Physics 511: Problem Set 5

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ABSTRACT: Due Friday Dec. 19

-TOC- Contents

Instructions:

1. Late problem sets are absolutely not accepted. Get started soon.
2. I strongly prefer getting a pdf copy. It would be nice if you could tex your answers. If you cannot do this write very clearly and give it to the NHETC secretary Diane Soyak, before she leaves work on Friday Dec. 19, so she can send me a pdf copy.
3. In general (but not always) each problem is graded so that each section counts for 10 points. You get an extra 30 points for sending me a texed pdf version with filename: Yourlastname-Yourfirstname-PS5.pdf. If you do not tex it you do not get the extra 30 points.

   (If you cannot make figures on the computer then draw them and scan to pdf. The NHETC secretary Diane Soyak can help you do that. If you make computer-drawn figures - great!)
4. There are many many more exercises in the notes. I strongly encourage you to do as many as you can! (But I will not grade them.)

1. Jordan's theorem

Suppose G is finite and acts transitively on a finite set X with more than one point. Show that there is an element $g \in G$ with no fixed points on X.

2. Induced representations

Let G be the symmetric group on $\{1, 2, 3\}$ and let $H = \{1, (12)\}$. Choose a representation of H with $V \cong \mathbb{C}$ and $\rho(\sigma) = +1$ or $\rho(\sigma) = -1$ where $\sigma = (12)$.

a.) Show that in either case, the induced representation $\mathrm{Ind}_H^G(V)$ is a three-dimensional vector space.

b.) Choose a basis for $\mathrm{Ind}_H^G(V)$ and compute the representation matrices of the elements of S_3 explicitly.

3. Useful formulae for homogeneous spaces

Let $\pi \in M_{n_1\times n_2}(\mathbb{C})$ be an $n_1 \times n_2$ complex matrix.

a.) Show that

$$\exp\begin{pmatrix} 0 & \pi \\ -\pi^\dagger & 0 \end{pmatrix} = \begin{pmatrix} \cos\left(\sqrt{\pi\pi^\dagger}\right) & \pi\frac{\sin\left(\sqrt{\pi^\dagger\pi}\right)}{\sqrt{\pi^\dagger\pi}} \\ -\pi^\dagger\frac{\sin\left(\sqrt{\pi\pi^\dagger}\right)}{\sqrt{\pi\pi^\dagger}} & \cos\left(\sqrt{\pi^\dagger\pi}\right) \end{pmatrix} \tag{3.1}$$

eq:CompactExp

b.) In particular, show that for z a complex number we have

$$\exp\begin{pmatrix} 0 & -\bar{z} \\ z & 0 \end{pmatrix} = \frac{1}{\sqrt{1+|u|^2}}\begin{pmatrix} 1 & -\bar{u} \\ u & 1 \end{pmatrix} \tag{3.2}$$

with

$$u = z\frac{\tan|z|}{|z|} \tag{3.3}$$

c.) Show that $SU(2)/U(1) \cong \mathbb{CP}^1$ by considering the linear action of $SU(2)$ on homogeneous coordinates of $\mathbb{CP}^1$. Using this action show that we can interpret the parameter u in part (b) as the stereographic projection of the sphere to the complex plane.

d.) Show that for $G = SO(n+1)$ taking $\pi = \sum_{i=1}^n \theta^i p_i$ we have

$$\exp[\pi]\vec{x}_0 = \begin{pmatrix} \frac{\sin|\theta|}{|\theta|}\theta^i \\ \cos|\theta| \end{pmatrix} \tag{3.4}$$

where $p_i = e_{i,n+1} - e_{n+1,i}$, $|\theta| = \sqrt{\sum(\theta^i)^2}$ and $\vec{x}_0$ has all components zero except for the last one, which is one.

e.) Show that

$$\exp\begin{pmatrix} 0 & \pi \\ \pi^\dagger & 0 \end{pmatrix} = \begin{pmatrix} \cosh\left(\sqrt{\pi\pi^\dagger}\right) & \pi\frac{\sinh\left(\sqrt{\pi^\dagger\pi}\right)}{\sqrt{\pi^\dagger\pi}} \\ \pi^\dagger\frac{\sinh\left(\sqrt{\pi\pi^\dagger}\right)}{\sqrt{\pi\pi^\dagger}} & \cosh\left(\sqrt{\pi^\dagger\pi}\right) \end{pmatrix} \tag{3.5}$$

eq:NonCompactExp

f.) Consider the Lorentz group on $\mathbb{M}^{1,d-1}$. Using (e) compute the Lorentz matrix corresponding to a boost of rapidity β in the $\hat{n}$ direction.

4. Sine-Gordon model

Consider the sine-Gordon model, a $1+1$-dimensional field theory with a field $\phi : \mathbb{M}^{1,1} \to \mathbb{R}$ with Hamiltonian

$$H = \int_{\mathbb{R}} dx\left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x\phi)^2 + \frac{m^2}{\beta^2}(1-\cos(\beta\phi))\right) \tag{4.1}$$

Here π is the momentum of the field. Under the Legendre transform to the Lagrangian formulation we have $\dot{\phi} = \pi$.

a.) Find the classical vacua, that is, the field configurations with $H[\phi] = 0$.

b.) Compute the topological charge between a pair of two such vacua.

c.) Expand the potential $U(\phi)$ around a vacuum. (The quadratic term gives the mass of the perturbative particles.)

d.) Find an exact formula for the minimum energy field configuration $\phi(x)$ interpolating between two consecutive vacua.

e.) Compute the energy density of the field configuration of part (d).

f.) Is there a time-independent solution with topological charge $4\pi/\beta$?

g.) Show that the sine-Gordon equation of motion for a field $\phi(x,t)$ is equivalent to the "zero-curvature condition":

$$[D_x, D_t] = 0 \tag{4.2}$$

where D_x and D_t are matrix-valued differential operators:

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + k\cos(\phi/2)\tau^1 + \omega\sin(\phi/2)\tau^2 + \frac{1}{2}(\partial_t\phi)\tau^3 \\ D_t &= \frac{\partial}{\partial t} + \omega\cos(\phi/2)\tau^1 + k\sin(\phi/2)\tau^2 + \frac{1}{2}(\partial_x\phi)\tau^3 \end{aligned} \tag{4.3}$$

Here $\omega^2 - k^2 = 1$, and $\tau^a = -\frac{i}{2}\sigma^a$ so that

$$[\tau^a, \tau^b] = \epsilon^{abc}\tau^c \tag{4.4}$$

Remark: This is an important observation and is the beginning of the application of integrable systems theory to the sine-Gordon model. See, for example, the book of Faddeev and Takhtadjan.