\nu} \tau_\mu \otimes \bar{\tau}_\nu \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}$ so that for all $x \in \mathbb{H}$

$$\sum_{\mu\nu} a_{\mu\nu} \tau_\mu x \bar{\tau}_\nu = 0 \tag{3.47}$$ `eq:allsnx`

By conjugation, if $a_{\mu\nu}$ satisfies (3.47) so does its transpose, so we can separate the equations into $a_{\mu\nu}$ symmetric and antisymmetric. Now note that the equation is $SO(4) \times SO(4)$ covariant, if (3.47) is satisfied then for any four unit quaternions q_1, q_2, p_1, p_2 we have

$$\sum_{\mu\nu} a_{\mu\nu} q_1 \tau_\mu \bar{q}_2 x p_1 \bar{\tau}_\nu \bar{p}_2 = 0 \tag{3.48}$$ `eq:allsnxi`

and hence if $a_{\mu\nu}$ satisfies (3.47) so does $R_1 a R_2$ where $R_1, R_2 \in SO(4)$. Thus for the symmetric case we can diagonalize a so that (3.47) becomes

$$\sum_{\mu} \lambda_\mu \tau_\mu x \bar{\tau}_\mu = 0 \qquad\qquad \forall x \in \mathbb{H} \tag{3.49}$$

Substituting $x = 1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ gives four linear equations which easily imply $\lambda_\mu = 0$. Similarly if $a_{\mu\nu}$ is anti-symmetric it can be skew-diagonalized and then it is easy to show that the skew eigenvalues vanish. Thus the kernel of the map $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \to \mathrm{End}(\mathbb{R}^4) = \mathbb{R}(4)$ is zero and since both domain and range have dimension 16 the map is an isomorphism.

Exercise
Show that

$$C\ell(3_+) = C\ell(1_-) \otimes C\ell(2_+) \cong \mathbb{C}(2) \tag{3.50}$$ `eq:clthrp`

$$C\ell(3-) = C\ell(1_+) \otimes C\ell(2_-) \cong \mathbb{H} \oplus \mathbb{H} \tag{3.51}$$ `eq:clthrm`

3.2.2 The periodicity theorem

Now we combine the above isomorphisms to produce some useful relations between the Clifford algebras:

On the one hand we can say

$$C\ell(r_+, s_-) \otimes C\ell(2_+) \otimes C\ell(2_-) \cong C\ell((s+2)_+, r_-) \otimes C\ell(2_-) \cong C\ell(r_+, (s+4)_-) \quad (3.52)$$ eq:pera

On the other hand we can say

$$C\ell(r_+, s_-) \otimes C\ell(2_-) \otimes C\ell(2_+) \cong C\ell(s_+, (r+2)_-) \otimes C\ell(2_+) \cong C\ell((r+4)_+, s_-) \quad (3.53)$$ eq:perb

Finally we note that

$$C\ell(2_+) \otimes C\ell(2_-) = \mathbb{R}(2) \otimes \mathbb{H} \cong \mathbb{H}(2) \quad (3.54)$$ eq:perc

Summarizing:

$$C\ell((r+4)_+, s_-) \cong C\ell(r_+, (s+4)_-) \cong C\ell(r_+, s_-) \otimes \mathbb{H}(2) \quad (3.55)$$ eq:perd

Thus, if we wish to understand the structure of $C\ell(r_+, s_-)$ we can use tensor product with $C\ell(1_+, 1_-)$ to reduce the algebra to the (Morita equivalent) form $C\ell(n_+)$ or $C\ell(n_-)$, depending on whether $r - s \geq 0$ or $r - s \leq 0$, respectively. Then we can use (3.55) to bring $0 \leq n_\pm \leq 3$. On the other hand, we have explicitly determined the algebras in this range using the above computations. In this way we can list the full set of algebras.

The mod-four periodicity under $\otimes_{\mathbb{R}}\mathbb{H}(2)$ can be iterated to produce a basic mod-eight periodicity which is a little easier to think about. Note that since $\mathbb{H}(2) \otimes \mathbb{H}(2) \cong \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{R}(4) \cong \mathbb{R}(16)$ the identities (3.55) immediately imply:

$$\boxed{\begin{array}{l} C\ell(r_+, (s+8)_-) \cong C\ell(r_+, s_-) \otimes \mathbb{R}(16) \\ C\ell((r+8)_+, s_-) \cong C\ell(r_+, s_-) \otimes \mathbb{R}(16) \end{array}} \quad (3.56)$$

Thus, combining the computation of the first four algebras with (3.55) we produce the following table:

$r =$	1	2	3	4	5	6	7	8
$C\ell(r_+)$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
$C\ell(r_-)$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$

Thus, implementing the above procedure we have the following result:

1. If $r \geq s$ and $r - s = \alpha + 8k$, $0 \leq \alpha \leq 7, k \geq 0$, then

$$C\ell(r_+, s_-) \cong C\ell(\alpha_+) \otimes \mathbb{R}(2^{\frac{1}{2}(d-\alpha)}) \tag{3.57}$$ eq:perioa

2. If $s \geq r$, and $s - r = \beta + 8k$, $0 \leq \beta \leq 7, k \geq 0$, then

$$C\ell(r_+, s_-) \cong C\ell(\beta_-) \otimes \mathbb{R}(2^{\frac{1}{2}(d-\beta)}) \tag{3.58}$$ eq:periob

(Note that $d - \alpha$ and $d - \beta$ are both even.)

Now, we can unify the two results (3.57) and (3.58) by defining the *type* of the Clifford algebra to be $\mathbb{R}, \mathbb{R} \oplus \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{H} \oplus \mathbb{H}$. That is, these are the basic Morita equivalence classes which appear.

Define the "transverse dimension" to be

$$d_T := s - r \tag{3.59}$$ eq:transversed

for $C\ell(r_+, s_-)$. Observe from the above table that the Morita equivalence class *only depends on* $d_T \mathrm{mod} 8$! For example $C\ell(1_+)$ with $d_T = -1$ is of the same type as $C\ell(7_-)$ with $d_T = +7$.

Figure 1: The Clifford stop sign. fig:clffstp

We can now list the real Clifford algebras $C\ell(r_+, s_-)$ in a unified way. The type of the algebra follows the basic pattern for $d_T = 0, 1, 2, \ldots$ of

$$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{H} \oplus \mathbb{H}, \mathbb{H}, \mathbb{C}, \mathbb{R}, \mathbb{R} \oplus \mathbb{R}, \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{H} \oplus \mathbb{H}, \mathbb{H}, \ldots \tag{3.60}$$

and can be visualized as in 1. The first three entries are easy to remember since you are adding squareroots of -1. If you remember the general pattern then you can easily reconstruct the type. Next you can construct the algebra since the total dimension over $\mathbb{R}$ is just 2^d where $d = r + s$ is the dimension.

To summarize. Define $\ell \equiv [d/2]$. Then:

$$
\begin{array}{rl}
(s-r)\bmod 8 & C\ell(r_+, s_-) \\
0 & \mathbb{R}(2^\ell) \\
1 & \mathbb{C}(2^\ell) \\
2 & \mathbb{H}(2^{\ell-1}) \\
3 & \mathbb{H}(2^{\ell-1}) \oplus \mathbb{H}(2^{\ell-1}) \\
4 & \mathbb{H}(2^{\ell-1}) \\
5 & \mathbb{C}(2^\ell) \\
6 & \mathbb{R}(2^\ell) \\
7 & \mathbb{R}(2^\ell) \oplus \mathbb{R}(2^\ell) \\
8 & \mathbb{R}(2^\ell)
\end{array}
\tag{3.61}
$$

eq:cliffalgs

Remarks

- A useful table of the Lorentzian signature Clifford algebras is

$d = s+1$	$C\ell(s_+, 1_-)$	$C\ell(1_+, s_-)$
0+1	$\mathbb{C}$	$\mathbb{R} \oplus \mathbb{R}$
1 +1	$\mathbb{R}(2)$	$\mathbb{R}(2)$
2 +1	$\mathbb{R}(2) \oplus \mathbb{R}(2)$	$\mathbb{C}(2)$
3 +1	$\mathbb{R}(4)$	$\mathbb{H}(2)$
4 +1	$\mathbb{C}(4)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$
5 +1	$\mathbb{H}(4)$	$\mathbb{H}(4)$
6 +1	$\mathbb{H}(4) \oplus \mathbb{H}(4)$	$\mathbb{C}(8)$
7 +1	$\mathbb{H}(8)$	$\mathbb{R}(16)$
8 +1	$\mathbb{C}(16)$	$\mathbb{R}(16) \oplus \mathbb{R}(16)$
9 +1	$\mathbb{R}(32)$	$\mathbb{R}(32)$
10 +1	$\mathbb{R}(32) \oplus \mathbb{R}(32)$	$\mathbb{C}(32)$
11 +1	$\mathbb{R}(64)$	$\mathbb{H}(32)$

•
Note in particular:

$$\begin{aligned} C\ell(1_+,3_-) &= \mathbb{H}(2) \qquad & C\ell(3_+,1_-) &= \mathbb{R}(4) \\ C\ell(0,4_-) &= \mathbb{H}(2) \qquad & C\ell(4_+,0) &= \mathbb{H}(2) \end{aligned} \tag{3.62}$$

Exercise
It appears that the convention of whether we assign time to directions of positive or negative norm leads to different Clifford algebras.
Why is this convention irrelevant when working with fermions in physics?

Exercise *Clifford algebras appearing in supergravity theories*
Verify the following isomorphisms:

$$\begin{aligned} C\ell(1_+,3_-) &= \mathbb{H}(2) \\ C\ell(0,4) &= \mathbb{H}(2) \\ C\ell(1,4) &= \mathbb{H}(2)\oplus\mathbb{H}(2) \\ C\ell(1_-,5_+) &= \mathbb{H}(2)\otimes\mathbb{R}(2)\cong\mathbb{H}(4) \\ C\ell(1_+,5_-) &= \mathbb{H}(2) \\ C\ell(0,6) &= \mathbb{R}(8) \\ C\ell(1_+,9_-) &= \mathbb{R}(32) \\ C\ell(9_+,1_-) &= \mathbb{R}(32) \\ C\ell(1_+,10_-) &= \mathbb{C}(16) \\ C\ell(10_+,1_-) &= \mathbb{R}(32)\oplus\mathbb{R}(32) \\ C\ell(11_+) &= \mathbb{C}(32) \\ C\ell(11_-) &= \mathbb{H}(16)\oplus\mathbb{H}(16) \end{aligned} \tag{3.63}$$ eq:clff

---

Exercise
Prove the isomorphisms:

$$C\ell(r_+,s_-)\cong C\ell((r+4)_+,(s-4)_-) \tag{3.64}$$

$$C\ell((r+1)_+,s_-)\cong C\ell((s+1)_+,r_-) \tag{3.65}$$

---

4. The Clifford algebras over $\mathbb{C}$

When we change the ground field from $\mathbb{R}$ to $\mathbb{C}$ the structure of the Clifford algebras simplifies considerably.

Now there is no need to take account of the signature. We can take $e_i e_j + e_j e_i = 2\delta_{ij}$. Or we could instead take $e_i e_j + e_j e_i = -2\delta_{ij}$. Both conventions lead to equivalent results. Since we are working over the complex numbers we could also send $e_i \to \sqrt{-1}e_i$.

