Physics 511: Problem Set 4

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ABSTRACT: Due Friday, Dec. 5, 2014

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Instructions:

1. Late problem sets are not accepted. Get started soon.
2. I strongly prefer getting a pdf copy. It would be nice if you could tex your answers. If you cannot do this write very clearly and hand it in by beginning of class December 5.
3. In general (but not always) each problem is graded so that each section counts for 10 points. You get an extra 30 points for sending me a texed pdf version with filename: Yourlastname-Yourfirstname-PS4.pdf. In the subject heading please put your name and "Physics511 PS4".

   (If you cannot make figures on the computer then draw them and scan to pdf. The NHETC secretary Diane Soyak can help you do that. If you make computer-drawn figures - great!)
4. There are many many more exercises in the notes. I strongly encourage you to do as many as you can! (But I will not grade them.)

1. The Hessian

Let $f : M \to \mathbb{R}$ where M is a smooth manifold and f is a differentiable function. Consider the matrix of second derivatives

$$\frac{\partial^2 \tilde{f}}{\partial x^i \partial x^j} \tag{1.1}$$

where $\tilde{f} = f \circ \phi^{-1}$ and (U, ϕ) is a coordinate chart near p.

a.) Compute how the matrix changes under a change of coordinates.

b.) How does the formula simplify when x is a critical point of f? The matrix of second derivatives at a critical point is known as the *Hessian* of f.

2. Immersions

a.) Show that the map $f : \mathbb{RP}^2 \to \mathbb{R}^3$ defined by

$$f([X^1 : X^2 : X^3]) = \frac{(X^2X^3, X^1X^3, X^1X^2)}{(X^1)^2 + (X^2)^2 + (X^3)^2} \tag{2.1}$$

is well-defined and smooth.

b.) Where does it fail to be an immersion?

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3. Structure constants

Let T^a be a vector space basis for a Lie algebra $\mathfrak{g}$. Then there must be elements $f^{ab}_c \in \kappa$ - known as *structure constants* - such that

$$[T^a, T^b] = f^{ab}_c T^c \tag{3.1}$$

a.) Show that $f^{ab}_c = -f^{ba}_c$

b.) Write out the identity on the f^{ab}_c implied by the Jacobi identity.

4. Lie groups from indefinite forms

In general, if Q is a quadratic form on a vector space over κ then $O(Q)$ is the automorphism group of the quadratic form. If h is an Hermitian form on a complex vector space then $U(h)$ is the group of complex linear automorphisms of h.

Consider the particular example of the matrix:

$$\eta_{p,q} = \begin{pmatrix} +\mathbf{1}_{p\times p} & 0 \\ 0 & -\mathbf{1}_{q\times q} \end{pmatrix} \tag{4.1}$$

where p, q are positive integers. Let $O(p, q)$ be the subgroup of $A \in GL(n; \mathbb{R})$ with $n = p+q$ such that $A\eta_{p,q}A^{tr} = \eta_{p,q}$.

a.) Show that $O(p, q)$ is a Lie group. The special case $O(1, 3)$, or $O(3, 1)$ is the Lorentz group in four-dimensional Minkowski spacetime.

Now let $U(p, q)$ be the subgroup of $A \in GL(n; \mathbb{C})$ such that $A\eta_{p,q}A^\dagger = \eta_{p,q}$.

b.) Show that $U(p, q)$ is a Lie group.

5. Gradient flow

Consider the gradient flow on $\mathbb{R}^n$ for a real-valued potential function U:

$$\frac{dx^i}{dt} = -\frac{\partial U}{\partial x^i} \tag{5.1}$$

$\boxed{\texttt{eq:Gradient-Flow}}$

a.) Show that U is nonincreasing on any flow.

b.) For which initial conditions is it strictly decreasing?

c.) Let $U(x)$ be a polynomial in one variable such that $U'(x)$ has n real roots. Describe the gradient flows.