Preprint typeset in JHEP style - HYPER VERSION

Physics 511: Problem Set 3

Gregory W. Moore

ABSTRACT: Due November 14, 2014

-TOC- Contents

Instructions:

1. Late problem sets are not accepted. Get started soon.
2. I strongly prefer getting a pdf copy. It would be nice if you could tex your answers. If you cannot do this write very clearly and hand it in by beginning of class Nov. 14
3. In general (but not always) each problem is graded so that each section counts for 10 points. You get an extra 30 points for sending me a texed pdf version with filename: Yourlastname-Yourfirstname-PS2.pdf

   (If you cannot make figures on the computer then draw them and scan to pdf. The NHETC secretary Diane Soyak can help you do that. If you make computer-drawn figures - great!)
4. There are many many more exercises in the notes. I strongly encourage you to do as many as you can! (But I will not grade them.)

1. Path-ordered exponential vs. ordinary exponential

Let $x(t) = 3t$ and suppose that

$$A(x) = \begin{cases} -\alpha\sigma^3 & 0 \le x \le 1 \\ \beta\sigma^1 & 1 \le x \le 2 \\ \alpha\sigma^3 & 2 \le x \le 3 \end{cases} \tag{1.1}$$

where α and β are complex numbers.

Evaluate the path-ordered exponential and the exponential of the integral

$$\exp[\int_0^1 A(x(t))\dot{x}(t)dt] \tag{1.2}$$

and compare the answers.

2. Holonomy and curvature

Let $A_\mu(x)$ be a collection of m complex $N \times N$ matrix-valued differentiable functions on $\mathbb{R}^m$.

Consider the path $\wp = \wp_1 \star \wp_2 \star \wp_3 \star \wp_4$ where

$$\begin{aligned} \wp_1(t) &= \vec{x}_0 + \epsilon_1(t)\vec{e}_\mu \\ \wp_2(t) &= (\vec{x}_0 + \epsilon_1\vec{e}_\mu) + \epsilon_2(t)\vec{e}_\nu \\ \wp_3(t) &= (\vec{x}_0 + \epsilon_2\vec{e}_\nu) + \epsilon_1(1-t)\vec{e}_\mu \\ \wp_4(t) &= \vec{x}_0 + +\epsilon_2(1-t)\vec{e}_\nu \end{aligned} \tag{2.1}$$

Here $\vec{e}_\mu$ is a unit vector pointing in the x^μ direction and $\epsilon_i := \epsilon_i(t=1)$. Show that, for small ϵ_1, ϵ_2 we have the expansion:

$$\boxed{\mathbb{U}(\wp) = 1 - \epsilon_1\epsilon_2 F_{\mu\nu}(\vec{x}_0) + \cdots} \tag{2.2}$$ eq:TransportCurva

$$\boxed{F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]} \tag{2.3}$$ eq:DefinitionCurv

and the higher order terms in (2.2) are all of order $\epsilon_1^a \epsilon_2^b$ with $a > 0$ and $b > 0$ and $a + b > 2$.

3. Gauge transformations

Let $A(x)$ be a matrix-valued $n \times n$ complex matrix on $\mathbb{R}$ and and $x \mapsto g(x)$ a differentiable map from $\mathbb{R}$ to $GL(n, \mathbb{C})$. Define a new matrix-valued function $\tilde{A}(x)$ by

$$A(x) = g(x)^{-1}\tilde{A}(x)g(x) + g(x)^{-1}\frac{d}{dx}g(x) \tag{3.1}$$

a.) Show that

$$d + \tilde{A} = g(x)(d + A)g(x)^{-1} \tag{3.2}$$

where $d = dx^\mu \frac{\partial}{\partial x^\mu} 1_{N\times N}$ is a first order differential operator and $A = dx^\mu A_\mu$.

b.) Show that, for any piecewise-differentiable path $x(t)$ from x_0 to x_1 we have

$$\text{Pexp}\left[-\int_0^1 \tilde{A}(x(t))\dot{x}(t)dt\right] = g(x_1)\text{Pexp}\left[-\int_0^1 A(x(t))\dot{x}(t)dt\right]g(x_0)^{-1} \tag{3.3}$$

c.) Show by direct computation that, if $\tilde{F}_{\mu\nu}$ is computed from $\tilde{A}_\mu$ then

$$F_{\mu\nu}(x) = g(x)^{-1}\tilde{F}_{\mu\nu}(x)g(x) \tag{3.4}$$

d.) Show that the commutator of matrix-valued first order differential operators gives the curvature:

$$[D_\mu, D_\nu] = F_{\mu\nu} \tag{3.5}$$

Use this to give another proof of the gauge transformation rule of part (c).

e.) Suppose $\Phi(x)$ and $\tilde{\Phi}(x)$ are matrix valued functions of x^μ related by $\tilde{\Phi}(x) = g(x)\Phi(x)g(x)^{-1}$. Show by direct computation that

$$\tilde{D}_\lambda\tilde{\Phi}(x) = g(x)D_\lambda\Phi(x)g(x)^{-1} \tag{3.6}$$

where $\tilde{D}_\lambda$ is the covariant derivative computed with $\tilde{A}_\lambda$.

f.) For a matrix-valued field $\Phi(x)$ define an operator $\mathrm{Ad}(\Phi)$ on matrix valued fields by

$$\mathrm{Ad}(\Phi) : \Psi \mapsto [\Phi, \Psi] \tag{3.7}$$

Show that, acting on such matrix valued fields Ψ we have

$$[D_\mu, \mathrm{Ad}(\Phi)] = \mathrm{Ad}(D_\mu \Phi) \tag{3.8}$$

Use this to give another proof of the gauge transformation rule in (e).

4. Connected Sum

The following operation on two manifolds M_1, M_2 of the same dimension is called the *connected sum*. Choose points $p_i \in M_i$ and remove a small ball B_i from around each point. Now $M_i - B_i$ is a manifold with boundary given by the sphere ∂B_i. Glue the two spheres together to produce a new manifold $M_1 \# M_2$.

a.) What is the dimension of $M_1 \# M_2$?

b.) If M_1 is of dimension n show that $M_1 \# S^n \cong M_1$.

c.) Show that a connected sum of a torus and $\mathbb{R}P^2$ is equivalent to a connected sum of three copies of $\mathbb{R}P^2$.

Remark: A beautiful and nontrivial theorem says that under connected sum all orientable 3-manifolds have a unique "prime decomposition." Reference: J. Milnor, A unique decomposition theorem for 3-manifolds, American Journal of Mathematics 84 (1962), 17.

5. Moduli space of hyperplanes

Consider the set of all n-dimensional linear subspaces of the vector space $\mathbb{F}^{n+1}$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$. (Two separate cases.)

a.) Show that these manifolds are diffeomorphic to certain manifolds discussed in the notes.

b.) What is the dimension of these manifolds? (Caution: Trick question.)

6. Orientability of the tangent bundle

Let M be a manifold of dimension n. Show that TM is an orientable manifold of dimension $2n$.