We have already shown that in low dimensions:

$$\begin{aligned} \mathbb{C}\ell(0) &= \mathbb{C} \\ \mathbb{C}\ell(1) &\cong \mathbb{C} \oplus \mathbb{C} \\ \mathbb{C}\ell(2) &\cong \mathbb{C}(2) \end{aligned} \tag{4.1}$$

Next, the periodicity is simplified considerably. We now have:

$$\mathbb{C}\ell(d+2) \cong \mathbb{C}\ell(d) \otimes_{\mathbb{C}} \mathbb{C}\ell(2) \cong \mathbb{C}\ell(d) \otimes_{\mathbb{C}} \mathbb{C}(2) \tag{4.2}$$

from which we obtain the basic result:

$d = 0\mathrm{mod}2$:

$$\mathbb{C}\ell(d) = \mathbb{C}(2^{[d/2]}) = \mathbb{C}(2^{d/2}) \tag{4.3}$$ eq:cplxev

$d = 1\mathrm{mod}2$:

$$\mathbb{C}\ell(d) = \mathbb{C}(2^{[d/2]}) \oplus \mathbb{C}(2^{[d/2]}) = \mathbb{C}(2^{(d-1)/2}) \oplus \mathbb{C}(2^{(d-1)/2}) \tag{4.4}$$ eq:cplxodd

Recall that we defined the *Clifford volume form* to be $\omega := e_1 e_2 \cdots e_d$, and showed moreover that

$$\omega^2 = (-1)^{\frac{1}{2}d(d-1)} = \begin{cases} +1 & d = 0,1\mathrm{mod}4 \\ -1 & d = 2,3\mathrm{mod}4 \end{cases} \tag{4.5}$$

(If we had taken $e_i e_j + e_j e_i = -2\delta_{ij}$ then we would have gotten $(-1)^{\frac{1}{2}d(d+1)}$.) When working over $\mathbb{C}$ we can always find a scalar ξ so that $\omega_c := \xi\omega$ satisfies $(\omega_c)^2 = 1$. Explicitly

$$\omega_c = \begin{cases} \omega & d = 0,1\mathrm{mod}\ 4 \\ i\omega & d = 2,3\mathrm{mod}\ 4 \end{cases} \tag{4.6}$$

Then we can form *projection operators*

$$P_\pm = \frac{1}{2}(1 \pm \omega_c) \tag{4.7}$$

When d is odd these projection operators are central - they commute with everything in the algebra - and this gives a decomposition of the algebra into a direct sum of subalgebras as in (4.4). That is elements of the form $P_+\phi$ form a sub*algebra*, not just a vector sub*space*, *because* P_+ is central. This subalgebra is isomorphic to $\mathbb{C}\ell(2^{[d/2]})$ and similarly for the subalgebra of elements of the form $P_-\phi$.

5. Representations of the Clifford algebras

Let us now consider representations of the Clifford algebras. This assigns $\phi \in C\ell \to \rho(\phi)$, where $\rho(\phi)$ is a linear transformation such that $\rho(\phi_1)\rho(\phi_2) = \rho(\phi_1\phi_2)$. The matrix representations of the generators $\rho(e^\mu) = \Gamma^\mu$ are called Γ-matrices in the physics literature.

We have seen how to write all the Clifford algebras in terms of matrix algebras over the division algebras. That is, they are all of the form $\mathbb{K}(n)$ or $\mathbb{K}(n) \oplus \mathbb{K}(n)$.

Now we only need a standard result of algebra which says that $\mathbb{K}(n)$ is a simple algebra (it has no nontrivial two-sided ideals) and hence has a unique representation (up to isomorphism) when $\mathbb{K}$ is a division algebra. To justify this recall

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FIX FOLLOWING: DID WE NOT COVER THIS IN AN EARLIER CHAPTER WITH ALGEBRAS?

****

If V is an irreducible representation then we have a homomorphism $A \to \mathrm{End}(V)$. Since V is irreducible the image cannot commute with any nontrivial projection operators and hence must be the full algebra $\mathrm{End}(V)$. On the other hand, the kernel would be a two-sided ideal in A. Now, if $A = M_n(D)$, where D is a division algebra then it is simple. To prove this: Consider any ideal $I \subset M_n(D)$. If $X = x_{ij}e_{ij}$ is in I with some $x_{kl} \neq 0$ then $e_{kk}Xe_{ll} \in I$, but this means $e_{kl} \in I$ but now the ideal generated by e_{kl} is all of $M_n(D)$.

The nature of the representation depends very much on the fields we are working over.

Let us consider first the complex Clifford algebras, $\mathbb{C}\ell(d)$. These are of the form $\mathbb{C}(n)$ or $\mathbb{C}(n) \oplus \mathbb{C}(n)$. The unique irrep of $\mathbb{C}(n)$ is just the defining representation space $V = \mathbb{C}^n$. Thus, $\mathbb{C}\ell(d)$ has a unique irrep for d even, $V = \mathbb{C}^n$ with

$$n = 2^{d/2} \qquad d \qquad \text{even} \tag{5.1}$$

While for d odd $\mathbb{C}\ell(d)$ has two inequivalent irreps $V_\pm$. As vector spaces $V_\pm \cong \mathbb{C}^n$ with

$$n = 2^{(d-1)/2} \qquad d \qquad \text{odd} \tag{5.2}$$

where $\rho(\phi_1 \oplus \phi_2) = \phi_1$ on V_+ and $\rho(\phi_1 \oplus \phi_2) = \phi_2$ on V_-. Another way to characterize these is to consider ω_c, with $(\omega_c)^2 = 1$. Then

$$\rho_+(\omega_c) = +1 \qquad on \qquad V_+ \tag{5.3}$$

$$\rho_-(\omega_c) = -1 \qquad on \qquad V_- \tag{5.4}$$

Now let us consider the Clifford algebras $C\ell(r_+, s_-)$ over $\mathbb{R}$. These are all of the form $\mathbb{K}(n)$ or $\mathbb{K}(n) \oplus \mathbb{K}(n)$. Now, one can form an $\mathbb{K}$-linear representation of $\mathbb{K}(n)$ by having the matrices act on $\mathbb{K}^n$. Here $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Note that for the quaternions we must take care that if the matrix multiplication acts from the left on a column vector then the scalar multiplication by $\mathbb{H}$ acts on the right. Again $\mathbb{K}(n)$ is a simple algebra and the unique irrep up to isomorphism is $\mathbb{K}^n$.

Of course, we can consider $\mathbb{K}^n$ to be a real vector space of real dimension $n, 2n, 4n$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

Conversely, a real vector space V is said to have a complex structure if there is a linear transformation $I \in End_{\mathbb{R}}(V)$ which satisfies $I^2 = -1$. With a choice of I, V can be made into a complex vector space of dimension $\frac{1}{2}\dim_{\mathbb{R}}(V)$.

Similarly, a real vector space V is said to have a quaternionic structure if there are linear transformations I, J, K with

$$I^2 = J^2 = K^2 = IJK = -1 \tag{5.5}$$

A choice of I, J, K makes V have a multiplication by quaternions, and defines an isomorphism to $\mathbb{H}^d$ with $d = \frac{1}{4}\dim_{\mathbb{R}}(V)$. For example on $\mathbb{H}^n$ the linear operators I, J, K would be *right*-multiplication by $\mathbf{i}, \mathbf{j}, \mathbf{k}$, respectively.

We say that the real representation V of $C\ell(r_+, s_-)$ has a complex structure I if $[\rho(\phi), I] = 0$. Similarly, we say that it has a quaternionic structure if

$$[\rho(\phi), I] = [\rho(\phi), J] = [\rho(\phi), K] = 0 \tag{5.6}$$

We can now trivially read off the basic properties of Clifford algebra representations from the above tables. Recall that $\ell := [d/2]$.

• If $d_T = s - r \neq 3\text{mod}4$ then $C\ell(r_+, s_-)$ is a simple algebra and there is a unique irrep. It is $\mathbb{K}(N)$ acting on $\mathbb{K}^N$ where $N = 2^\ell$ or $N = 2^{\ell-1}$.

• If $d_T = s - r = 3\text{mod}4$ then $C\ell(r_+, s_-)$ is a sum of simple algebras and there are two inequivalent irreps $\rho_\pm$. Note that this is the case where the volume element satisfies $\omega^2 = 1$ *and* ω is central. So we can characterize the representations as $\rho_\pm(\omega) = \pm 1$. Since ω is central these rep's cannot be equivalent.

• For $d_T = 6, 7, 8\text{mod}8$, $\dim_{\mathbb{R}} V = 2^\ell$. For $d_T = 6, 7, 8\text{mod}8$ we can represent Γ^μ by $2^\ell \times 2^\ell$ *real* matrices. In physics this is called a *Majorana representation* of the Clifford algebra.

• Multiplying Γ matrices by a factor of i changes the signature. Therefore for $d_T = 0, 1, 2\text{mod}8$ we can represent Γ^μ by $2^\ell \times 2^\ell$ *pure imaginary* matrices. In physics this is called a *Pseudo-Majorana representation*.

• For $d_T = 1, 5\text{mod}8$ (i.e. $d_T = 1\text{mod}4$) $\dim_{\mathbb{C}} V = 2^\ell$ and hence $\dim_{\mathbb{R}} V = 2^{\ell+1}$, *and* V carries a complex structure commuting with the Clifford action. But if we use $2^\ell \times 2^\ell$ matrices they must be complex.

• For $d_T = 2, 3, 4\text{mod}8$, $\dim_{\mathbb{H}} V = 2^{\ell-1}$, so $\dim_{\mathbb{C}} V = 2^\ell$, and hence $\dim_{\mathbb{R}} V = 2^{\ell+1}$, *and* V carries a quaternionic structure commuting with the Clifford action. Thus we can write a reresentation by $2^{\ell-1} \times 2^{\ell-1}$ quaternionic matrices. If we choose to write the quaternions as 2×2 complex matrices A such that $A^* = \sigma_2 A \sigma_2$ then we can represent the Γ^μ by $2^\ell \times 2^\ell$ complex matrices such that

$$\Gamma^{\mu,*} = J\Gamma^\mu J^{-1} \tag{5.7}$$

with $J = i\sigma^2 \oplus \cdots \oplus i\sigma^2$. Note that $JJ^* = J^2 = -1$ and $J^{tr} = -J$.

****

confusions:

2. Traubenberg has a notion of Pseudo-Symplectic Majorana. Does this make sense?
****

Thus the pattern of representations is the following: $\nu=$ number of irreps of the algebra, $d=$ real dimension of the irrep,

s	$C\ell(s_-)$	$\nu(s)$	$d(s)$	Structure	K	$\dim_{\mathbb{C}}$(Irreps of $\mathbb{C}\ell(s)$)
1	$\mathbb{C}$	1	2	$\mathbb{C}$	$\mathbb{Z}$	1
2	$\mathbb{H}$	1	4	$\mathbb{H}$	$\mathbb{Z}$	2
3	$\mathbb{H}\oplus\mathbb{H}$	2	4	$\mathbb{H}$	$\mathbb{Z}\oplus\mathbb{Z}$	2
4	$\mathbb{H}(2)$	1	8	$\mathbb{H}$	$\mathbb{Z}$	4
5	$\mathbb{C}(4)$	1	8	$\mathbb{C}$	$\mathbb{Z}$	4
6	$\mathbb{R}(8)$	1	8	$\mathbb{R}$	$\mathbb{Z}$	8
7	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	2	8	$\mathbb{R}$	$\mathbb{Z}\oplus\mathbb{Z}$	8
8	$\mathbb{R}(16)$	1	16	$\mathbb{R}$	$\mathbb{Z}$	16
9	$\mathbb{C}(16)$	1	32	$\mathbb{C}$	$\mathbb{Z}$	16
10	$\mathbb{H}(16)$	1	64	$\mathbb{H}$	$\mathbb{Z}$	32
11	$\mathbb{H}(16)\oplus\mathbb{H}(16)$	2	64	$\mathbb{H}$	$\mathbb{Z}\oplus\mathbb{Z}$	32
12	$\mathbb{H}(32)$	1	128	$\mathbb{H}$	$\mathbb{Z}$	64

$d=s+1$	$C\ell(s_+,1_-)$	$\nu(s)$	$d(s)$	Structure	K
0+1	$\mathbb{C}$	1	2	$\mathbb{C}$	$\mathbb{Z}$
1 +1	$\mathbb{R}(2)$	1	2	$\mathbb{R}$	$\mathbb{Z}$
2 +1	$\mathbb{R}(2)\oplus\mathbb{R}(2)$	2	2	$\mathbb{R}$	$\mathbb{Z}\oplus\mathbb{Z}$
3 +1	$\mathbb{R}(4)$	1	4	$\mathbb{R}$	$\mathbb{Z}$
4 +1	$\mathbb{C}(4)$	1	8	$\mathbb{C}$	$\mathbb{Z}$
5 +1	$\mathbb{H}(4)$	1	16	$\mathbb{H}$	$\mathbb{Z}$
6 +1	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	2	16	$\mathbb{H}$	$\mathbb{Z}\oplus\mathbb{Z}$
7 +1	$\mathbb{H}(8)$	1	32	$\mathbb{H}$	$\mathbb{Z}$
8 +1	$\mathbb{C}(16)$	1	32	$\mathbb{C}$	$\mathbb{Z}$
9 +1	$\mathbb{R}(32)$	1	32	$\mathbb{R}$	$\mathbb{Z}$
10 +1	$\mathbb{R}(32)\oplus\mathbb{R}(32)$	2	32	$\mathbb{R}$	$\mathbb{Z}\oplus\mathbb{Z}$
11 +1	$\mathbb{R}(64)$	1	64	$\mathbb{R}$	$\mathbb{Z}$

Note that for $C\ell(s_-)$, $d(s+8k) = 16^k d(s)$.

5.1 Representations and Periodicity: Relating Γ-matrices in consecutive even and odd dimensions

The mod-two periodicity of the Clifford algebras over $\mathbb{C}$ is reflected in the representation theory as follows:

If γ^i is an irrep of $\mathbb{C}\ell(2n-1)$, and hence $\gamma^1 \cdots \gamma^{2n-1}$ is a scalar, then we get irreps of $\mathbb{C}\ell(2n)$ and $\mathbb{C}\ell(2n+1)$ by defining new representation matrices:

$$\begin{aligned} \Gamma^i &= \gamma^i \otimes \sigma^1 = \begin{pmatrix} 0 & \gamma^i \\ \gamma^i & 0 \end{pmatrix} \qquad i = 1, \dots, 2n-1 \\ \Gamma^{2n} &= 1_{2^{n-1}} \otimes \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \Gamma^{2n+1} &= 1_{2^{n-1}} \otimes \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{5.8}$$

eq:evenodd

Iterating this procedure gives an explicit matrix representation of $\mathbb{C}\ell(d)$ in terms of $2^{[d/2]} \times 2^{[d/2]}$ complex matrices. Note that if we start with $\mathbb{C}\ell(1)$ with $\gamma^1 = 1$ then the matrices we generate will be both Hermitian and unitary, satisfying $\{\Gamma^\mu, \Gamma^\nu\} = 2\delta^{\mu\nu}$.

Of course, this by no means a unique way of relating representations in dimensions $d, d+1, d+2$. In our discussion of the relation to oscillators below we will see another one.

Similarly, working over $\mathbb{R}$, if γ^μ is an irrep of $C\ell(r_+, s_-)$ then

$$\begin{aligned} &\gamma^\mu \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &1 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{5.9}$$

eq:oneoneclf

gives an irrep of $C\ell((r+1)_+, (s+1)_-)$.
****

See Van Proeyen, Tools for susy ; Or Trubenberger for explicit formulae of this type.
****

Remarks

• A representation of the form (5.8) with off-diagonal Γ matrices and diagonal volume element is known as a chiral basis of complex gamma matrices. When we study the spin representations the generators of spin are block diagonal and $\Gamma^1, \dots, \Gamma^{2n}$ exchange chirality.

• If $\gamma^1 \cdots \gamma^{2n-1} = \xi_{2n-1}$ then

$$\Gamma^1 \cdots \Gamma^{2n+1} = i\xi_{2n-1} \tag{5.10}$$

• There are two distinct irreps of $\mathbb{C}\ell(2n+1)$. We can get the other one by switching the sign of an odd number of Γ matrices above.

6. Comments on a connection to topology

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THIS SECTION IS OUT OF PLACE - You should go directly to the description of representations in terms of free fermions.

This section is actually more allied with the Z2-graded structure section and should go before or after.

****

Consider a representation of $\mathbb{C}\ell(d)$ by anti-Hermitian gamma matrices Γ^μ such that $\{\Gamma^\mu, \Gamma^\nu\} = -2\delta^{\mu\nu}$, where $\mu = 1, \dots, d$.

Suppose x_0, x_μ, $\mu = 1, \dots, d$ are coordinates on the unit sphere S^d embedded in $\mathbb{R}^{d+1}$.

Consider the function

$$T(x) := x_0 1 + x_\mu \Gamma^\mu \tag{6.1}$$

eq:tachfld

Note that

$$T(x)T(x)^\dagger = 1 \tag{6.2}$$

and therefore $T(x)$ is a unitary matrix for every $x \in S^{d-1}$. We can view $T(x)$ as describing a map $T : S^d \to U(2^{[d/2]})$. Now, sometimes this map is topologically trivial and sometimes it is topologically nontrivial. By nontrivial we mean that it represents a nontrivial element of the homotopy group $\pi_d(U(2^{[d/2]}))$.

For example: If $d = 1$ then one of the two irreducible representations is $\Gamma = i$. If $x_0^2 + x_1^2 = 1$ then

$$T(x) = x_0 + ix_1 \tag{6.3}$$

is a map $S^1 \to U(1)$ of winding number 1. If $d = 3$ then we may choose $\Gamma^i = \sqrt{-1}\sigma^i$ and

$$T(x) = x_0 + x_i\tau_i \tag{6.4}$$

is our representation of $SU(2)$. Thus the map $T : S^3 \to SU(2)$ is the identity map. It has winding number one and generates $\pi_3(SU(2)) = \mathbb{Z}$.

Here is one easy criterion for triviality: Suppose we can introduce another anti-Hermitian $2^{[d/2]} \times 2^{[d/2]}$ gamma matrix, Γ so that $(\Gamma)^2 = -1$ and $\{\Gamma, \Gamma^\mu\} = 0$. Now consider the unit sphere

$$S^{d+1} = \{(x_0, x_\mu, y) | x_0^2 + \sum_{\mu=1}^{d} x_\mu x_\mu + y^2 = 1\} \subset \mathbb{R}^{d+2} \tag{6.5}$$

Then we can define

$$\tilde{T}(x, y) = x_0 + x_\mu \Gamma^\mu + y\Gamma \tag{6.6}$$

Figure 2: The map on the equator extends to the northern hemisphere, and is therefore homotopically trivial. `fig:clffstpa`

When restricted to $S^{d+1} \subset \mathbb{R}^{d+2}$, $\tilde{T}$ is also unitary and maps $S^{d+1} \to U(2^{[d/2]})$. Moreover $\tilde{T}(x,0) = T(x)$ while $\tilde{T}(0,1) = \Gamma$. Thus $\tilde{T}(x,y)$ provides an explicit homotopy of $T(x)$ to the constant map.

Thus, we can see that *if the irreducible representation of* $\mathbb{C}\ell(d)$ *is the restriction of an irreducible representation of* $\mathbb{C}\ell(d+1)$ *to* $\mathbb{C}\ell(d)$ *then* $T(x)$ *is topologically trivial.*

Amazingly, it turns out that the converse is also true. If the irrep cannot be extended, then $T(x)$ is homotopically nontrivial. In fact, it represents a generator of $\pi_{d-1}(U(2^{[d/2]}))$.

Now, let us look at the Clifford representation theory. When $d = 2p+1$ is odd there are two irreps of complex dimension 2^p. Restricting to the action of $\mathbb{C}\ell_{2p}$ on either irrep gives the 2^p-dimensional representation. On the other hand, if $d = 2p$ is even, then $\mathbb{C}\ell_{2p-1}$ has irreps of complex dimension 2^{p-1}. Thus, the action of $\mathbb{C}\ell_{2p-1}$ on the complex 2^p-dimensional irrep of $\mathbb{C}\ell_{2p}$ does not give an irrep. Or, put differently, the 2^{p-1}-dimensional irrep of $\mathbb{C}\ell_{2p-1}$ does *not* extend to an irrep of $\mathbb{C}\ell_{2p}$.

These facts are compatible with the statement in topology that

$$\pi_{2p-1}(U(N)) = \mathbb{Z} \qquad N \geq p \tag{6.7}$$ `eq:piodd`

$$\pi_{2p}(U(N)) = 0 \qquad N > p \tag{6.8}$$ `eq:pieven`

Note that these equations say that for N sufficiently large, the homotopy groups do not depend on N. These are called the *stable homotopy groups* of the unitary groups and can be denoted $\pi_k(\mathbf{U})$. The mod two periodicity of $\pi_k(\mathbf{U})$ as a function of k is known as Bott periodicity.

Similarly, there are stable homotopy groups $\pi_k(\mathbf{O})$ and $\pi_k(\mathbf{Sp})$ which are mod-8 periodic in k.

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Explain about vector fields on spheres

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References:

1. Atiyah, Bott, and Shapiro,

2. Lawson and Michelson, *Spin Geometry*

7. Free fermion Fock space

7.1 Left regular representation of the Clifford algebra

The Clifford algebra acts on itself, say, from the left. On the other hand, it is a vector space. Thus, as with any algebra, it provides a representation of itself, called the left-regular representation.

Note that this representation is 2^d dimensional, and hence rather larger than the $\sim 2^{[d/2]}$ dimensional irreducible representations. Hence it is highly reducible. In order to find irreps we should "take a squareroot" of this representation.

We will now describe some ways in which one can take such a "squareroot."

7.2 The Exterior Algebra as a Clifford Module

We have noted that

$$C\ell(r_+, s_-) \cong \Lambda^* \mathbb{R}^d \tag{7.1}$$ eq:vspi

as a vector spaces. Also, while the exterior algebra $\Lambda^* \mathbb{R}^d$ is an algebra we stressed that (7.1) is *not* an algebra isomorphism.

Nevertheless, since (7.1) is a vector space isomorphism this means that $\Lambda^*(\mathbb{R}^d)$ must be a Clifford *module*, that is, a representation space of the Clifford algebra. We now describe this representation.

If $v \in \mathbb{R}^{r,s}$ then we can define the contraction operator by

$$\iota(v)(v_{i_1} \wedge \cdots \wedge v_{i_k}) := \sum_{s=1}^{k} (-1)^{s-1} Q(v, v_{i_s}) v_{i_1} \wedge \cdots \wedge \widehat{v_{i_s}} \wedge \cdots \wedge v_{i_k} \tag{7.2}$$ eq:contract

where the hat superscript means we omit that factor. Similarly, we can define the wedge operator by

$$w(v)(v_{i_1} \wedge \cdots \wedge v_{i_k}) := v \wedge v_{i_1} \wedge \cdots \wedge v_{i_k} \tag{7.3}$$ eq:wedgeoper

These operators are easily shown to satisfy the algebra:

$$\begin{aligned} \{\iota(v_1), \iota(v_2)\} &= 0 \\ \{w(v_1), w(v_2)\} &= 0 \\ \{\iota(v_1), w(v_2)\} &= Q(v_1, v_2) \end{aligned} \tag{7.4}$$ eq:rccrs

Using these relations we see that we can represent Clifford multiplication by v by the operator:

$$\rho(v) = \iota(v) + w(v) \tag{7.5}$$

Since the $v \in V$ generate the Clifford algebra we can then extend this to a representation of the entire Clifford algebra by taking $\rho(\phi_1 \cdot \phi_2) = \rho(\phi_1)\rho(\phi_2)$.

7.3 Representations from maximal isotropic subspaces

We still need to take a "squareroot" of our representation. One example where this can be done is the following.

Let W be a real vector space and consider $V = W \oplus W^*$. Note that V admits a natural nondegenerate quadratic form of signature $(+1^n, -1^n)$ where we take W, W^* to be isotropic and use the pairing $W \times W^* \to \mathbb{R}$. That is, if we choose a basis w_i for W and a dual basis $\hat{w}^i$ for W^* then with respect to this basis

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7.6}$$

The resulting Clifford algebra is $C\ell(n_+, n_-) \cong \mathbb{R}(2^n)$ where $n = \mathrm{dim} W$.

One way to construct the irrep of dimension 2^n is by taking the representation space to be $\Lambda^* W^*$.

Now $\rho(\hat{w})$ for $\hat{w} \in W^*$ is defined by wedge product, $\hat{w}\wedge$ and $\rho(w)$ for $w \in W$ is defined by $\rho(w) = \iota(w)$ where $\iota(w)$ is the contraction operator:

$$\iota(w)(\hat{w}^{i_1} \wedge \cdots \wedge \hat{w}^{i_n}) = \sum_{j=1}^{n} (-1)^{j-1} \langle w, \hat{w}^{i_j} \rangle \hat{w}^{i_1} \wedge \cdots \wedge \hat{w}^{i_{j-1}} \wedge \hat{w}^{i_{j+1}} \wedge \cdots \wedge \hat{w}^{i_n} \tag{7.7}$$ eq:contracta

A simple computation shows that

$$\begin{aligned} \{\rho(w), \rho(w')\} &= 0 \\ \{\rho(\hat{w}), \rho(\hat{w}')\} &= 0 \\ \{\rho(w), \rho(\hat{w}')\} &= \langle w, \hat{w}' \rangle \end{aligned} \tag{7.8}$$ eq:sensf

and thus the Clifford relations are satisfied.

Example: For example, if M is a manifold we can consider $TM \oplus T^*M$ which has a natural quadratic form of signature (n, n) since TM and T^*M are dual spaces. Note that $W = TM$ a maximal isotropic subspace, and a natural choice of complementary isotropic subspace is $U = T^*M$. Then the Clifford algebra acts on the DeRham complex $\Lambda^* T^* M$. $\rho(v^i) = dx^i \wedge$ acts by wedge product, and $\rho(w_i) = \iota(\frac{\partial}{\partial x^i})$ acts by contraction. These represent the Clifford algebra:

$$\begin{aligned} \{v^i, v^j\} &= 0 \\ \{v^i, w_j\} &= \delta^i{}_j \\ \{w_i, w_j\} &= 0 \end{aligned} \tag{7.9}$$ eq:tensors

The above construction can be generalized as follows:

Suppose V is $2n$-dimensional with a nondegenerate metric of signature (n, n). Thus $C\ell(n_+, n_-) \cong \mathbb{R}(2^n)$ and we wish to construct the 2^n-dimensional irrep. Suppose we have a decomposition of V into two maximal isotropic subspaces $V = W \oplus U$ where W, U are maximal isotropic. That is, with respect to this decomposition we have

$$Q = \begin{pmatrix} 0 & q \\ q^\dagger & 0 \end{pmatrix} \tag{7.10}$$

where $q : U \to W$ is an isomorphism.

Note that there is a family of such decompositions, parametrized by some kind of Grassmannian.

Then, we claim, the exterior algebra $\Lambda^*(V/W)$ is naturally a 2^n dimensional representation of the Clifford algebra on V.

$u \in U$ acts on $\Lambda^*(V/W)$ by wedge product: Note that V/W acts via wedge product. Since U is a subspace of V it descends to a subspace of V/W and hence it acts by wedge product. On the other hand, $w \in W$ acts by contraction

$$\iota(w)([v_{i_1}] \wedge \cdots \wedge [v_{i_n}]) = \sum_{j=1}^{n} (-1)^{j-1} Q(w, v_{i_j}) [v_{i_1}] \wedge \cdots \wedge [v_{i_{j-1}}] \wedge [v_{i_{j+1}}] \wedge \cdots \wedge [v_{i_n}] \tag{7.11}$$ eq:contractaa

Note that the expression $Q(w, v_{i_j})$ is unambiguous because W is isotropic.

There is an alternative description of the same representation since one can show that $V/W \cong W^*$. To see this note that given v, $\ell_v : w \mapsto (v, w)$ is an element of W^* and $\ell_v = \ell_{v+w}$ for $w \in W$ (since W is isotropic). Thus we could also have represented the Clifford algebra on $\Lambda^* W^*$. Elements of W act by contraction and elements of U act by wedge product (where one needs to use the isomorphism $V/W \cong W^*$.)

Remarks

• The above construction appears to depend on a *choice* of decomposition of a vector space into a sum of isotropic subspaces. In finite dimensions all these representations are equivalent. But this is not so in infinite dimensions.

•

The choice of W is a choice of "Dirac sea." Different choices are related by "Bogoliubov transformation."

Exercise

Describe the analogous representation on $\Lambda^* TM$.

7.4 Splitting using a complex structure

Another way we can split the space V into half-dimensional spaces is by using a complex structure. Our goal in this section is to construct a representation of $\mathbb{C}\ell(d)$ for $d = 2n$ using the ideas of the previous two sections.

For definiteness, let us take $V = \mathbb{R}^{2n}$ with Euclidean signature:

$$Q(e^i, e^j) = +\delta^{ij} \tag{7.12}$$

and choose a complex structure compatible with the metric:

Given a choice of complex structure compatible with the metric there exists a basis e^i, $i = 1, \dots, 2n$ of V such that

$$\begin{aligned} Ie^{2j-1} &= -e^{2j} \\ Ie^{2j} &= e^{2j-1} \qquad\qquad j = 1, \dots, n \end{aligned} \tag{7.13}$$

That is, in this basis:

$$I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{7.14}$$

(with n summands).

By extending scalars to $\mathbb{C}$ we can diagonalize I. Put differently, $V \otimes_{\mathbb{R}} \mathbb{C} \cong W \oplus \bar{W}$:

$$\begin{aligned} Iv^j &= iv^j \\ Iw_j &= -iw_j \qquad\qquad j = 1, \dots, n \end{aligned} \tag{7.15}$$

where $v^j = \frac{1}{2}(e^{2j-1} + ie^{2j})$, $w_j = \frac{1}{2}(e^{2j-1} - ie^{2j})$.

Moreover, if we extend the metric $\mathbb{C}$-linearly then it becomes:

$$\begin{aligned} Q(v^j, v^k) &= 0 \\ Q(w_j, w_k) &= 0 \\ Q(v^j, w_k) &= \delta^j{}_k \qquad\qquad j, k = 1, \dots, n \end{aligned} \tag{7.16}$$

eq:nnmetr

so that on the complex vector space we have a metric of signature (n, n).

Thus, the Clifford algebra elements corresponding to v^j, w_k satisfy the canonical commutation relations of fermionic oscillators. That is, if we define:

$$\begin{aligned} \bar{a}_j = v^j &= \frac{1}{2}(e^{2j-1} + ie^{2j}) \\ a_j = w_j &= \frac{1}{2}(e^{2j-1} - ie^{2j}) \end{aligned} \tag{7.17}$$

Then the above relations are the standard fermion CCR's:

$$\begin{aligned} \{a_j, a_k\} &= 0 \\ \{\bar{a}_j, \bar{a}_k\} &= 0 \\ \{a_j, \bar{a}_k\} &= \delta_{jk} \end{aligned} \tag{7.18}$$

We will now use the above insight about the importance of isotropic subspaces to construct representations of the Clifford algebras over $\mathbb{C}$. For example we could take the representation space to be $\Lambda^* \bar{W}$ with $\bar{a}_j$ acting by wedge product and a_j acting by contraction.

Our construction can be related to the standard construction by identifying the "vacuum state" $|0\rangle$ with $1 \in \Lambda^* \bar{W}$. Then

$$a_j|0\rangle = 0 \tag{7.19}$$ eq:vacchoice

The vacuum state is also known as a *Dirac sea* or a *Clifford vacuum.* We can build a representation of the Clifford algebra by acting on the Clifford vacuum with creation operators $\bar{a}_j$. In this way we obtain a natural basis for the representation of the Clifford algebra by creating fermionic states:

$$\bar{a}_{j_1} \cdots \bar{a}_{j_s}|0\rangle \tag{7.20}$$

Thus, the representation is isomorphic to $\Lambda^* \bar{W} \cong \mathbb{C}^{2^n}$, and hence is an irrep of $\mathbb{C}\ell_{2n}$.

We can put a Hermitian structure on the representation space and then $\bar{a}_j = (a_j)^\dagger$.

Remarks

Repeat for other signatures –

7.5 Explicit matrices and intertwiners in the Fock representations

Let us use the oscillator approach to form an explicit basis for the irreducible representations of $\mathbb{C}\ell(d)$.

Then

$$\begin{aligned} a_j^\dagger &= \frac{1}{2}(e_{2j-1} + ie_{2j}) \\ a_j &= \frac{1}{2}(e_{2j-1} - ie_{2j}) \qquad j = 1, \ldots, n \end{aligned} \tag{7.21}$$

so $\{a_j, a_k^\dagger\} = \delta_{jk}$.

Consider first the case $n = 1$. we use the representation

$$|0\rangle := |-\frac{1}{2}\rangle \qquad a^\dagger|0\rangle := |+\frac{1}{2}\rangle \tag{7.22}$$

This labelling will be useful later for representations of the spin group.

Now taking

$$x_1|+\rangle + x_2|-\rangle \to \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{7.23}$$

we have the representation

$$\rho(e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \rho(e_2) = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \tag{7.24}$$

Now, with n oscillator pairs we choose a basis for a 2^n dimensional vector space

$$(a_n^\dagger)^{s_n+\frac{1}{2}}(a_{n-1}^\dagger)^{s_{n-1}+\frac{1}{2}} \cdots (a_1^\dagger)^{s_1+\frac{1}{2}}|0\rangle \tag{7.25}$$

where $s_i = \pm\frac{1}{2}$. We identify these states with the basis for the tensor product of representations

$$|s_n, s_{n-1}, \dots, s_1\rangle = |s_n\rangle \otimes |s_{n-1}\rangle \otimes \cdots \otimes |s_1\rangle \tag{7.26}$$

(The vector $(s_n, \dots, s_1)$ is what is called a *spinor weight*. In the theory of representations of semi-simple Lie algebras the space is graded by the action of the Cartan subalgebra and the grading is called the weight. The vectors $(s_n, \dots, s_1)$ are the weights of the spinor representations of $so(2n;\mathbb{C})$. See Chapter *** below.)

Let $\Gamma^j_{(n-1)}$ be the $2^{n-1} \times 2^{n-1}$ representation matrices of e_j for a collection of $(n-1)$ oscillators. Then when we add the n^{th} oscillator pair we get

$$\begin{aligned}
\rho_n(e_j) = \Gamma^j_{(n)} &= \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \otimes \Gamma^j_{(n-1)} \qquad\qquad j = 1, \dots, 2n-2 \\
\rho_n(e_{2n-1}) = \Gamma^{2n-1}_{(n)} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes 1_{2^{n-1}} \\
\rho_n(e_{2n}) = \Gamma^{2n}_{(n)} &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes 1_{2^{n-1}}
\end{aligned} \tag{7.27}$$

We take the complex volume form to be

$$\Gamma_\omega = (-i)^n \Gamma^1 \cdots \Gamma^{2n} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.28}$$

eq:cplxvol

where there are n factors.

For $d = 2n+1$ we still take n pairs of oscillators and set $\Gamma^{2n+1} = \Gamma_\omega$.

8. Bogoliubov Transformations and the Choice of Clifford vacuum

It is important to note that our decomposition into creation and annihilation operators depends on a *choice* of complex structure.

On $\mathbb{R}^{2n}$ we can produce other complex structures from $I' = RIR^{-1}$ with $R \in GL(2n, \mathbb{R})$. The complex structure will be compatible with the Euclidean metric if $R \in O(2n)$.

If we use I' rather than I then the new oscillators $b_j, \bar{b}_j$ will be related to the old ones by a *Bogoliubov transformation*:

$$\begin{aligned}
\bar{b}_i &= A_{ij}\bar{a}_j + B_{ij}a_j \\
b_i &= C_{ij}\bar{a}_j + D_{ij}a_j
\end{aligned} \tag{8.1}$$

eq:bogoliub

For a general complex linear combination (8.1) the CCR's are preserved iff the matrix

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{8.2}$$

satisfies

$$g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g^{tr} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{8.3}$$

That is, iff

$$\begin{aligned} AD^{tr} + BC^{tr} &= 1 \\ BA^{tr} + AB^{tr} &= 0 \\ CD^{tr} + DC^{tr} &= 0 \end{aligned} \tag{8.4}$$

That is, $g \in O(n,n;\mathbb{C})$.

****

SO: WHAT IS THE POINT? HOW IS g RELATED TO R ?

****

The new Clifford vacuum is of course the same if g amounts to a complex transformation of the a_j to the b_j and the $\bar{a}_j$ to the $\bar{b}_j$. Thus, $B = C = 0$ and $AD^{tr} = 1$. Thus, the space of Dirac vacua is $O(n,n;\mathbb{C})/GL(n,\mathbb{C})$, a homogeneous space of complex dimension $n(n-1)$.

Remark: Note $O(n,n;\mathbb{C}) \cong O(2n;\mathbb{C})$. Indeed it is useful to make this isomorphism explicit. Let

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \tag{8.5}$$

then

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = TT^{tr} \tag{8.6}$$

so that $\hat{g} = T^{-1} g T$ is in $O(2n;\mathbb{C})$, i.e. $\hat{g}\hat{g}^{tr} = 1$.

If we impose a conjugation operation $a_j \to \bar{a}_j$ which is preserved by the Bogoliubov transformation then $A = D^*$ and $B = C^*$ and we get AB^{tr} is antisymmetric and $AA^\dagger + BB^\dagger = 1$. Note that in this case g is a unitary transformation $g \in U(2n)$. Now, the matrix T defined above is also unitary so in this case $\hat{g} \in U(2n) \cap O(2n;\mathbb{C}) \cong O(2n;\mathbb{R})$. Moreover, the group preserving the Clifford vacuum is the subgroup with $B = 0$ is indeed isomorphic to $U(n)$. Thus, the space of Clifford vacua obtained by unitary transformations is $O(2n;\mathbb{R})/U(n)$, as space of real dimension $n(n-1)$. This is the space of complex structures on $\mathbb{R}^{2n}$ compatible with the Euclidean metric.

We will discuss this further after we have introduced the Spin group.

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To add:

1. More Discussion of polarization. see Pressley and Segal, *Loop Groups*, Oxford, p. 233. Segal's Stanford notes.

2. J. Strathdee, Int. J. Mod. Phys. **A2** 273 (1987)

9. Comments on Infinite-Dimensional Clifford Algebras

In quantum field theories of fermions we encounter infinite-dimensional Clifford algebras. The construction of the irreducible representations are a little different from the finite-dimensional case explained above.

Here we follow some notes of G. Segal. ("Stanford Lectures")

The typical situation is that we have a vector space of one-particle wavefunctions of fermions on a spatial slice. Call this E. It could be, for example, the L^2-spinors on a spatial slice. Canonical quantization calls for a representation of the infinite-dimensional Clifford algebra $C\ell(E \oplus E^*)$. It turns out to be physically wrong to consider $\Lambda^* E$ or $\Lambda^* E^*$. Rather, the physically correct construction is more subtle.

Typically, there is a self-adjoint operator D on E with spectrum of eigenvalues λ_k which goes to $\pm\infty$ for $k \to \pm\infty$. (In the physical applications, D would be the spatial Dirac operator.) Now one wants to consider a "Dirac sea." This is a formal element of $\Lambda^* E$ given by taking the wedge-product with all the negative energy levels. Let e_k be an ON basis of eigenvectors of D. Let us assume that $\lambda = 0$ is not in the spectrum, and let us label the eigenvectors so that e_k has $\lambda_k > 0$ for $k > 0$ and $\lambda_k < 0$ for $k < 0$.

Now we try to define the "Fock vacuum."

$$\Omega = e_0 \wedge e_{-1} \wedge e_{-2} \wedge \cdots \tag{9.1}$$

This is a "semi-infinite form" The Fock space will then be spanned by elements $e_{\vec{k}}$ where $\vec{k} = \{k_0, k_{-1}, k_{-2}, \cdots\}$ is a strictly decreasing series of integers $k_0 > k_{-1} > k_{-2} > \cdots$ which only differs from $\{0, -1, -2, \cdots\}$ by a finite number of elements. The resulting space is even a Hilbert space and is acted on by $C\ell(E \oplus E^*)$ in the way we have described above.

However, there is a problem problem with this definition. The problem is that the resulting vector space is only well-defined as a *projective space*. The reason is that we could choose a different eigenbasis $\tilde{e}_k = u_k e_k$, with $|u_k| = 1$. In general $\prod u_k$ is not well defined.

Now, this becomes a real problem when we consider *families* of operators D. For example, we could consider families of metrics on our space(time), or we could consider the Dirac operator coupled to a gauge field and consider the family parametrized by the gauge field. We will find in such situations that there is no unambiguous way to choose a well-defined Fock vacuum throughout the entire family.

One mathematical approach to making the construction well-defined is the following. We first introduce the notion of a polarization on a vector space. This is a *family* of decompositions $E = E^+ \oplus E^-$ where $E^\pm$ are – very roughly speaking – half of E and different decompositions in the family are "close" to one another. We certainly want to consider two decompositions $E = E^+ \oplus E^-$ and $E = \tilde{E}^+ \oplus \tilde{E}^-$ to be in the same polarization if they differ by finite-dimensional spaces.

Definition: A *coarse polarization*, denoted $\mathcal{J}$, of E is a class of operators $P : E \to E$ such that

1. $P^2 = 1$ modulo compact operators.

2. For any two elements $P, P' \in \mathcal{J}$, $P' - P$ is compact.
3. $\mathcal{J}$ does not contain ± 1.

****

COMPACT OR HILBERT-SCHMIDT ?

****

Given a polarization, one defines the restricted Grassmannian $Gr(E)$ to be the set of spaces E^- which arise in the decompositions allowed by the polarization.

In our example of spinors on spacetime, an operator such as D above defines a polarization by considering the projection onto positive and negative eigenvalues of D to be in $\mathcal{J}$. If Y is the boundary of some spacetime $Y = \partial X$ then the boundary values of solutions of the Dirac equation on X will define an element of the corresponding restricted Grassmannian. As we vary, say, the metric on X we will obtain a family of vector spaces in $Gr(E)$.

Now, the polarization defined by D let $E = E^+ \oplus E^-$ be an allowed decomposition. Then we can make the above Dirac sea precise by considering the Fock space

$$\mathcal{F}_{E^-}(E) := \Lambda^*((E^-)^*) \otimes \Lambda^*(E/E^-) \tag{9.2}$$

where now $C\ell(E \oplus E^*)$ acts as follows: $e \in E$ can be decomposed as $e = e_- \oplus e_+$ and it acts by

$$\rho(e) = 1 \otimes (e_+ \wedge) + \iota(e_-) \otimes 1 \tag{9.3}$$

while $\hat{e} \in E^*$ has a decomposition $\hat{e}_- \oplus \hat{e}_+$ and it acts as

$$\rho(\hat{e}) = (\hat{e}_- \wedge) \otimes 1 + 1 \otimes \iota(\hat{e}_+) \tag{9.4}$$

Now the vector space $\mathcal{F}_{E^-}(E)$ has a canonically defined vacuum, namely, 1, just as in finite dimensions. However, the price we pay is that for different elements E_1^- and E_2^- in the Grassmannian the isomorphism

$$\mathcal{F}_{E_1^-}(E) \to \mathcal{F}_{E_2^-}(E) \tag{9.5}$$

is only defined up to a scalar. The line bundle of Fock vacua will be nontrivial.

****

Explain the last two statements.

****

In finite dimensions there is a unique irrep of $C\ell(E \oplus E^*)$. Indeed, we constructed an isomorphism between different representations constructed using different complex structures, or using different maximal isotropic subspaces using a Bogoliubov transformations. But in infinite dimensions different polarizations can lead to inequivalent representations of $C\ell(E \oplus E^*)$.

10. Properties of Γ-matrices under conjugation and transpose: Intertwiners

In physical applications of Clifford algebras it is often important to have a good understanding of the Hermiticity, unitarity, (anti-)symmetry and complex conjugation properties of the gamma matrices.

The reason we care about this apparently dull question is that these properties are very important for:

1. Imposing reality conditions on fermions - needed to get proper numbers of degrees of freedom in various theories, especially supersymmetric theories.

2. Forming action principles for theories of fermions.

3. Constructing supersymmetry algebras.

4. Group theory manipulations such as decomposition of tensor products of spinor representations. This is crucial for understanding what super-Poincare and superconformal algebras one can construct.

10.1 Definitions of the intertwiners

A useful ref. for this section is Kugo-Townsend pp. 360-375

Given a representation provided by gamma matrices Γ^μ we always have another representations:

$$\Gamma^\mu \to -\Gamma^\mu \tag{10.1}$$

Moreover, we can take the transpose

$$\Gamma^\mu \to (\Gamma^\mu)^{tr} \tag{10.2}$$

For representations by matrices over the complex numbers or over the quaternions we can take the conjugation:

$$\Gamma^\mu \to (\Gamma^\mu)^* \tag{10.3}$$

Finally, we can combine these operations. For example

$$\pm(\Gamma^\mu)^\dagger \tag{10.4}$$

is another representation.

These representations might, or might not be equivalent, depending on dimension and signature.

When the the representations are equivalent (and this is guaranteed when there is a unique irrep, and hence when $d = r + s$ is even) Schur's lemma guarantees that we can define intertwiners:

$$\begin{aligned}
-\Gamma^\mu &= \Gamma\Gamma^\mu\Gamma^{-1} \\
\xi(\Gamma^\mu)^\dagger &= A_\xi\Gamma^\mu A_\xi^{-1} \\
\xi(\Gamma^\mu)^* &= B_\xi\Gamma^\mu B_\xi^{-1} \\
\xi(\Gamma^\mu)^{tr} &= C_\xi\Gamma^\mu C_\xi^{-1}
\end{aligned} \tag{10.5}$$

where $\xi = \pm 1$. Here $\Gamma = \rho(\omega) = \Gamma^1 \cdots \Gamma^d$ is the volume element. These equations define A, B, C only up to a nonzero scalar multiple. If we know two of the intertwiners we can easily obtain a third, e.g. we could take $A_+ = C_\xi^* B_\xi$ etc.

The Γ^μ can be chosen to be unitary. If this is done then...
****

Be more systematic about the relations between the interwiners.
****

10.2 The charge conjugation matrix for Lorentzian signature

As a simple example of why we wish to know about symmetry properties of the transpose let us consider the special case of $C\ell(s_+, 1_-)$ appropriate to Lorentzian signature spacetime.

As we will see we can (????? NOT FOR ALL ODD DIMENSIONS! ???) define the charge conjugation matrix C with

$$-(\Gamma^\mu)^{tr} = C\Gamma^\mu C^{-1} \tag{10.6}$$

eq:chargeconj

It follows from explicit constructions above that we can, and will, choose a basis such that Γ^0 is anti-hermitian and Γ^i are Hermitian, where 0 denotes the negative signature direction. For *real* gamma matrices we have:

$$(\Gamma^0)^{tr} = -\Gamma^0 \qquad\qquad (\Gamma^i)^{tr} = +\Gamma^i \tag{10.7}$$

eq:cconja

where 0 denotes the direction with $\eta^{00} = -1$ and $i = 1, \dots, s$. Now, using the properties of the Clifford algebra we can take $C = \Gamma^0$ Note that $C^{-1} = -C$, $C^{tr} = -C$.

Moreover, we have

$$(C\Gamma^\mu)^{tr} = +(C\Gamma^\mu) \tag{10.8}$$

eq:cconjb

That is, the symmetric product of the real representation of the Clifford algebra contains the vector. This is important: It means that when we work with real fields we can make a real action:

$$\int \mathrm{vol}\Psi^{tr} C\Gamma^\mu \partial_\mu \Psi + \cdots \tag{10.9}$$

Note that it is important that Ψ is anti-commuting. If $C\Gamma^\mu$ were anti-symmetric with anti-commuting Ψ, or if $C\Gamma^\mu$ were symmetric, with commuting Ψ then the action density would be a total derivative.

The symmetry of $C\Gamma^\mu$ also allows the definition of supersymmetry algebras:

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^\mu)_{\alpha\beta} P_\mu + \cdots \tag{10.10}$$

Exercise
Show that in this situation

$$C\Gamma^{\mu_1 \cdots \mu_k} = \begin{cases} \text{Symmetric} & fork = 1,2 \text{mod } 4 \\ \text{Antisymmetric} & fork = 3,4 \text{mod } 4 \end{cases} \tag{10.11}$$

10.3 General Intertwiners for $d = r + s$ even

10.3.1 Unitarity properties

When representing $C\ell(r_+, s_-)$ by complex matrices we can always choose Γ^μ to be unitary. To prove this, note that we constructed such representations for $C\ell(1_\pm)$, $C\ell(2_\pm)$ and $C\ell(1_+, 1_-)$ and $C\ell(3\pm)$. Then it follows from the tensor product construction.

The Clifford relations imply $(\Gamma^\mu)^{-1} = \eta^{\mu\mu}\Gamma^\mu$, and hence if we choose our matrices to be unitary then

$$(\Gamma^\mu)^\dagger = \eta^{\mu\mu}\Gamma^\mu = \begin{cases} +\Gamma^\mu & \mu = 1, \dots, r \\ -\Gamma^\mu & \mu = r+1, \dots, r+s \end{cases} \tag{10.12}$$ eq:herm

with no sum on μ. Thus, as opposed to representations of $\mathbb{C}\ell(d)$, we are *not* free to choose the Hermiticity properties.

Exercise
Define

$$U_+ = \Gamma^1 \dots \Gamma^r \tag{10.13}$$

$$U_- = \Gamma^{r+1} \dots \Gamma^{r+s} \tag{10.14}$$

Show that

$$(\Gamma^\mu)^\dagger = \begin{cases} U_+\Gamma^\mu U_+^{-1} & r = 1\text{mod}2 \\ U_-\Gamma^\mu U_-^{-1} & s = 0\text{mod}2 \end{cases} \tag{10.15}$$

$$-(\Gamma^\mu)^\dagger = \begin{cases} U_+\Gamma^\mu U_+^{-1} & r = 0\text{mod}2 \\ U_-\Gamma^\mu U_-^{-1} & s = 1\text{mod}2 \end{cases} \tag{10.16}$$

The $U_\pm$ are used to construct intertwiners for complex conjugation and transpose below.

10.3.2 General properties of the unitary intertwiners

We will write some explicit intertwiners using our oscillator representation betlow. In this section we derive some general properties of the intertwiners.

First, by taking the Hermitian conjugate of the defining relations and applying Schur's lemma we see that $A^\dagger A, B^\dagger B, C^\dagger C$ must be proportional to the unit matrix. Moreover that scalar must be positive and therefore, WLOG we can always take A, B, C to be unitary.

Second, by iterating the equations for B and C we can see that

Now, again by Schur,

$$C_\xi^{-1} C_\xi^{tr} = \epsilon 1 \tag{10.17}$$ eq:ceexirel

and

$$B_\xi^* B_\xi = \epsilon' 1 \tag{10.18}$$ eq:beexirel

for some scalars ϵ and ϵ'. Note, moreover that consistency of (10.17) means $\epsilon = \pm 1$ and, if we also use unitarity of B_ξ then $\epsilon' = \pm 1$. Moreover, if we make a definite choice of A_+ and use this to relate C_ξ and B_ξ then ϵ and ϵ' are not independent.

It turns out that ϵ cannot be chosen arbitrarily, but is determined by the dimension $d$mod8 and ξ.

Using the definition of C_ξ we get:

$$(C_\xi \Gamma^{\mu_1 \cdots \mu_k})^{tr} = \xi^k (-1)^{\frac{1}{2}k(k-1)} C_\xi \Gamma^{\mu_1 \cdots \mu_k} C_\xi^{-1} C_\xi^{tr} \tag{10.19}$$ eq:symmetry

It follows that all the matrices

$$C_\xi \Gamma^{\mu_1 \cdots \mu_k} \tag{10.20}$$

are either symmetric or antisymmetric:

$$(C_\xi \Gamma^{\mu_1 \cdots \mu_k})^{tr} = \epsilon \xi^k (-1)^{\frac{1}{2}k(k-1)} C_\xi \Gamma^{\mu_1 \cdots \mu_k} \tag{10.21}$$ eq:sympar

Now, we are working with Clifford matrices over $\mathbb{C}$ and so the Clifford algebra is just $\mathbb{C}(2^{d/2})$. That algebra contains:

- $2^{d-1} + 2^{\frac{d}{2}-1}$ *symmetric matrices*
- $2^{d-1} - 2^{\frac{d}{2}-1}$ *antisymmetric matrices*

On the other hand, we can enumerate the number of symmetric or antisymmetric matrices combining (10.21) with the sums:

$$\begin{aligned}
\sum_{k=0(4)} \binom{d}{k} &= 2^{d-2} + 2^{\frac{1}{2}d-1}\cos(\frac{\pi d}{4}) \\
\sum_{k=1(4)} \binom{d}{k} &= 2^{d-2} + 2^{\frac{1}{2}d-1}\sin(\frac{\pi d}{4}) \\
\sum_{k=2(4)} \binom{d}{k} &= 2^{d-2} - 2^{\frac{1}{2}d-1}\cos(\frac{\pi d}{4}) \\
\sum_{k=3(4)} \binom{d}{k} &= 2^{d-2} - 2^{\frac{1}{2}d-1}\sin(\frac{\pi d}{4})
\end{aligned} \tag{10.22}$$

(This can be proved by applying the binomial expansion to $(1+\zeta)^d$ for the four distinct fourth roots of 1. It holds for d even or odd.)

The result is:

$\xi = +1, \epsilon = +1$: $d = 0, 2$mod8

$\xi = +1, \epsilon = -1$: $d = 4, 6$mod8

$\xi = -1, \epsilon = +1$: $d = 0, 6$mod8

$\xi = -1, \epsilon = -1$: $d = 2, 4$mod8

For examples, if we want $\xi = +1, \epsilon = +1$ then we can get the number of symmetric matrices by summing on $k = 0, 1\text{mod}4$, this will correctly give the number of symmetric matrices if

$$\cos(\frac{\pi d}{4}) + \sin(\frac{\pi d}{4}) = 1 \tag{10.23}$$

which is the case for $d = 0, 2\text{mod}8$ but not for $d = 4, 6\text{mod}8$.

****

GIVE CONDITIONS ON ϵ'

****

10.3.3 Intertwiners for $d = r + s$ odd

In this case there are two distinct irreducible representations of the same dimension. Now Γ^μ and $-\Gamma^\mu$ are *not* equivalent.

Similarly, the other intertwiners only exist in certain dimensions modulo 8:

1. Since $\Gamma^1 \cdots \Gamma^d$ is represented as a scalar $\xi (\Gamma^\mu)^{tr}$ can only be equivalent if $\xi^d (-1)^{\frac{1}{2}d(d-1)} = 1$ That is:

$\xi = +1$ and $d = 1\text{mod}4$

$\xi = -1$ and $d = 3\text{mod}4$

****

CONTINUE

****

10.4 Constructing Explicit Intertwiners from the Free Fermion Rep

Let us return to the Free fermion representation constructed in equations **** and **** above.

In our explicit basis Γ^i are real and symmetric for i odd, and imaginary ($= i\times$ real) and antisymmetric for i even. Our explicit intertwiners are

$$\begin{aligned} A\Gamma^i A^{-1} &= (\Gamma^i)^\dagger \\ B_\pm \Gamma^i B_\pm^{-1} &= \pm(\Gamma^i)^* \\ C_\pm \Gamma^i C_\pm^{-1} &= \pm(\Gamma^i)^{tr} \end{aligned} \tag{10.24}$$

We can take $A = 1$. Note that in this basis we can take $B_\pm = C_\pm$.

Let $U := \Gamma^2\Gamma^4 \cdots \Gamma^{2n}$. Then we have

$$C_+ = B_+ = \begin{cases} U & neven \\ \Gamma_\omega U & nodd \end{cases} \tag{10.25}$$

eq:plusinter

$$C_- = B_- = \begin{cases} \Gamma_\omega U & neven \\ U & nodd \end{cases} \tag{10.26}$$

eq:plusintera

To see this note that

$$\begin{aligned} U\Gamma^{2j}U^{-1} &= (-1)^{n-1}\Gamma^{2j} = (-1)^n(\Gamma^{2j})^* \\ U\Gamma^{2j-1}U^{-1} &= (-1)^n\Gamma^{2j-1} = (-1)^n(\Gamma^{2j-1})^* \end{aligned} \tag{10.27}$$

It is now straightforward to compute

$$UU^* = U^*U = \begin{cases} +1 & n = 0, 3 \text{ mod} 4 \\ -1 & n = 1, 2 \text{ mod} 4 \end{cases} \tag{10.28}$$

$$(\Gamma_\omega U)(\Gamma_\omega U)^* = (\Gamma_\omega U)^*(\Gamma_\omega U) = \begin{cases} +1 & n = 0, 1 \text{ mod} 4 \\ -1 & n = 2, 3 \text{ mod} 4 \end{cases} \tag{10.29}$$

Recall that for $d = 2n + 1$ we take $\Gamma^{2n+1} = \Gamma_\omega$. Then

$$U\Gamma^{2n+1}U^{-1} = (-1)^n\Gamma^{2n+1} = (-1)^n(\Gamma^{2n+1})^* \tag{10.30}$$

and also

$$(\Gamma_\omega U)\Gamma^{2n+1}(\Gamma_\omega U)^{-1} = (-1)^n\Gamma^{2n+1} = (-1)^n(\Gamma^{2n+1})^* \tag{10.31}$$

Now, if n is even we *must* use B_+, while if n is odd we *must* use B_-.

That is:

• For the Dirac representation the intertwiner B_+ exists for $d = 0, 1, 2\text{mod}4$, but *not* for $d = 3\text{mod}4$.

• For the Dirac representation the intertwiner B_- exists for $d = 0, 2, 3\text{mod}4$, but *not* for $d = 1\text{mod}4$.

• The sign of $B_\xi B_\xi^*$, computed above, depends on $d$mod8. In this way we compute:

$d$mod8	$B_+^* B_+$	$B_-^* B_-$
0	$+1$	$+1$
1	$+1$	*
2	$+1$	-1
3	*	-1
4	-1	-1
5	-1	*
6	-1	$+1$
7	*	$+1$

The * is in the entry where the matrix does not exist.

10.5 Majorana and Symplectic-Majorana Constraints

The above representations are complex. However, we know that the irreducible representations are sometimes real. How do we see that using intertwiners in the Dirac representation?

10.5.1 Reality, or Majorana Conditions

We attempt to define a real vector subspace invariant under the Clifford action defined above. The real subspace will be defined to be the vectors satisfying the constraint

$$\psi^* = B\psi \tag{10.32}$$ eq:majorana

Notice that this equation is only consistent if $B^* B = 1$.

Mathematically, we are trying to introduce a *real structure*. That is, a $\mathbb{C}$-*anti*-linear operator $\mathcal{C}$ such that $\mathcal{C}^2 = 1$. We would define $\mathcal{C} : \psi \mapsto B\psi^*$ and the real subspace is the $+1$ eigenspace of $\mathcal{C}$.

The subspace $\mathcal{C} = 1$ defined by (10.32) is a *real* vector space. Next, to get a real representation of the Clifford algebra we need to know that it is preserved by the Clifford action. Thus we need to check that if ψ satisfies (10.32) then

$$(\Gamma^i \psi)^* \stackrel{?}{=} B\Gamma^i \psi \tag{10.33}$$

Equivalently, we want the Clifford action to commute with $\mathcal{C}$. This will only be the case if we choose B_+.

A glance at the table for $C\ell(r_+)$ confirms that we only expect such representations for $d = 0, 1, 2 \mathrm{mod} 8$.

This can be confirmed using the explicit free fermion representation. Consistency of (10.32), or equivalently, $\mathcal{C}^2 = 1$ can be checked by computing the following:

$$B_+^* B_+ = \begin{cases} +1 & d = 0, 1, 2 \bmod 8 \\ -1 & d = 4, 5, 6 \bmod 8 \end{cases} \tag{10.34}$$ eq:majcond

$$B_-^* B_- = \begin{cases} +1 & d = 6, 7, 8 \bmod 8 \\ -1 & d = 2, 3, 4 \bmod 8 \end{cases} \tag{10.35}$$ eq:majcondi

As shown in table *** above. (THIS IS REDUNDANT!)

10.5.2 Quaternionic, or Symplectic-Majorana Conditions

In the case of $d_T = 3, 4, 5 \mathrm{mod} 8$ we have a representation by quaternionic matrices. We can always represent a quaternion as a 2×2 complex matrix A such that $A^* = \sigma^2 A \sigma^2$.

In this way we convert $2^N \times 2^N$ quaternionic matrices to $2^{N+1} \times 2^{N+1}$ complex matrices such that

$$\Gamma^{\mu,*} = J \Gamma^\mu J^{-1} \tag{10.36}$$

with $J = i\sigma^2 \oplus \cdots \oplus i\sigma^2$. Thus, the complex conjugate representation matrices are equivalent to conjugation with J. This is the case of $\xi = +1$ and $B_\xi^* B_\xi = -1$.

Now, vectors in the irreducible representation are N-component vectors of quaternions where $N = 2^{\ell-1}$. Therefore, when we write in terms of complex vectors we have

$$\begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} \rightarrow \begin{pmatrix} z_1 & w_1 \\ -\bar{w}_1 & \bar{z}_1 \\ \vdots & \vdots \\ z_N & w_N \\ -\bar{w}_N & \bar{z}_N \end{pmatrix} \tag{10.37}$$

Thus, we get a *pair* of complex spinors of dimension 2^{N+1} from the two columns. Moreover, denoting these two columns by ψ_1 and ψ_2 we see they are related by

$$\psi_1^* = J\psi_2 \tag{10.38}$$

We can introduce a spinor-index notation for the $SU(2)$ subgroup of the quaternions with a, b running over $1, 2$ and then we can write our condition as

$$(\psi_{i,a})^* = \epsilon^{ab} J_{ij} \psi_{jb} \tag{10.39}$$

eq:sympmajorar

where $\epsilon^{12} = +1$. Note that $J = J^*$, and $JJ^* = J^2 = -1$, as is required by consistency with (10.39).

Conversely, when there is an intertwiner $B_\xi B_\xi^* = -1$ we can introduce a "symplectic Majorana" spinor by taking a *pair* of spinors ψ_a, $a = 1, 2$, and imposing the reality condition:

$$(\psi^*)^a := (\psi_a)^* = \epsilon^{ab} B_\xi \psi_b \tag{10.40}$$

eq:sympmajii

Consistency of this equation requires $B_\xi B_\xi^* = -1$. Again, the subspace defined by (10.40) will only be preserved by Γ^μ, that is

$$(\Gamma^\mu \psi_a)^* = \epsilon^{ab} B_\xi \Gamma^\mu \psi_b, \tag{10.41}$$

if we choose $\xi = +1$. As we see from (10.39) above, this is the same as saying that the representation is quaternionic.

Remark: More generally, in the physics literature if Ω^{ab}, $a, b = 1, \dots, u$ is an anti-symmetric symplectic matrix, so $\Omega^2 = -1$ we can impose a condition:

$$(\psi^*)^a := (\psi_a)^* = \Omega^{ab} B \psi_b \tag{10.42}$$

and these are also referred to as symplectic Majorana spinors. This is useful in 6d supersymmetry for reasons connected with the tensor product of spin reps we come to next.

10.5.3 Chirality Conditions

When we come to representations of the *Spin group* we will see that we want to represent just the *even* Clifford algebra. Then we might wish to impose both a Majorana *and* a Weyl condition. This is only possible if $d = 2n$ is even. In this case we have

$$P_\pm = \frac{1}{2}(1 \pm \Gamma_\omega) \tag{10.43}$$

and we compute

$$BP_\pm B^{-1} = \begin{cases} P_\pm & d = 0 \text{ mod} 4 \\ P_\mp & d = 2 \text{ mod} 4 \end{cases} \tag{10.44}$$

eq:bonjc

Combining with (10.34)(10.35) we see that we can impose Majorana and Weyl for $d = 0 \text{mod} 8$, that is, the chiral representation is self-conjugate.

If $d = 2, 6 \text{mod} 8$ then complex conjugation switches the chirality.

As we have explained above, when working with field theories of fermions one also needs to know the transpose properties because it is sometimes important to know the (anti)symmetry of $C_{\pm}\Gamma^{\mu_1\cdots\mu_k}$. These are easily obtained from

$$U^{tr} = (-1)^{\frac{1}{2}n(n+1)}U \tag{10.45}$$

$$(\Gamma_{\omega}U)^{tr} = (-1)^{\frac{1}{2}n(n-1)}\Gamma_{\omega}U \tag{10.46}$$

Note that the signs depend on $n$mod4 and hence on $d$mod8.

Exercise
Repeat this exercise for Lorentzian signature.

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11. $\mathbb{Z}_2$ graded algebras and modules

11.1 $\mathbb{Z}_2$ grading on the algebra

As we mentioned at the beginning of this chapter, there is a natural $\mathbb{Z}_2$ grading defined on generators by $\lambda : e^{\mu} \to -e^{\mu}$, and then extending λ so that it is an algebra homomorphism. This makes the Clifford algebra a $\mathbb{Z}_2$-graded algebra (i.e. a superalgebra):

$$C\ell(Q) \cong C\ell^0(Q) \oplus C\ell^1(Q) \tag{11.1}$$

where $C\ell^0(Q)$ is the subalgebra generated by products of even numbers of e^{μ}'s

Let us stress that for d odd the decomposition

$$\mathbb{C}\ell(d) \cong \mathbb{C}(2^{[d/2]}) \oplus \mathbb{C}(2^{[d/2]}) \tag{11.2}$$ eq:dirsumalg

is *not* compatible with the $\mathbb{Z}_2$-grading, that is, with the decomposition

$$\mathbb{C}\ell(d) \cong (\mathbb{C}\ell(d))^0 \oplus (\mathbb{C}\ell(d))^1 \tag{11.3}$$ eq:dirsumgrad

The volume form ω_c is odd, and therefore $P_{\pm}$ do not commute with the grading.

Using the form (11.2) of the Clifford algebra we can consider to be the subalgebra of $\mathbb{C}(2N)$ where $N = 2^{[d/2]}$ of matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \tag{11.4}$$

where $a, b \in \mathbb{C}(N)$. In this block decomposition we have

$$\omega_c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11.5}$$

Since ω_c is odd, we see this block decomposition is *not* (11.3). In the block decomposition (11.2) the grading operator is given by *conjugation* with

$$\begin{pmatrix} 0 & 1_N \\ 1_N & 0 \end{pmatrix} \tag{11.6}$$

and hence the even subalgebra consists of matrices with $a = b$ and the odd subspace consists of matrices of the form $a = -b$.

Sometimes we prefer to use the block decomposition (11.3). In this decomposition the Clifford algebra consists of matrices of the form

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \tag{11.7}$$

eq:gradcd

with $A = \frac{1}{2}(a+b) \in \mathbb{C}(N)$ and $B = \frac{1}{2}(a-b) \in \mathbb{C}(N)$. Note that the set of matrices (11.7) is indeed a subalgebra of $\mathbb{C}(2N)$:

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \begin{pmatrix} A' & B' \\ B' & A' \end{pmatrix} = \begin{pmatrix} AA' + BB' & AB' + BA' \\ BA' + AB' & AA' + BB' \end{pmatrix} \tag{11.8}$$

The even subalgebra corresponds to the matrices of the type

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \tag{11.9}$$

and is indeed isomorphic to $\mathbb{C}(N)$.

In the block decomposition (11.3) the volume form ω_c is odd and we can take it to be

$$\omega_c = \begin{pmatrix} 0 & 1_N \\ 1_N & 0 \end{pmatrix} \tag{11.10}$$

where $N = 2^{[d/2]}$, While the grading operator is *conjugation* with $\Pi = (-1)^{\deg}$ where

$$\Pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11.11}$$

We now explore some consequences of this $\mathbb{Z}_2$-grading.

11.2 The even subalgebra $C\ell^0(r_+, r_-)$

The first important remark to make is that the even subalgebra is isomorphic to another Clifford algebra. To see this, choose any basis element e^{μ_0} and consider the algebra generated by

$$\tilde{e}^\nu = e^{\mu_0\nu} \qquad \nu \neq \mu_0 \tag{11.12}$$

Note that

$$\tilde{e}^\nu \tilde{e}^\rho + \tilde{e}^\rho \tilde{e}^\nu = -2\eta^{\mu_0\mu_0}\eta^{\nu\rho} \qquad \nu, \rho \neq \mu_0 \tag{11.13}$$

and therefore [4]

$$\begin{aligned} C\ell^0(r_+, s_-) &\cong C\ell(r_+, (s-1)_-) \qquad s \geq 1 \\ C\ell^0(r_+, s_-) &\cong C\ell(s_+, (r-1)_-) \qquad r \geq 1 \end{aligned} \tag{11.14}$$ eq:twoevens

Over the complex numbers we have $\mathbb{C}\ell^0(d) \cong \mathbb{C}\ell(d-1)$.

Example: The even subalgebra of $C\ell(2,0)$ is the algebra of matrices:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{11.15}$$

and is isomorphic to $\mathbb{C}$.

Remarks

• This observation is useful when discussing $\mathbb{Z}_2$-graded representations and when we discuss representations of the Spin group.

•

Exercise

Show that when both $r \geq 1$ and $s \geq 1$ then the two equations in (11.14) are compatible.

Exercise

Show that

$$(C\ell(r_+, s_-))^0 \cong (C\ell(s_+, r_-))^0 \tag{11.16}$$

11.3 $\mathbb{Z}_2$ graded tensor product of Clifford algebras

Since the Clifford algebra is $\mathbb{Z}_2$ graded we can also define a graded tensor product. Of course

$$(C\ell(Q_1)\hat{\otimes}C\ell(Q_2))^0 \cong (C\ell(Q_1))^0 \otimes (C\ell(Q_2))^0 \oplus (C\ell(Q_1))^1 \otimes (C\ell(Q_2))^1 \tag{11.17}$$

as vector spaces (with a similar formula for the odd part).

---

[4] Note that this implies that we must have $C\ell((r+1)_+, s_-) \cong C\ell((s+1)_+, r_-)$ for all $r, s \geq 0$. One can indeed prove this is so using the periodicity isomorphisms and the observation that $C\ell(2_+) \cong C\ell(1_+, 1_-) \cong \mathbb{R}(2)$. Nevertheless, at first site this might seem to be very unlikely since the transverse dimensions are $r-s-1$ and $s-r-1$ and in general are *not* equal modulo 8. Note that the sum of the transverse dimensions is $-2 = 6\text{mod}8$. Thus, we have the pairs $(0,6)$, $(1,5)$, $(2,4)$, and $(3,3)$. One can check from the table that these all do in fact have the same Morita type! Of course, the dimensions are the same, so they must in fact be isomorphic.

The important new point is the sign rule in the Clifford multiplication on the graded tensor product. We define the graded Clifford multiplication to be:

$$(\phi\hat{\otimes}\xi)\cdot(\phi'\hat{\otimes}\xi') := (-1)^{\deg\xi\cdot\deg\phi'}\phi\phi'\hat{\otimes}\xi\xi' \tag{11.18}$$ eq:gradedcliff

Quite generally, if $V = V_1 \oplus V_2$ and $Q = Q_1 \oplus Q_2$ then

$$C\ell(Q) = C\ell(Q_1)\hat{\otimes}C\ell(Q_2) \tag{11.19}$$ eq:grdedtenspr

It follows that

$$C\ell(r_+, s_-) = C\ell(1_+)^{\hat{\otimes}r}\hat{\otimes}C\ell(1_-)^{\hat{\otimes}s} \tag{11.20}$$

Note that (11.19) is *not* true for the ungraded tensor product.

The graded tensor product is not as useful for constructing representations because when one takes a tensor product of matrices one normally uses the ordinary tensor product. This is why we used the ordinary tensor product to construct the structure of the algebras and discuss their representations above.

11.4 $\mathbb{Z}_2$ graded modules

A $\mathbb{Z}_2$-graded module M for the Clifford algebra is a representation which itself a $\mathbb{Z}_2$-graded vector space

$$M = M^0 \oplus M^1 \tag{11.21}$$

such that the Clifford action respects the $\mathbb{Z}_2$ grading. Thus,

$$C\ell(Q)^A \cdot M^B \subset M^{A+B\mathrm{mod}2} \tag{11.22}$$

A $\mathbb{Z}_2$-graded module said to be of graded-dimension $(\mathrm{dim}M^0|\mathrm{dim}M^1)$.

One useful way to think about a grading is the following: A $\mathbb{Z}_2$ grading means we have an operator Π on M with $\Pi^2 = 1$ such that Π anti-commutes with all the Clifford generators γ_i. So, Π can be considered to be another gamma-matrix!

Let us now examine the $\mathbb{Z}_2$-graded modules of the Clifford algebras. We begin with the complex Clifford algebras for simplicity.

It is very useful to speak of the free abelian group generated by Clifford modules. The direct sum of modules is a module so we can let

$$nM := M^{\oplus n} \tag{11.23}$$

for $n > 0$. Then we introduce formally an additive inverse so that we can write

$$nM + n'M = (n + n')M \tag{11.24}$$

for *all* integers $n, n' \in \mathbb{Z}$.

Let $\mathcal{M}_k^c$ denote the free abelian group of graded $\mathbb{C}\ell_k$ modules and $\mathcal{N}_k^c$ denote the free abelian group of ungraded $\mathbb{C}\ell_k$ modules

Before giving the general construction let us look at some low-dimensional examples. It will also be interesting to consider the following problem: We can consider $\mathbb{C}\ell_k^c$ to be a subalgebra of $\mathbb{C}\ell^c_{k+1}$ by identifying the first k generators of the latter algebra to be the generators $e_1, \dots, e_k$ of the former. Therefore, every $\mathbb{C}\ell^c_{k+1}$ module can be considered to be a $\mathbb{C}\ell^c_k$ module, and hence we have a group homomorphism $\iota^* : \mathcal{M}^c_{k+1} \to \mathcal{M}^c_k$. The problem – of great utility in topology - is to determine the abelian group

$$\mathcal{M}^c_k/\iota^*(\mathcal{M}^c_{k+1}) \tag{11.25}$$

(For some idea of why this might be so, see section *** above and section *** below.)

We begin with zero dimensions. $\mathbb{C}\ell(0) = \mathbb{C}$ has one (irreducible) ungraded module, $\mathbb{C}$. We will refer to it as N_0 below. However, $\mathbb{C}\ell(0)$ has *two* irreducible graded modules $M_{0,+}$ of dimension $(1|0)$ and $M_{0,-}$ of dimension $(0|1)$. Thus,

$$\begin{aligned} \mathcal{N}_0 &\cong \mathbb{Z} \\ \mathcal{M}_0 &\cong \mathbb{Z} \oplus \mathbb{Z} \end{aligned} \tag{11.26}$$

Now consider $\mathbb{C}\ell(1) \cong \mathbb{C} \oplus \mathbb{C}$. Choosing the generator so $e_1 e_1 = -1$ the two subalgebras are obtained by the projection operators $P_\pm = \frac{1}{2}(1 \pm ie)$. Thus note that neither subalgebra has homogeneous grading.

Now, $\mathbb{C}\ell(1)$ has two inequivalent ungraded modules, denoted $N_{1,\pm}$ depending on whether we represent e by $\mp i$. That is $N_{1,\pm} = P_\pm C\ell^c_1$. Each module $N_{1,\pm}$ is a one-dimensional complex vector space. On the other hand, $\mathbb{C}\ell(1)$ has only one irreducible graded module. We will denote it M_1. An even vector must be mapped by e_1 to an odd vector, and vice versa. Thus M_1 has dimension $(1|1)$ and is $\mathbb{C} \oplus \mathbb{C}$ as a vectorspace with e_1 represented by

$$\rho(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{11.27}$$

Now the even subalgebra $(\mathbb{C}\ell(1))^0$ is just $\mathbb{C}$, and is generated by the scalars.

It is useful to note that we could define M_1 by the formula

$$M_1 = \mathbb{C}\ell(1) \otimes_{(\mathbb{C}\ell(1))^0} N_0 \tag{11.28}$$

A vector space basis is $v_1 = 1 \otimes 1$ generating the even subspace $(M_1)^0$ and $v_2 = e \otimes 1$, generating the odd subspace $(M_1)^1$, and it is clear that in this basis

$$e \to \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \tag{11.29}$$

Note that, as is required, e is an odd operator - it is off-diagonal. Thus,

$$\begin{aligned} \mathcal{N}_1 &\cong \mathbb{Z} \oplus \mathbb{Z} \\ \mathcal{M}_1 &\cong \mathbb{Z} \end{aligned} \tag{11.30}$$

We can also determine the homomorphism $\iota^* : \mathcal{M}_1 \to \mathcal{M}_0$. Since the generator M_1 has dimension $(1|1)$ the map $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is just $n \to (n, n)$.

Now let us turn to $\mathbb{C}\ell(2) \cong \mathbb{C}(2)$. Clearly, there is only one irreducible ungraded module $N_2 \cong \mathbb{C}^2$. However, there are two inequivalent $\mathbb{Z}_2$ graded modules: $M_2^\pm$. To see this first note that the even subalgebra is $(\mathbb{C}\ell(2))^0 \cong \mathbb{C} \oplus \mathbb{C}$ the isomorphism being given by $P_\pm = \frac{1}{2}(1 \mp i e_1 e_2)$. But there are two inequivalent ungraded modules for $(\mathbb{C}\ell(2))^0$ and these induce inequivalent graded modules via

$$M_2^\pm = \mathbb{C}\ell(2) \otimes_{(\mathbb{C}\ell(2))^0} N_{1,\pm} \tag{11.31}$$

As a vector space $M_2^\pm \cong \mathbb{C}^2$. We can choose a vector space basis $v_1 = 1 \otimes 1$, generating the even part $(M_2^\pm)^0$ and $v_2 = e_1 \otimes 1$, generating the odd part $(M_2^\pm)^1$. Note $e_2 \otimes 1$ is not a linearly independent vector because

$$\begin{aligned} e_2 \otimes 1 &= -e_1(e_1 e_2) \otimes 1 \\ &= -e_1 \otimes e_1 \cdot 1 \qquad \text{definition of tensor product} \\ &= e_1 \otimes \mp i \\ &= \mp i e_1 \otimes 1 \end{aligned} \tag{11.32}$$

A small computation shows that in this basis

$$e_1 \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad\qquad e_2 \to \begin{pmatrix} 0 & \mp i \\ \mp i & 0 \end{pmatrix} \tag{11.33}$$

eq:cliffacttwo

If we normalize the volume form so it squares to 1, $\omega = i e_1 e_2$, then, although both the graded and ungraded modules are $\mathbb{C}^2$ as vector spaces, the two inequivalent graded modules are distinguished by the sign of ω acting on the ungraded component $(M_2^\pm)^0$.

Thus

$$\begin{aligned} \mathcal{N}_2 &\cong \mathbb{Z} \\ \mathcal{M}_2 &\cong \mathbb{Z} \oplus \mathbb{Z} \end{aligned} \tag{11.34}$$

Moreover, we can determine $\iota^* : \mathcal{M}_2 \to \mathcal{M}_1$ easily since (11.33) gives the action of e_1, on $M_{2,\pm}$ which is identical to that of $\mathbb{C}\ell_1^c$ on M_1. Thus the map is $(1,0) \to 1$ and $(0,1) \to 1$.

Finally, let us look at $\mathbb{C}\ell(3) \cong \mathbb{C}(2) \oplus \mathbb{C}(2)$, with projectors $P = \frac{1}{2}(1 \mp e_1 e_2 e_3)$. The even subalgebra $(\mathbb{C}\ell(3))^0 \cong \mathbb{C}(2)$ is generated by, say, $\hat{e}_1 = e_1 e_2$ and $\hat{e}_2 = e_2 e_3$. Clearly $\mathbb{C}\ell(3)$ has two ungraded modules, isomorphic to $\mathbb{C}^2$ as vector spaces. We can call these $N_{3,\pm}$. Explicitly, they are represented by $e_j \to \pm i \sigma^j$ where σ^j are Pauli matrices. The two different modules differ by the *sign* of the volume element. On the other hand, there is only one graded module, which we can again take to be

$$M_3 = \mathbb{C}\ell(3) \otimes_{(\mathbb{C}\ell(3))^0} N_2 \tag{11.35}$$

M_3 is $\mathbb{C}^4$ as a vector space. to be explicit we can choose a basis v_1, v_2 for N_2 as above, then we have a representation of $\hat{e}_1 = e_1 e_2, \hat{e}_2 = e_2 e_3$, as above. Then we choose

$$w_1 = 1 \otimes v_1, w_2 = 1 \otimes v_2, w_3 = e_1 \otimes v_1, w_4 = e_1 \otimes v_2 \tag{11.36}$$

Note that

$$e_2 \otimes v_1 = -e_1 e_1 e_2 \otimes v_1 = -e_1 \otimes (e_1 e_2) v_1 = -e_1 \otimes v_2 \tag{11.37}$$

and in a similar fashion we get

$$\begin{aligned} e_2 \otimes v_2 &= e_1 \otimes v_1 \\ e_3 \otimes v_1 &= -ie_1 \otimes v_1 \\ e_3 \otimes v_2 &= -ie_1 \otimes v_2 \end{aligned} \tag{11.38}$$

so we compute

$$\begin{aligned} e_1 : w_1 &\to w_3 \\ w_2 &\to w_4 \\ w_3 &\to -w_1 \\ w_4 &\to -w_2 \end{aligned} \tag{11.39}$$

etc.

Exercise
Show that restriction of M_3 as a graded $\mathbb{C}\ell(2)$ module is

$$\iota^*(M_3) = M_{2,+} \oplus M_{2,-} \tag{11.40}$$

Now we turn to the general story:
$\mathbb{C}\ell(2n) \cong \mathbb{C}(2^n)$ has
1. Unique ungraded irrep N_{2n}, $\cong \mathbb{C}^{2^n}$ as a vector space.
2. Two inequivalent graded irreps $M_{2n,\pm}$, $\cong \mathbb{C}^{2^{n+1}}$ as vector spaces. [??? Aug. 28, 2011: Dimension seems wrong. ???].
while $\mathbb{C}\ell(2n+1) = \mathbb{C}(2^n) \oplus \mathbb{C}(2^n)$ has
1. Two inequivalent ungraded irreps $N_{2n+1,\pm}$, $\cong \mathbb{C}^{2^n}$ as vector spaces.
2. A unique graded irrep M_{2n+1}, $\cong \mathbb{C}^{2^{n+1}}$ as a vector space.
That is:

$$\begin{aligned} \mathcal{N}_{2k} &\cong \mathbb{Z} \\ \mathcal{N}_{2k+1} &\cong \mathbb{Z} \oplus \mathbb{Z} \end{aligned} \tag{11.41}$$

$$\begin{aligned} \mathcal{M}_{2k} &\cong \mathbb{Z} \oplus \mathbb{Z} \\ \mathcal{M}_{2k+1} &\cong \mathbb{Z} \end{aligned} \tag{11.42}$$

Moreover, if $\iota : \mathbb{C}\ell(n) \to \mathbb{C}\ell(n+1)$ then we examine $\iota^*(M)$ or $\iota^*(N)$, the restriction of the module to the Clifford subalgebra, and we have

$$\iota^*(M_{2n+1}) = M_{2n,+} \oplus M_{2n,-} \tag{11.43}$$

$$\iota^*(M_{2n,\pm}) = M_{2n-1} \tag{11.44}$$

If M is a graded module for $\mathbb{C}\ell(k+1)$ then M^0 is a graded module for $(\mathbb{C}\ell(k+1))^0 \cong \mathbb{C}\ell(k)$. The inverse relation is that if N is an ungraded module for $\mathbb{C}\ell(k)$ then we can produce a graded module for $\mathbb{C}\ell(k+1)$ by

$$M = \mathbb{C}\ell(k+1) \otimes_{(\mathbb{C}\ell(k+1))^0} N \tag{11.45}$$

Remarks

• To make contact with our discussion of representations in section ****, note the following: For d odd, ω^c is central and there are two inequivalent representations of $\mathbb{C}\ell(d)$ depending on the sign of ω^c. For d even, there is only one inequivalent representation. These are the ungraded representations. For the graded representations we can look at the even subalgebra acting on the even space M^0. Now for d even, the even subalgebra has a volume form ω_c^0 which is central in the even subalgebra and hence acting on the even part of the module, M^0, has ω_c is represented by a scalar ± 1. This defines the two kinds of inequivalent graded modules. For d odd the even subalgebra has no such central term and there is only one inequivalent graded irreducible graded module.

• There is an analogous, but more intricate, discussion for the modules of the real Clifford algebra.

11.5 K-theory over a point

****

Explain about brane/antibrane and how invertible tachyon fields produce annihilation.
****

One can thus consider $\mathbb{Z}_2$-graded $\mathbb{C}\ell_n^c$-modules with an odd anti-hermitian operator.

Need to mod out by invertible operators, and homotopy relation....

Thus we consider the space

$$K^{-j}(pt) = \mathcal{M}_j/\iota^*(\mathcal{M}_{j+1}) \tag{11.46}$$

So $M_{2n,+} \sim -M_{2n,-}$ in K-theory

From the above

$$K^{-j}(pt) = \begin{cases} \mathbb{Z} & j = 0 \mathrm{mod} 2 \\ 0 & j = 1 \mathrm{mod} 2 \end{cases} \tag{11.47}$$

11.6 Graded tensor product of modules and the ring structure

and $M_{2n,+} \cong (M_{2,+})^{\otimes n}$ as tensor products of $\mathbb{Z}_2$-graded modules. So $M_{2,+}$ is the Bott element $\pm u^{-1}$.

Moreover, as we have seen $C\ell^0((r+1)_+) \cong C\ell(r_+)$. Thus we can list the free abelian groups generated by irreducible $\mathbb{Z}_2$ graded modules as:

11.6.1 The Grothendieck group

Now describe Z_2-graded modules modulo those which extend to Z_2 graded modules one dimension higher.

Tensor product of $\mathbb{Z}_2$-graded modules. Ring structure of the Grothendieck group.

12. Clifford algebras and the division algebras

There is an intimate relation between Clifford algebras and the division algebras. See Kugo-Townsend. Also see

A. Sudbury, “Division algebras, (pseudo)orthogonal groups and spinors,” J. Phys. A Math. Gen. **17** (1984) 939.

Interesting constructions of the Clifford algebras for $Spin(7)$ and $Spin(8)$ using the mutliplication table of the octonions can be found in:

M. Gunaydin and F. Gursey, J. Math. Phys. **14** (1973)1651

R. Dundarer, F. Gursey, and C.H. Tze, J. Math. Phys. **25** (1984)1496

13. Some sources

Some references:

1. E. Cartan, *The theory of Spinors*

2. Chevalley,

2'. P. Deligne, “Notes on spinors,” in *Quantum Fields and Strings: A Course for Mathematicians*

3. One of the best treatments is in Atiyah, Bott, and Shapiro, “Clifford Modules”

4. A textbook version of the ABS paper can be found in Lawson and Michelson, *Spin Geometry*, ch.1

5. Freund, Introduction to Supersymmetry

6. M. Sohnius, “Introducing Supersymmetry” Phys. Rept.

7. T. Kugo and P. Townsend, “Supersymmetry and the division algebras,” Nuc. Phys. **B221** (1983)357.

8. M. Rausch de Traubenberg, “Clifford Algebras in Physics,” arXiv:hep-th/0506011.